R-Mode Constraints On Neutron Star Equation of State

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Outline

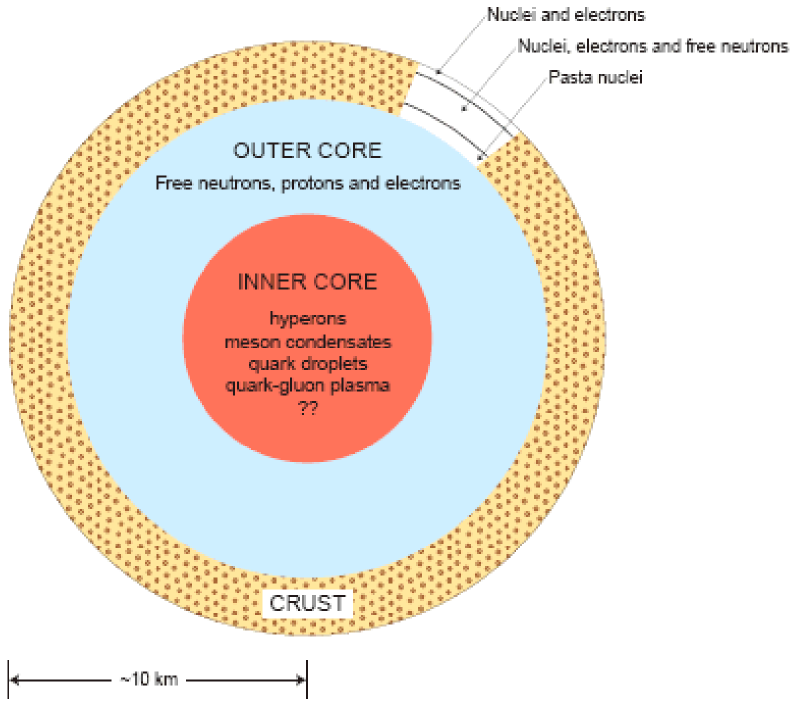
A. Introduction on the r-mode instability
of neutron stars

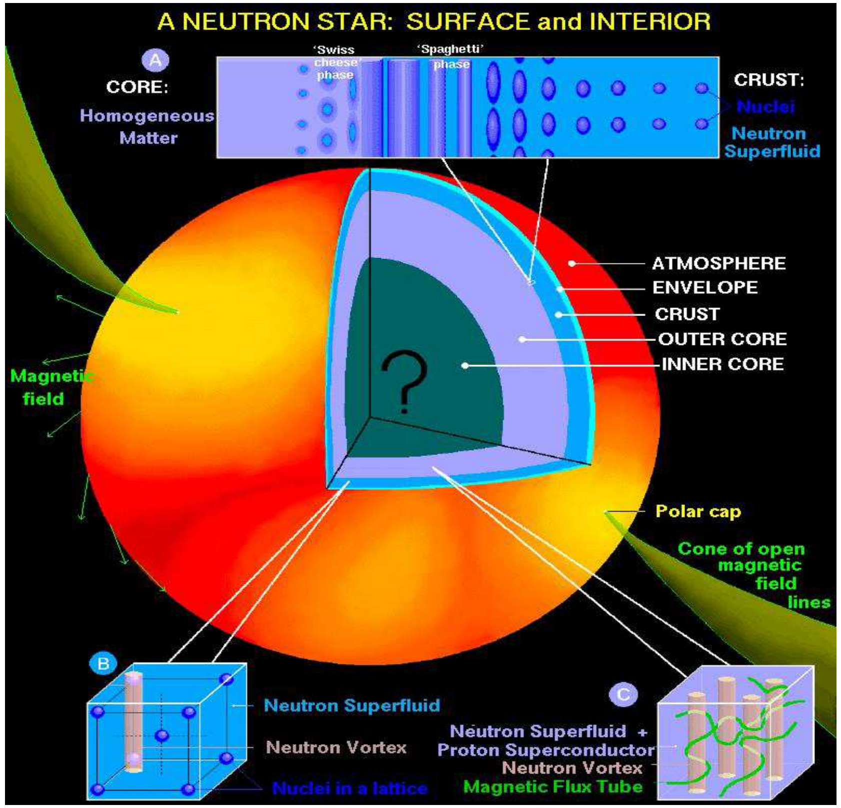
B. R-mode instability formalism

C. Nuclear equation of state relative to r-mode
studies

D. Analytical solutions of Einstein equations

E. Presentation of the results





Properties of Neutron Stars

Radius: R~10km
Mass: M~1.4–2.5Msun
Mean density: ρ(r)~4x10^14 g/cm3
Period: T~ms– a few s
Magnetic field: B~108–1015 G

Schematic mechanism for spin down of rotating neutron star

Rotating newborn neutron star:

→ **random density or velocity oscillation in nuclear fluid**

→ **r-mode instability (if r-mode unstable growth)**

→ **emission of gravitational waves (curry away angular momentum)**

→ **rotating neutron star spin down over time**

The time dependence of an r-mode instability is given by

$$e^{i\omega t - t/\tau} \tag{39}$$

where ω is the frequency of the mode and τ is the overall time scale of the mode which describes both its exponential growth, driven by the CFS mechanism and its decay due to viscus damping and can be written as

$$\frac{1}{\tau(\Omega,T)} = \frac{1}{\tau_{GR}(\Omega)} + \frac{1}{\tau_v(\Omega,T)} \tag{40}$$

The r-mode will be unstable only when $1/\tau$ is negative that is when τ_{GR} is shorter than τ_v.

The critical angular velocity Ω_c defined by the equation

$$\frac{1}{\tau(\Omega_c,T)} = \frac{1}{\tau_{GR}(\Omega_c)} + \frac{1}{\tau_v(\Omega_c,T)} = 0 \tag{41}$$

In total

$$\begin{array}{ll} \Omega > \Omega_c & \mathrm{unstable\ r-mode} \\ \Omega < \Omega_c & \mathrm{stable\ r-mode} \end{array}$$

The relevant time scales

$$\frac{1}{\tau(\Omega,T)} = \frac{1}{\tau_{GR}(\Omega)} + \frac{1}{\tau_{EL}(\Omega,T)} + \frac{1}{\tau_{BV}(\Omega,T)} + \frac{1}{\tau_{SV}(\Omega,T)} + \text{additional terms}$$

$$\frac{1}{\tau(\Omega,T)} = \frac{1}{\tilde{\tau}_{GR}}\left(\frac{\Omega}{\Omega_0}\right)^{2l+2} + \frac{1}{\tilde{\tau}_{SV}}\left(\frac{10^9K}{T}\right)^{2} + \frac{1}{\tilde{\tau}_{BV}}\left(\frac{T}{10^9K}\right)^{6}\left(\frac{\Omega}{\Omega_0}\right)^{2} + \frac{1}{\tilde{\tau}_{EL}}\left(\frac{10^8K}{T}\right)\left(\frac{\Omega}{\Omega_0}\right)^{1/2}$$

$$\frac{1}{\tau_i} \equiv -\frac{1}{2E}\left(\frac{dE}{dt}\right)_i.$$

$$E = \frac{1}{2}\alpha^2 R^{-2m+2}\Omega^2 \int_0^R \rho(r) r^{2m+2} dr,$$

The relevant time scales

$$\frac{1}{\tau(\Omega,T)} = \frac{1}{\tau_{GR}(\Omega)} + \frac{1}{\tau_{EL}(\Omega,T)} + \frac{1}{\tau_{BV}(\Omega,T)} + \frac{1}{\tau_{SV}(\Omega,T)} + \text{additional terms}$$

$$\frac{1}{\tau_{GR}} = -\frac{32\pi G\Omega^{2l+2}}{c^{2l+3}} \frac{(l-1)^{2l}}{[(2l+1)!!]^2} \left(\frac{l+2}{l+1}\right)^{2l+2} \int_0^R \rho(r) r^{2l+2} dr.$$

$$\frac{1}{\tau_{BV}} = \frac{4\pi}{690} \left(\frac{\Omega}{\Omega_0}\right)^4 R^{2l-2} \left(\int_0^R \rho(r) r^{2l+2} dr\right)^{-1} \int_0^R \xi_{BV} \left(\frac{r}{R}\right)^6 \left[1 + 0.86 \left(\frac{r}{R}\right)^2\right] r^2 dr$$

$$\xi_{BV} = 6.0 \times 10^{-59} \left(\frac{l+1}{2}\right)^2 \left(\frac{\mathrm{Hz}}{\Omega}\right)^2 \left(\frac{\rho}{\mathrm{gr\ cm^{-3}}}\right)^2 \left(\frac{T}{\mathrm{K}}\right)^6 \quad (\mathrm{gr\ cm^{-1}\ s^{-1}})$$

$$\frac{1}{\tau_{SV}} = (l-1)(2l+1) \left(\int_0^R \rho(r) r^{2l+2} dr\right)^{-1} \int_0^R \eta_{SV} r^{2l} dr, \quad (\mathrm{s^{-1}})$$

$$\eta_{nn} = 347 \left(\frac{\rho}{\mathrm{gr\ cm^{-3}}}\right)^{9/4} \left(\frac{T}{\mathrm{K}}\right)^{-2}, \qquad (\mathrm{g\ cm^{-1}\ s^{-1}}).$$

$$\eta_{ee} = 6.0 \cdot 10^6 \left(\frac{\rho}{\mathrm{gr\ cm^{-3}}}\right)^2 \left(\frac{T}{\mathrm{K}}\right)^{-2}, \qquad (\mathrm{g\ cm^{-1}}\ s^{-1}),$$

The case without crust consideration

$$-\left(\frac{\Omega_c}{\mathrm{Hz}}\right)^6 + a\left(\frac{\Omega_c}{\mathrm{Hz}}\right)^2 + b = 0$$

$$\Omega_c = \begin{cases} \left(\frac{b}{2}\right)^{1/6}\left[\left(1+\sqrt{1-\mathcal{Y}}\right)^{1/3} + \left(1-\sqrt{1-\mathcal{Y}}\right)^{1/3}\right]^{1/2}, & \mathcal{Y} \le 1 \\ \left(4b\sqrt{\mathcal{Y}}\right)^{1/6}\sqrt{\cos\left[\frac{1}{3}\tan^{-1}\left(\sqrt{\mathcal{Y}-1}\right)\right]}, & \mathcal{Y} \ge 1 \end{cases}$$

where $\mathcal{Y} = \frac{4a^3}{27b^2}$ and also

$$a = 3.237\cdot 10^4 \left(\frac{10\mathrm{km}}{R}\right)^3 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{T}{10^9\mathrm{K}}\right)^6 \frac{\mathcal{I}_2}{\mathcal{I}_1^2}$$

$$b = 2.265\cdot 10^{15} \left(\frac{10^9\mathrm{K}}{T}\right)^2 \left(\frac{10\mathrm{km}}{R}\right)^9 \frac{1}{\mathcal{I}_1^2}\left[\left(\frac{\rho_c}{10^{16}\mathrm{gr\ cm^{-3}}}\right)^{1/4}\mathcal{I}_3^{nn} + 1.729\mathcal{I}_3^{ee}\right]$$

The case with crust

$$\tau_{EL} = \frac{1}{2\Omega}\frac{2^{l+3/2}(l+1)!}{l(2l+1)!!C_l}\sqrt{\frac{2\Omega R_c^2\rho_{cr}}{\eta_{cr}}}\int_0^{R_c}\frac{\rho(r)}{\rho_{cr}}\left(\frac{r}{R_c}\right)^{2l+2}\frac{dr}{R_c},$$

$$-\left(\frac{\Omega_c}{\mathrm{Hz}}\right)^6 + \tilde{a}\left(\frac{\Omega_c}{\mathrm{Hz}}\right)^2 + \tilde{d}\left(\frac{\Omega_c}{\mathrm{Hz}}\right)^{1/2} + \tilde{b} = 0$$

We study the case of the crust elasticity via the slippage factor S:

- **S=1 (complete rigid crust)**
- **S<1 (the crust exhibits elasticity)**
- **Typical theoretical values of S=0.05-0.1**

The S factor included via the replacement

$$\tau_{EL}^{S} \rightarrow \frac{\tau_{EL}}{S^2}$$

Nuclear equation of state

EOS

$$E(n,x) \simeq E(n,x=0) + E_{sym}(n)(1-2x)^2,$$

$$E(n,x=0) \simeq -16 + \frac{\mathcal{K}}{18}\left(1-\frac{n}{n_s}\right)^2 + \frac{\mathcal{L}}{27}\left(1-\frac{n}{n_s}\right)^3$$

$$E_e = (3/4)\hbar c x (3\pi^2 n x^4)^{1/3},$$

$$\mathcal{E}(n,x) = E(n,x) + E_e(n,x)$$

$$P(n,x) = n^2 \frac{\partial \mathcal{E}}{\partial n}$$

$$P(n_s,x_s) = n_s\left[\frac{L}{3}(1-2x_s)^2 + x_s J(1-2x_s)\right]$$

$$J = E_{sym}(n_s)$$

$$P(n_s,x_s) \simeq n_s \frac{L}{3}$$

$$L = 3n_s \frac{\partial E_{sym}(n)}{\partial n}|_{n_s}$$

Lattimer and Prakash empirical relation

$$R(M) = C(n,M)\left[\frac{P(n)}{\mathrm{MeV\ fm^{-3}}}\right]^{1/4}$$

$$P(n) \simeq \left[\frac{R}{C(n,M)}\right]^4$$

$$C(n_s, 1.4M_\odot) = 9.52 \pm 0.49\ \mathrm{km}$$

Analytical solutions of Einstein equations

For a static spherical symmetric system, the metric can be written as follow

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

$$\frac{8\pi G}{c^2}\rho(r) = \frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\lambda'(r)}{r}$$

$$\frac{8\pi G}{c^4}P(r) = -\frac{1}{r^2}\left(1 - e^{-\lambda(r)}\right) + e^{-\lambda(r)}\frac{\nu'(r)}{r}$$

Tolman-Oppenheimer-Volkoff equations (TOV)

$$\frac{dP(r)}{dr} = -\frac{G\rho(r)M(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi P(r)r^3}{M(r)c^2}\right)\left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2\rho(r)$$

Neutron star solutions

Tolman VII

$$\rho(r) = \rho_c \left(1 - \frac{r^2}{R^2}\right), \quad \rho_c = \frac{15M}{8\pi R^3}$$

Buchdahl's solution

$$\rho = 12\sqrt{P^* P} - 5P$$

Nariai IV solution

$$\frac{G}{c^2}\rho(r') = \frac{\sqrt{3\beta}}{4\pi R'^2(1-2\beta)} \frac{C^2}{E^2} \left[3 \sin \tilde{f}(r') \cos \tilde{f}(r') - \sqrt{\frac{3\beta}{4}} \left(\frac{r'}{R'}\right)^2 (3 - \cos^2 \tilde{f}(r')\right]$$

Quark star like solutions

Uniform density:

$$\rho = \frac{3M}{4\pi R^3} = \text{constant}$$

Tolman VI variant (N=1):

$$\rho(r) = \frac{3M}{8\pi R^3} \frac{(2-3\beta)(1-3\beta) + \beta(3-7\beta)x^2 + 2\beta^2 x^4}{(1-3\beta+2\beta x^2)^2}.$$

Tolman VI variant (N=2) :

$$\rho(r) = \frac{M}{4\pi R^3} \frac{(2-2\beta)^{2/3}(6-15\beta+5\beta x^2)}{(2-5\beta+3\beta x^2)^{5/3}}.$$

Matese-Whitman I:

$$\rho(r) = \frac{3M}{4\pi R^3} \frac{1-2\beta+2\beta x^2/3}{(1-2\beta+2\beta x^2)^2}$$

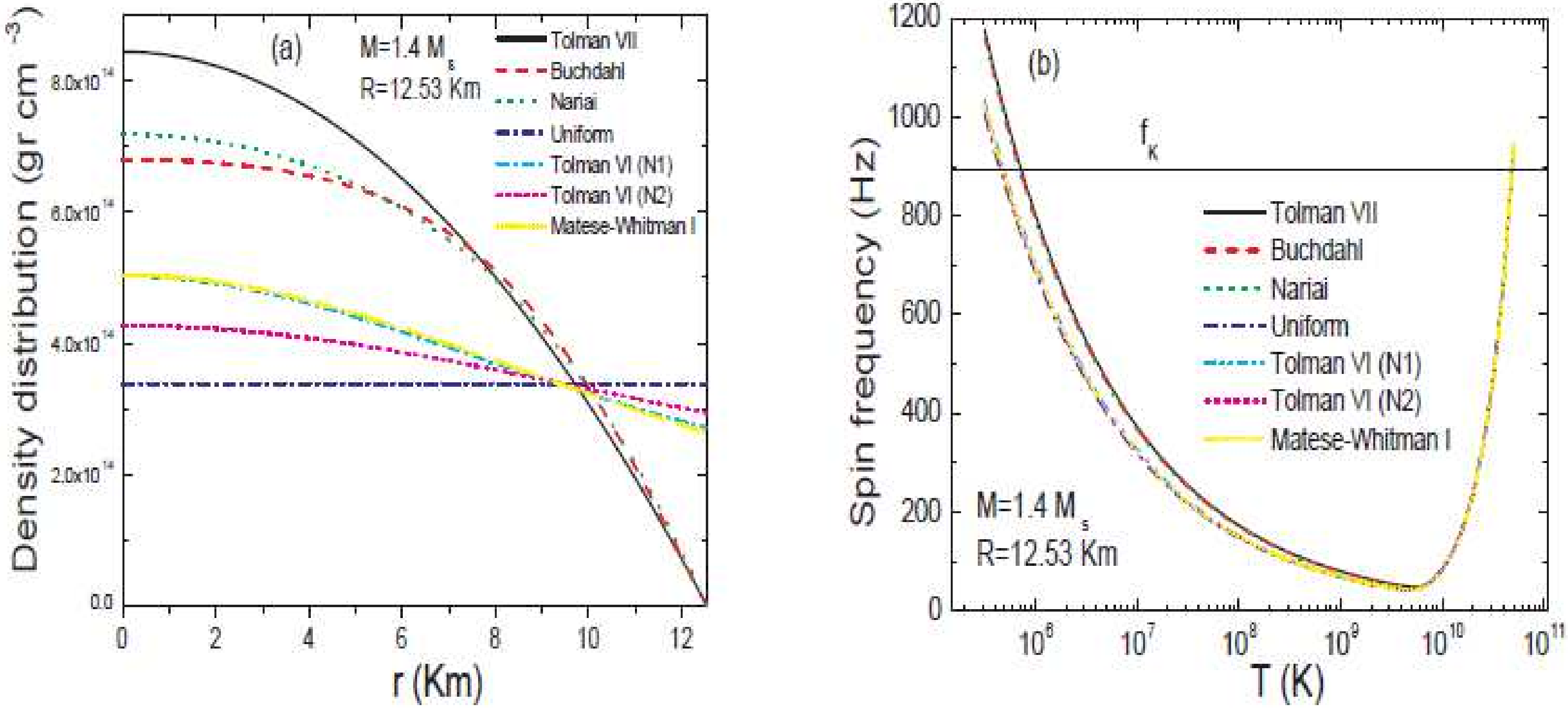


Figure 1: (a) The density distribution $\rho(r)$ for the seven selected analytical solution of the TOV equations for neutron star $M = 1.4\ M_{\odot}$ and radius $R = 12.53$ Km. (b) The instability window for the seven selected analytical solution of the TOV equations for neutron star $M = 1.4\ M_{\odot}$ and radius $R = 12.53$ Km. The thin solid line corresponds to the Kepler frequency $\mathrm{f_K} = 893$ Hz.

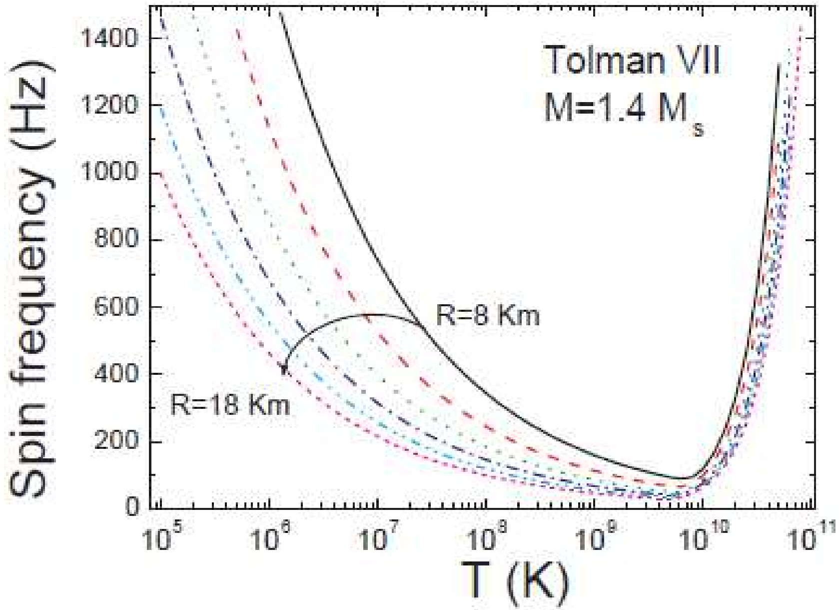


Figure 3: The instability window for the Tolman VII solution for $M = 1.4\ M_\odot$ and various values of the radius R.

$$\Omega_c \simeq 706.88 \left(\frac{10^9\mathrm{K}}{T}\right)^{1/3} \left(\frac{10\mathrm{km}}{R}\right)^{3/2}$$

$$\Omega_c \simeq 13.4 \left(\frac{10\mathrm{km}}{R}\right)^{3/4} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{T}{10^9\mathrm{K}}\right)^{3/2} \mathcal{J}_2^{1/4}, \quad \mathcal{J}_2 = \frac{\mathcal{I}_2}{\mathcal{I}_1^2}$$

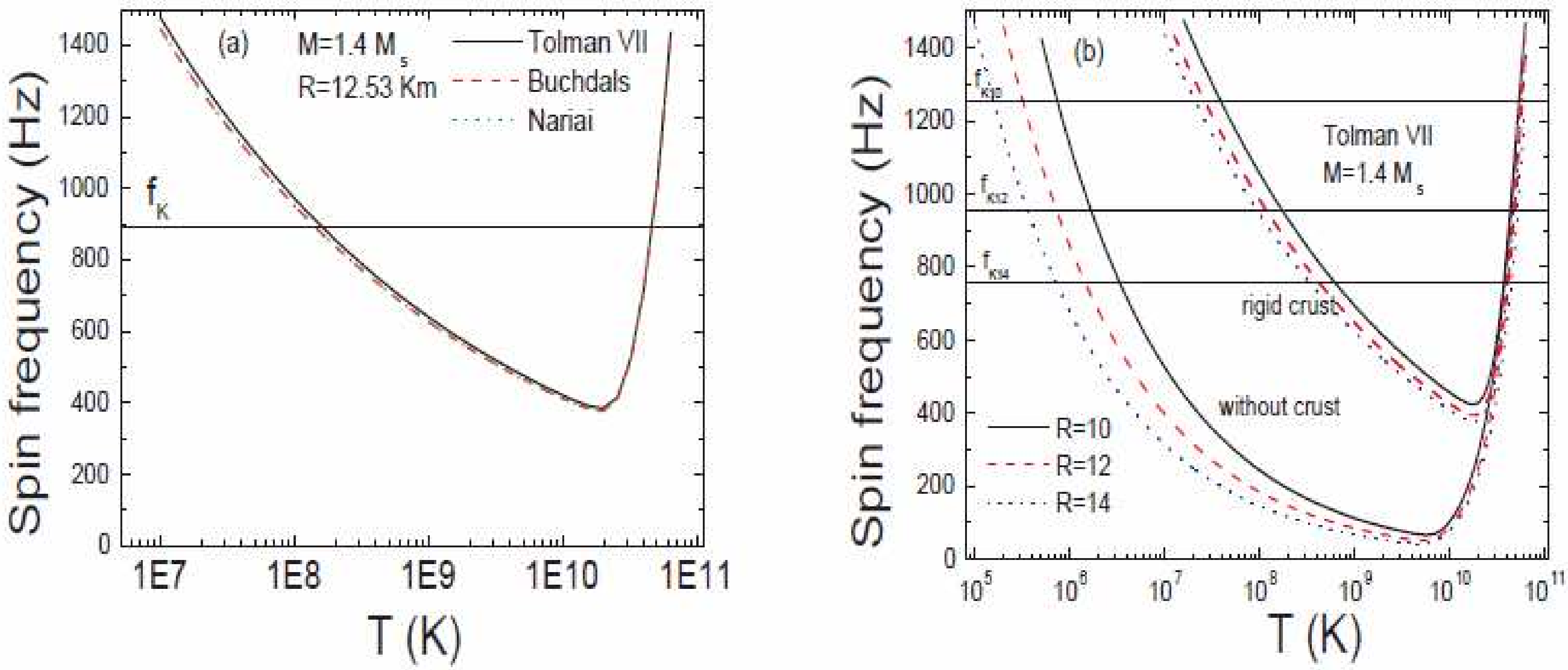


Figure 4: (a) The instabilty window for the three reliable analytical solutions when the effect of the crust has been included. (b) The instabiliy window for the Tolman VII solution, with and without crust effects, for three diferent values of R. The corresponding Kepler frequencies have included also for comparison.

$$\Omega_c \simeq 4360 \left(\frac{10^9\mathrm{K}}{T}\right)^{2/11} \left(\frac{10\mathrm{Km}}{R}\right)^{4/11}$$

$$R \simeq 1.02 \cdot 10^{11} \left(\frac{10^9\mathrm{K}}{T}\right)^{1/2} \left(\frac{\mathrm{Hz}}{\Omega_c}\right)^{11/4} \quad (\mathrm{Km})$$

$$\Omega_c \simeq (5794 \pm 108.348) \left(\frac{10^9\mathrm{K}}{T}\right)^{2/11} \left(\frac{\mathrm{MeV}}{L}\right)^{1/11}$$

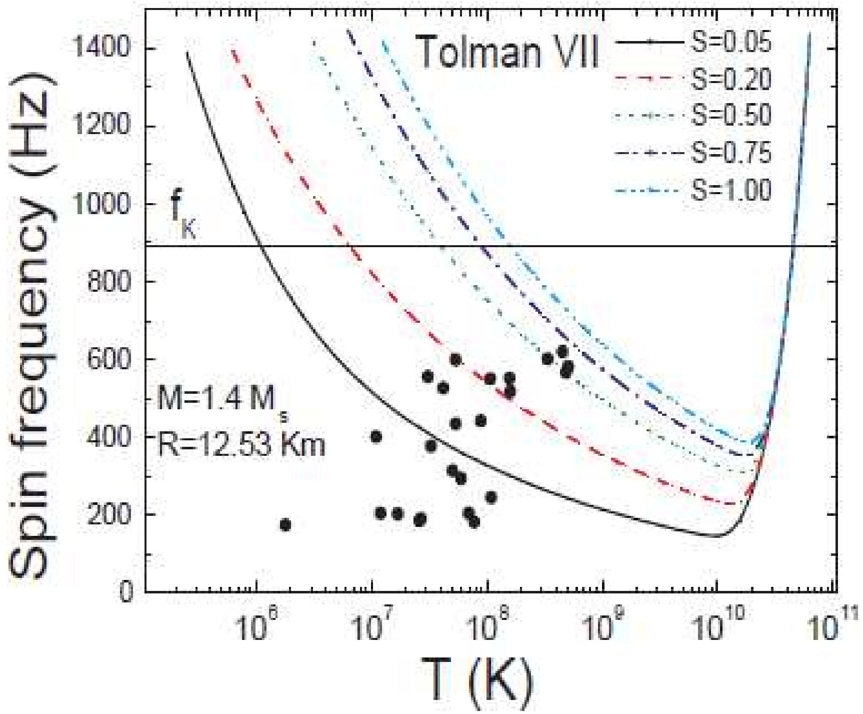


Figure 5: The instability window for the Tolman VII solution when the elasticity of the crust is taken into account via the slippage factor S. The observed cases of LMXBs and MSRPs from Haskell *et al.* [35] are also included for comparison.

$$\Omega_c \simeq 4360\ S^{4/11} \left(\frac{10^9 \mathrm{K}}{T}\right)^{2/11} \left(\frac{10 \mathrm{Km}}{R}\right)^{4/11} \qquad S \simeq 9.8 \cdot 10^{-11} \left(\frac{\Omega_c}{\mathrm{rad} \cdot \mathrm{s}^{-1}}\right)^{11/4} \left(\frac{T}{10^9 \mathrm{K}}\right)^{1/2} \left(\frac{R}{10 \mathrm{Km}}\right)$$

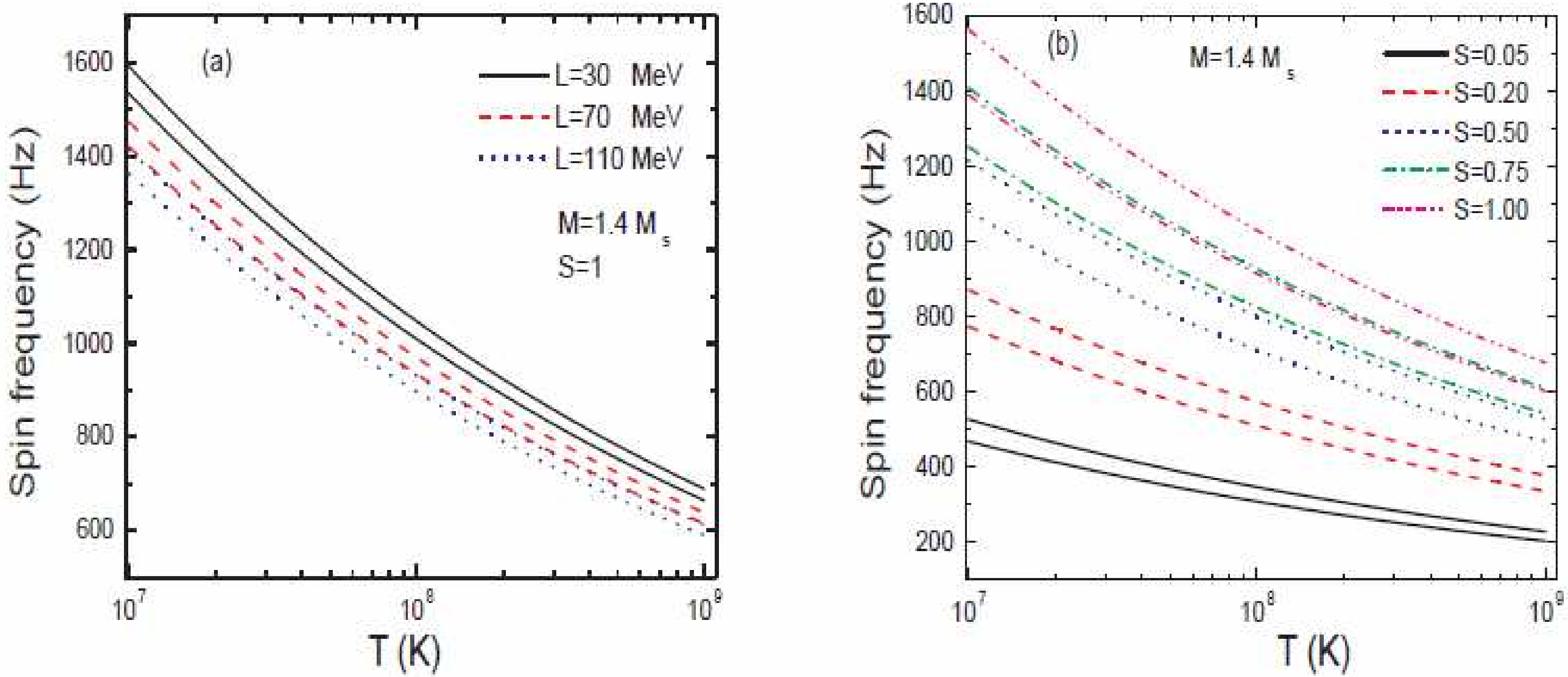


Figure 6: (a) The instability window for rigid crust (slippage factor $S = 1$) and for various values of the slope parameter L. In each case the alike curves corresponds to the lower and higher limits. (b) The instability window for various values of the factor S. In each case the lower and higher limits corresponds to the value $L = 30$ MeV and $L = 110$ MeV respectivelly.

$$\Omega_c \simeq (5794 \pm 108.348)\ S^{4/11} \left(\frac{10^9\mathrm{K}}{T}\right)^{2/11} \left(\frac{\mathrm{MeV}}{L}\right)^{1/11}$$

Conclusions-Outlook

- **In the first case (without crust consideration) the instability window depends mainly on neutron star size (radius) (the mass and density distribution effects are negligible)**
- **In the second case (with crust) the main dependence originated from the elasticity properties of the crust. The factor S regulate the size of the instability windows. The equation of state effect (via the slope parameter L) is less important but not negligible.**
- **In general the crust effects is very important and must to taken into account in order ensure the reliability of the theoretical predictions.**
- **The accurate measures of Ωc and temperature T may impose constraints on the measure of the radius R**
- **The accurate measures of Ωc, T and R may also impose constraints on the measure of the slippage factor S**
- **A combination of theoretical predictions and observational measures may impose significant constraints in bulk neutron star properties, in nuclear equation of state and the structure of the crust**