Hamiltonian construction of translationally symmetric extended MHD with equilibrium applications

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Abstract

The noncanonical Hamiltonian structure of translationally symmetric extended MHD (XMHD) [1, 2, 3] with barotropic ion and electron fluids, is obtained by employing a method of Hamiltonian reduction [5] on the three-dimensional noncanonical Poisson bracket of XMHD [2]. The existence of the continuous spatial translation symmetry allows the introduction of the so-called poloidal representation for the magnetic field and an analogous Clebsch-like representation for the velocity field, consistent with the Helmholtz decomposition theorem. Upon employing the chain rule for functional derivatives, the 3D Poisson bracket is reduced to its translationally symmetric counterpart. Using this symmetric version of the noncanonical Poisson bracket, the families of extended MHD Casimir invariants are identified and used to obtain Energy-Casimir variational principles for generalized XMHD equilibrium equations with arbitrary macroscopic flows. The obtained set of equilibrium equations is cast into one of the Grad-Shafranov-Bernoulli (GSB) type. Hall MHD equilibria with finite ion flow but neglected electron inertia is studied as a special case. The barotropic Hall MHD equilibrium equations are derived as a limiting case of the XMHD GSB system and they are consistent with those derived for axisymmetric plasmas in [6] via direct projection of the 3D equilibrium equations. In addition, we present a numerically computed equilibrium with D-shaped boundary, that plausibly shows the separation of ion flow from electron-magnetic surfaces, since in the framework of Hall MHD the magnetic field is frozen into the electron fluid.

The XMHD model

By extended MHD (XMHD) we mean the one-fluid model obtained by reduction of the standard two-fluid plasma model, when the quasineutrality assumption is imposed and expansion in the smallness of the electron mass is performed (e.g. [1]). The resulting model has a generalized Ohm's law that contains Hall drift and electron inertia physics. The dynamical equations of the XMHD model, written in the standard Alfvén units, are the following:

$$\partial_t\rho = -\nabla\cdot(\rho\mathbf{v})\,, \tag{1}$$

$$\partial_t\mathbf{v} = \mathbf{v}\times\nabla\times\mathbf{v} - \nabla\left(h+\frac{v^2}{2}\right) + \rho^{-1}\mathbf{J}\times\mathbf{B}^* - d_e^2\nabla\left(\frac{|\nabla\times\mathbf{B}|^2}{2\rho^2}\right)\,, \tag{2}$$

$$\partial_t\mathbf{B}^* = \nabla\times(\mathbf{v}\times\mathbf{B}^*) - d_i\nabla\times\left[\frac{\mathbf{J}\times\mathbf{B}^*}{\rho}\right] + d_e^2\nabla\times\left[\frac{\mathbf{J}\times(\nabla\times\mathbf{v})}{\rho}\right]\,, \tag{3}$$

where

$$\mathbf{J} = \nabla\times\mathbf{B}\,, \qquad \mathbf{B}^* = \mathbf{B} + d_e^2\nabla\times\left(\frac{\nabla\times\mathbf{B}}{\rho}\right)\,. \tag{4}$$

Here, a barotropic equation of state has been assumed, which means the enthalpy h is related to pressure by $\nabla h = \rho^{-1}\nabla p$, and the parameters d_i and d_e are the normalized ion and electron skin depths, respectively, with $d_s = c/(\omega_{ps}L)$ and $s = i, e$.

Hamiltonian structure

The equations (1)-(3) can be cast into the following Hamiltonian form [2, 3]

$$\partial_t\mathbf{u} = \{\mathbf{u}, \mathcal{H}\}\,, \tag{5}$$

with $\mathbf{u} = (\rho, \mathbf{v}, \mathbf{B}^*)$, $\mathcal{H}$ being a Hamiltonian functional and $\{F, G\}$ a non-canonical Poisson bracket, given below

$$\mathcal{H} = \frac{1}{2}\int_V d^3x\,\left[\rho v^2 + 2\rho U(\rho) + \mathbf{B}\cdot\mathbf{B}^*\right]\,, \tag{6}$$

$$\begin{aligned}\{F,G\} = &\int_V d^3x\,\big\{G_\rho\nabla\cdot F_{\mathbf{v}} - F_\rho\nabla\cdot G_{\mathbf{v}} + \rho^{-1}(\nabla\times\mathbf{v})\cdot(F_{\mathbf{v}}\times G_{\mathbf{v}})\\ &+ \rho^{-1}\mathbf{B}^*\cdot\left[F_{\mathbf{v}}\times(\nabla\times G_{\mathbf{B}^*}) - G_{\mathbf{v}}\times(\nabla\times F_{\mathbf{B}^*})\right] - d_i\rho^{-1}\mathbf{B}^*\cdot\left[(\nabla\times F_{\mathbf{B}^*})\times(\nabla\times G_{\mathbf{B}^*})\right]\\ &+ d_e^2\rho^{-1}(\nabla\times\mathbf{v})\cdot\left[(\nabla\times F_{\mathbf{B}^*})\times(\nabla\times G_{\mathbf{B}^*})\right]\big\}\,,\end{aligned} \tag{7}$$

Reduction to translationally symmetric formulation

Assuming continuous translational symmetry and adopting a Cartesian system (x, y, z) the fields $\mathbf{B}^*$ and $\mathbf{v}$ can be written in the following Clebsch representation [4, 5]

$$\mathbf{B}^* = B_z^*(x,y,t)\hat{z} + \nabla\psi^*(x,y,t)\times\hat{z}\,, \tag{8}$$

$$\mathbf{v} = v_z(x,y,t)\hat{z} + \nabla\chi(x,y,t)\times\hat{z} + \nabla\Upsilon(x,y,t)\,, \tag{9}$$

Functional derivatives with respect to the new variables

$$F_{\mathbf{v}} = F_{v_z}\hat{z} + \nabla F_\Omega\times\hat{z} - \nabla F_w\,, \qquad F_{\mathbf{B}^*} = F_{B_z^*}\hat{z} - \nabla\left(\Delta^{-1}F_{\psi^*}\right)\times\hat{z}\,, \tag{10}$$

$$\nabla\times F_{\mathbf{B}^*} = F_{\psi^*}\hat{z} + \nabla F_{B_z^*}\times\hat{z}\,. \tag{11}$$

Translationally symmetric Poisson bracket of barotropic XMHD:

$$\begin{aligned}\{F,G\}_{TS}^{XMHD} = &\int_D d^2x\,\Big\{F_\rho\Delta G_w - G_\rho\Delta F_w + \rho^{-1}\Omega\big([F_\Omega, G_\Omega] + [F_w, G_w]\\ &+\nabla F_w\cdot\nabla G_\Omega - \nabla F_\Omega\cdot\nabla G_w\big) + v_z\big([F_\Omega, \rho^{-1}G_{v_z}] - [G_\Omega, \rho^{-1}F_{v_z}]\\ &+\nabla(\rho^{-1}G_{v_z})\cdot\nabla F_w - \nabla(\rho^{-1}F_{v_z})\cdot\nabla G_w\\ &+\rho^{-1}F_\Upsilon G_{v_z} - \rho^{-1}G_\Upsilon F_{v_z}\big) + \psi^*\big([F_\Omega, \rho^{-1}G_{\psi^*}] - [G_\Omega, \rho^{-1}F_{\psi^*}]\\ &+[F_{B_z^*}, \rho^{-1}G_{v_z}] - [G_{B_z^*}, \rho^{-1}F_{v_z}] + \nabla F_w\cdot\nabla(\rho^{-1}G_{\psi^*}) - \nabla G_w\cdot\nabla(\rho^{-1}F_{\psi^*})\\ &+\rho^{-1}F_\Upsilon G_{\psi^*} - \rho^{-1}G_\Upsilon F_{\psi^*}\big) + \rho^{-1}B_z^*\big([F_\Omega, G_{B_z^*}] - [G_\Omega, F_{B_z^*}]\\ &+\nabla F_w\cdot\nabla G_{B_z^*} - \nabla G_w\cdot\nabla F_{B_z^*}\big) + d_i\psi^*\big([G_{B_z^*}, \rho^{-1}F_{\psi^*}] - [F_{B_z^*}, \rho^{-1}G_{\psi^*}]\big)\\ &-d_i\rho^{-1}B_z^*[F_{B_z^*}, G_{B_z^*}] + d_e^2\rho^{-1}\Omega[F_{B_z^*}, G_{B_z^*}] + d_e^2 v_z\big([F_{B_z^*}, \rho^{-1}G_{\psi^*}] - [G_{B_z^*}, \rho^{-1}F_{\psi^*}]\big)\Big\}\,,\end{aligned} \tag{12}$$

where $[a, b] := (\nabla a\times\nabla b)\cdot\hat{z} = (\partial_x a)(\partial_y b) - (\partial_x b)(\partial_y a)$.

Casimir invariants and equilibrium variational principle

The Casimirs satisfy $\{F, \mathcal{C}\} = 0, \forall F$. For the bracket (12) this gives

$$\mathcal{C}_1 = \int_D d^2x\,(B_z^* + \mu\Omega)\mathcal{A}(\psi^* + \mu v_z)\,, \quad \mathcal{C}_2 = \int_D d^2x\,(B_z^* + \lambda^{-1}\Omega)\mathcal{G}(\psi^* + \lambda^{-1}v_z)\,, \tag{13}$$

$$\mathcal{C}_3 = \int_D d^2x\,\rho\mathcal{K}(\psi^* + \mu v_z)\,, \quad \mathcal{C}_4 = \int_D d^2x\,\rho\mathcal{M}(\psi^* + \lambda^{-1}v_z)\,, \tag{14}$$

where the parameters λ and μ are either $(\lambda, \mu) = (\lambda_+, \mu_+)$ or $(\lambda, \mu) = (\lambda_-, \mu_-)$, with $\mu_\pm := d_i - \lambda_\pm^{-1} = \lambda_\mp^{-1}$ and $\lambda_\pm = (-d_i \pm \sqrt{d_i^2 + 4d_e^2})/(2d_e^2)$.

The Energy-Casimir variational principle reads as follows

$$\delta\left(\mathcal{H} - \mathcal{C}_1 - \mathcal{C}_2 - \mathcal{C}_3 - \mathcal{C}_4\right) = 0 \tag{15}$$

For the first variation to vanish, the coefficients of the arbitrary variations must separately vanish, yielding the following conditions:

$$\begin{aligned}\delta\rho\,:\,&\frac{v^2}{2} + [\rho U(\rho)]_\rho - \mathcal{M}(\phi) - \mathcal{K}(\varphi) - \frac{d_e^2}{\rho^2}\Big\{\frac{1}{2}(\Delta\psi)^2 + \frac{1}{2}|\nabla B_z|^2 - \nabla B_z\cdot\nabla\left[\mathcal{A}(\varphi) + \mathcal{G}(\phi)\right]\\ &+\Delta\psi\big[(B_z^* + \mu\Omega)\mathcal{A}'(\varphi) + (B_z^* + \lambda^{-1}\Omega)\mathcal{G}'(\phi) + \rho(\mathcal{M}'(\phi) + \mathcal{K}'(\varphi))\big]\Big\} = 0\,,\end{aligned} \tag{16}$$

$$\delta v_z\,:\,\rho v_z - \lambda^{-1}\rho\mathcal{M}'(\phi) - \mu\rho\mathcal{K}'(\varphi) - \mu(B_z^* + \mu\Omega)\mathcal{A}'(\varphi) - \lambda^{-1}(B_z^* + \lambda^{-1}\Omega)\mathcal{G}'(\phi) = 0\,, \tag{17}$$

$$\delta\chi\,:\,\nabla\cdot(\rho\nabla\chi) - [\rho, \Upsilon] = \mu\Delta\mathcal{A}(\varphi) + \lambda^{-1}\Delta\mathcal{G}(\phi)\,, \tag{18}$$

$$\delta\Upsilon\,:\,\nabla\cdot(\rho\nabla\Upsilon) = [\chi, \rho]\,, \quad \delta B_z^*\,:\,B_z = \mathcal{A}(\varphi) + \mathcal{G}(\phi)\,, \tag{19}$$

$$\delta\psi^*\,:\,\Delta\psi + \rho\mathcal{M}'(\phi) + \rho\mathcal{K}'(\varphi) + (B_z^* + \mu\Omega)\mathcal{A}'(\varphi) + (B_z^* + \lambda^{-1}\Omega)\mathcal{G}'(\phi) = 0\,, \tag{20}$$

where $\phi := \psi^* + \lambda^{-1}v_z$, $\varphi := \psi^* + \mu v_z$ and $'$ denotes the derivative with respect to argument. For the derivation of the equilibrium equations above we used the expressions for B_z^*, ψ^* in terms of the ordinary magnetic field variables ψ and B_z according to $\mathbf{B}^* := \mathbf{B} + d_e^2\nabla\times(\rho^{-1}\nabla\times\mathbf{B}) = B_z^*(x,y)\hat{z} + \nabla\psi^*(x,y)\times\hat{z}$ with $\mathbf{B} = B_z(x,y)\hat{z} + \nabla\psi(x,y)\times\hat{z}$:

$$B_z^* = B_z - d_e^2\nabla\cdot(\rho^{-1}\nabla B_z)\,, \quad \psi^* = \psi - d_e^2\rho^{-1}\Delta\psi\,. \tag{21}$$

Grad-Shafranov-Bernoulli equilibrium equations and special equilibria

We can show that the system (16)-(20), can be written conveniently as a Grad-Shafranov-Bernoulli system:

$$\begin{aligned}&\alpha_1\mathcal{A}'(\varphi)\nabla\cdot\left(\frac{\mathcal{A}'(\varphi)}{\rho}\nabla\varphi\right) + \alpha_2\rho(\varphi - \phi) - \alpha_3\frac{\rho}{d_e^2}\left(\psi - \frac{\varphi - \lambda\mu\phi}{1 - \lambda\mu}\right) = \left[\mathcal{A}(\varphi) + \mathcal{G}(\phi)\right]\mathcal{A}'(\varphi) + \rho\mathcal{K}'(\varphi)\,,\\ &\gamma_1\mathcal{G}'(\phi)\nabla\cdot\left(\frac{\mathcal{G}'(\phi)}{\rho}\nabla\phi\right) + \gamma_2\rho(\varphi - \phi) + \gamma_3\frac{\rho}{d_e^2}\left(\psi - \frac{\varphi - \lambda\mu\phi}{1 - \lambda\mu}\right) = \left[\mathcal{A}(\varphi) + \mathcal{G}(\phi)\right]\mathcal{G}'(\phi) + \rho\mathcal{M}'(\phi)\,,\\ &\Delta\psi = \frac{\rho}{d_e^2}\left(\psi - \frac{\varphi - \lambda\mu\phi}{1 - \lambda\mu}\right)\,, \quad \tilde{P}(\rho) = \rho\left[\mathcal{M}(\phi) + \mathcal{K}(\varphi)\right] - \rho\frac{v^2}{2} - \frac{d_e^2}{2\rho}\left[(\Delta\psi)^2 + |\nabla B_z|^2\right]\,,\end{aligned} \tag{22}$$

where

$$\alpha_1 = \mu^2 + d_e^2\,, \quad \alpha_2 = \frac{\lambda^2}{(1 - \lambda\mu)^2}\,, \quad \alpha_3 = \frac{1}{1 - \lambda\mu}\,, \quad \gamma_1 = \lambda^{-2} + d_e^2\,, \quad \gamma_2 = -\alpha_2\,, \quad \gamma_3 = \lambda\mu\,\alpha_3\,. \tag{23}$$

HMHD equilibria

For $d_e \to 0$ we have $\mu \to 0$ and $\lambda^{-1} \to d_i$; therefore, the independent flux functions are the poloidal magnetic flux function ψ and the ion flow function $\phi := \psi + d_i v_z$. One can find that

$$v_z = d_i^{-1}(\phi - \psi)\,, \quad \mathbf{v}_p = \frac{d_i}{\rho}\mathcal{G}'(\phi)\mathbf{B}_{ip}\,, \qquad \text{where} \qquad \mathbf{B}_{ip} := \nabla\phi\times\hat{z}\,. \tag{24}$$

Next, with $d_e = 0$ Eqs. (22) reduce to

$$d_i^2\mathcal{G}'(\phi)\nabla\left(\frac{\mathcal{G}'(\phi)}{\rho}\nabla\phi\right) + \frac{\rho}{d_i^2}(\phi - \psi) - \left[\mathcal{G}(\phi) + \mathcal{A}(\psi)\right]\mathcal{G}'(\phi) - \rho\mathcal{M}'(\phi) = 0\,, \tag{25}$$

$$\Delta\psi + \frac{\rho}{d_i^2}(\phi - \psi) + \rho\mathcal{K}'(\psi) + \left[\mathcal{G}(\phi) + \mathcal{A}(\psi)\right]\mathcal{A}'(\psi) = 0\,, \tag{26}$$

$$\tilde{P}(\rho) = \rho\big[\mathcal{K}(\psi) + \mathcal{M}(\phi) - \frac{(\phi - \psi)^2}{2d_i^2}\big] - \frac{d_i^2}{2\rho}\left(\mathcal{G}'(\phi)\right)^2|\nabla\phi|^2\,. \tag{27}$$

The system above is in agreement with the system derived in [6] for axisymmetric HMHD equilibria.

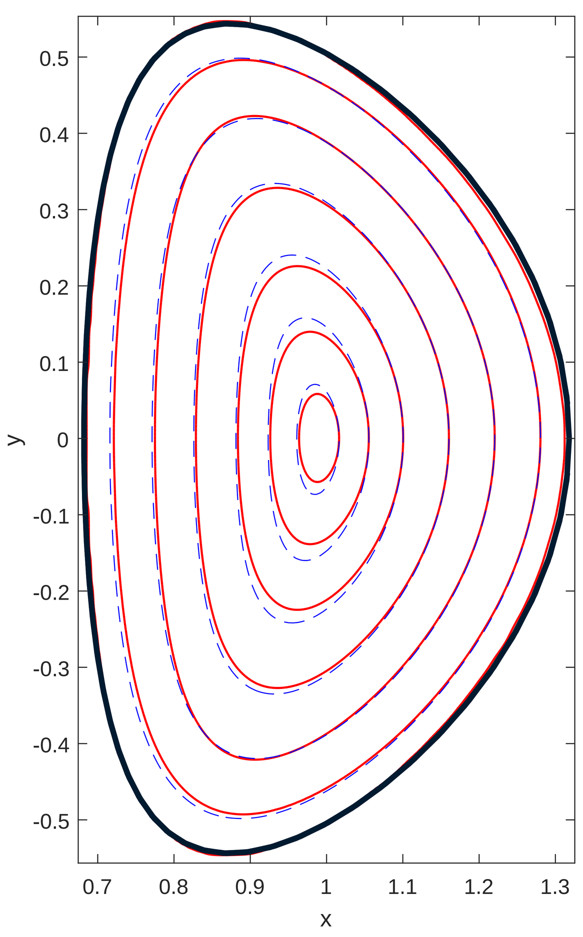

Figure 1: Numerical Hall MHD equilibrium with $\epsilon = 1.7$, $\delta = 0.33$, $a_0 = 0.325$ with dimensionless Hall parameter $d_i = 0.03$ (normalized ion skin depth). The solid black line represents the boundary, the solid-red contours are the ion flow surfaces ($\phi = constant$) and the dashed-blue contours are magnetic surfaces (dashed-blue) ($\psi = constant$). Departure of the flow surfaces from the magnetic surfaces due to the Hall term in Ohm's law is observed, with a separation distance of the order of $0.04\,L_0$.

Conclusion

- The Hamiltonian formulation of translationally symmetric barotropic extended magnetohydrodynamics is presented.
- We derived the symmetric Casimir integrals of motion and produced the Energy-Casimir variational principle for obtaining the generalized equilibrium equations, which govern XMHD stationary states. These states may be particularly interesting for the study of 2D collisionless reconnection configurations.
- The equilibrium system of equations were shown to be a Grad-Shafranov-Bernoulli type, and we studied special case of HMHD equilibria with arbitrary flow.
- We computed a numerical HMHD equilibrium on a D-Shaped domain, relevant to fusion experiments. The resulting configuration is representative of the predicted separation of the ion-flow and magnetic surfaces.
- Extension of the present study to cases of arbitrary symmetry, as done for MHD in [7], in particular for helically symmetric configurations, is in progress.

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