# Analytic anisotropic-pressure equilibria with incompressible flow in helically symmetric geometry

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## Introduction

In this project we derive a generalized Grad-Shafranov equation (GGSE) that governs the equilibrium states of an MHD helically symmetric plasma in the presence of pressure anisotropy and incompressible flow of arbitrary direction. This equation generalizes previous equations obtained both for axisymmetric and translationally symmetric equilibria. Through the most general linearizing ansatz for the various free functions involved therein, we construct equilibrium solutions and study their properties. It turns out that pressure anisotropy can act either paramagnetically or diamagnetically, the parallel flow induces paramagnetism, while the non-parallel component of the flow associated with the electric field has a diamagnetic effect. Also, pressure anisotropy and flow noticeably affect the current density.

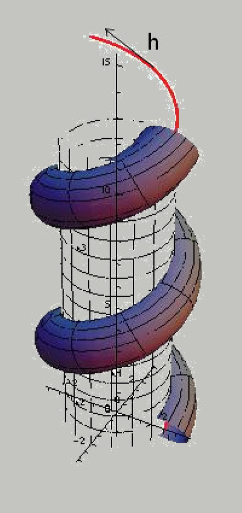
## Model and Geometry

CGL Pressure Tensor [1]

$$\overleftrightarrow{\mathbb{P}} = p_\perp \overleftrightarrow{\mathbb{I}} + \frac{\sigma_d}{\mu_0}\vec{B}\vec{B}$$

Measure of Anisotropy

$$\sigma_d \equiv \mu_0 \frac{p_\parallel - p_\perp}{|\vec{B}|^2}$$



Ideal MHD

$$\vec{\nabla}\cdot(\varrho\vec{v}) = 0 \quad (1)$$

$$\varrho(\vec{v}\cdot\vec{\nabla})\vec{v} = \vec{J}\times\vec{B} - \vec{\nabla}\cdot\overleftrightarrow{\mathbb{P}} \quad (2)$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi \quad (3)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad (4)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (5)$$

$$\vec{E} + \vec{v}\times\vec{B} = 0 \quad (6)$$

The expected plasma configuration consists of a helically symmetric vessel composed of nested helicoidal surfaces of arbitrary poloidal cross-section. The innermost of these surfaces denotes the magnetic axis. The shape of the cross-section remains invariant along the helical direction.

## Derivation Procedure

- Employ helical coordinates: $(r = \rho,\ u = m\phi - kz,\ \xi = z)$. The condition of helical symmetry implies that equilibrium quantities depend only on $r$ and $u$.
- On account of helical symmetry express the divergence-free fields $\vec{B}$, $\varrho\vec{v}$, $\vec{J}$ into a helical and a poloidal component, in terms of scalar functions.
- Project the Ohm's law (6) onto $\vec{h}$, $\vec{B}$, $\vec{\nabla}\psi$, and Eq. (2) along the helical direction $\vec{h}$ to obtain four integrals of the system involving functions of the magnetic flux $\psi$.
- Project the momentum density equation (2) along $\vec{B}$ and $\vec{\nabla}\psi$ to obtain the GGSE together with a Bernoulli equation for the effective pressure $\bar{p} := (p_\perp + p_\parallel)/2$. These two equations valid for generic (compressible) flows are coupled through the density $\varrho$ and the anisotropy function $\sigma_d$.
- Consider incompressibility, $\varrho = \varrho(\psi)$, and assume that the anisotropy function is uniform on the magnetic surfaces, $\sigma_d = \sigma_d(\psi)$ [2]. In this case the aforementioned equations decouple to obtain a single Grad-Shafranov equation for the magnetic flux $\psi$.
- Adopt the generalized transformation, $U(\psi) = \int_0^\psi \sqrt{1 - \sigma_d(g) - M_p^2(g)}dg$ [3], where $U$ relabels the magnetic surfaces in the place of $\psi$, to obtain a GGSE which can be solved analytically for linear choices of the surface functions included.

## Generalized Grad-Shafranov Equation

$$\mathcal{L}U + \frac{2kmqX}{(1-\sigma_d - M_p^2)^{1/2}} + \frac{1}{2}\frac{d}{dU}\left[\frac{X^2}{1-\sigma_d - M_p^2}\right] + \frac{\mu_0}{q}\frac{d}{dU}\left[\bar{p}_s - X\left(\frac{dF}{dU}\right)\left(\frac{d\Phi}{dU}\right)\right] + \frac{\mu_0}{2q^2}\frac{d}{dU}\left[(1-\sigma_d)\varrho\left(\frac{d\Phi}{dU}\right)^2\right] = 0 \quad (7)$$

- $\Phi(U)$ is the electrostatic potential, $F(U)$ labels the velocity surfaces, $\bar{p}_s(U)$ is the static effective pressure (in the absence of flow), $M_p^2(U)$ is the poloidal Alfvén Mach function, and $X(U)$ is related with the helical magnetic field; $\mathcal{L} \equiv (1/q)\vec{\nabla}\cdot(q\vec{\nabla})$, $q := (k^2r^2 + m^2)^{-1}$.
- The form of the GGSE indicates that in the absence of the electric field term pressure anisotropy through $\sigma_d$ and parallel flow through $M_p^2$ have an additive effect on equilibrium.
- Eq. (7) recovers the respective axisymmetric and translationally symmetric ones, either with pressure anisotropy and flow or not, as particular cases [3]-[5].

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## Equilibrium

Under the linearizing ansatz (8) a solution for $U(r,u)$ is obtained in terms of series expansion around the geometric center $r_0$.

$$\frac{X}{(1-\sigma_d - M_p^2)^{1/2}} = X_0 + X_1(U - U_b)$$
$$\varrho(1-\sigma_d)\left(\frac{d\Phi}{dU}\right)^2 = 2G_1(U-U_b) + G_2(U-U_b)^2$$
$$\bar{p}_s - X\frac{dF}{dU}\frac{d\Phi}{dU} = 2P_1(U-U_b) + P_2(U-U_b)^2 \quad (8)$$

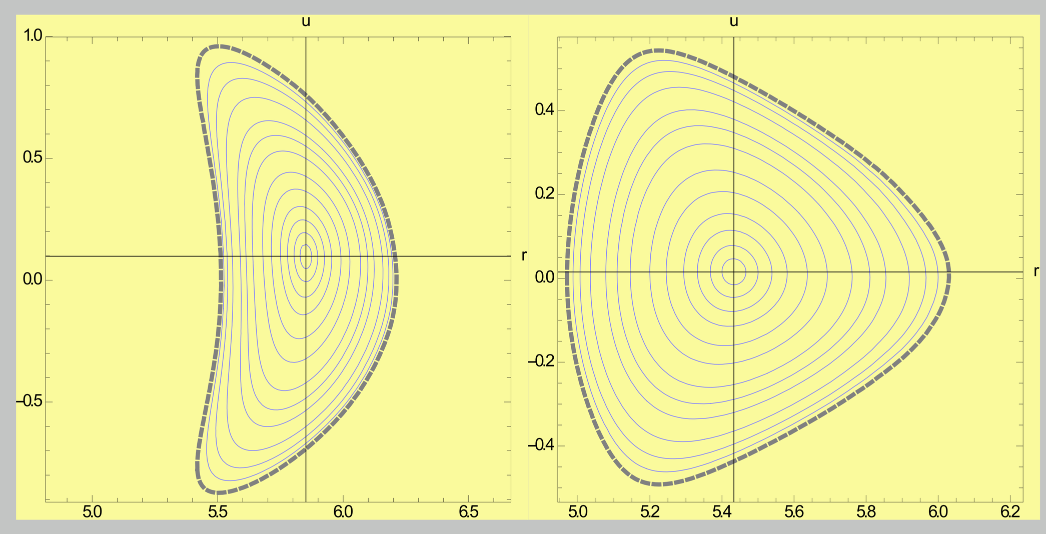

Figure 1: *Equilibrium configurations of banana and triangular shape obtained for Wendelstein 7-X geometrical characteristics: major radius, $r_0 = 5.5$ m, and minor radius, $\alpha = 0.53$ m. The cross-sections remarkably resemble the ones of the actual 3-D device for toroidal angles 0° and 36° respectively [6]-[7].*

## Impact of Flow and Pressure Anisotropy

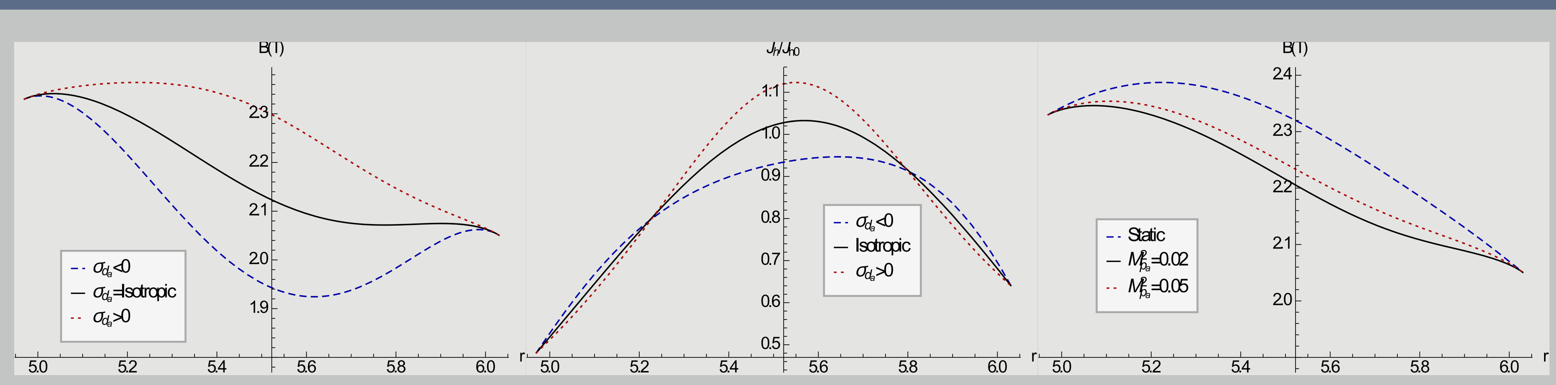

Figure 2: *Pressure anisotropy has a paramagnetic effect for $p_\parallel > p_\perp$ and a diamagnetic one for $p_\parallel < p_\perp$ (left figure), while it affects the current density mainly in the central plasma region (centered figure). The parallel flow induces paramagnetism (right figure), since it acts in an additive way with pressure anisotropy.*

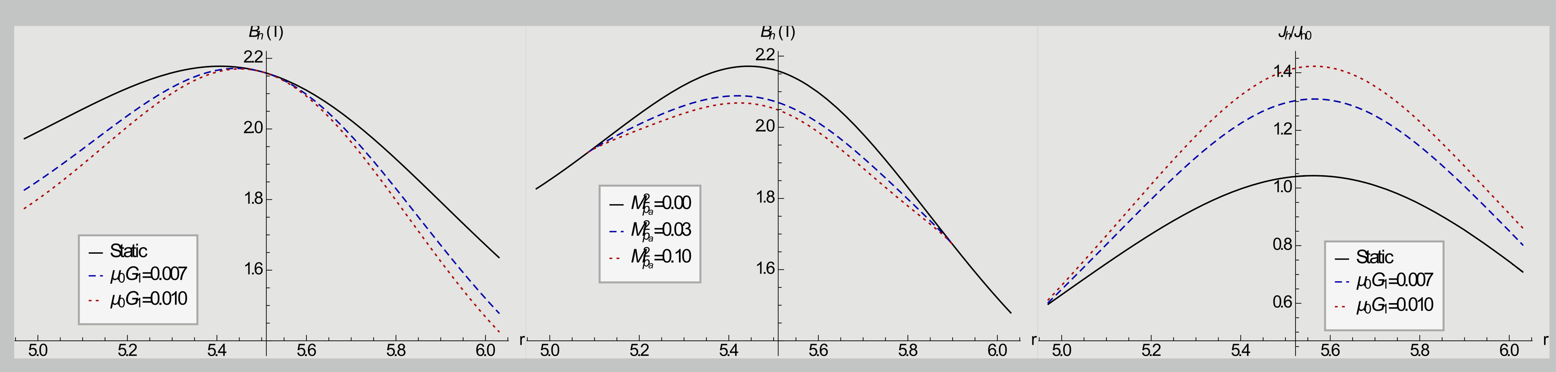

Figure 3: *The non-parallel flow has a diamagnetic impact on equilibria (left figure), while the stronger this diamagnetic effect is the higher $M_p^2$ (centered figure). Also, the non-parallel flow affects the helicoidal current density in a broader region than pressure anisotropy.*

## Conclusions

- A generalized Grad-Shafranov equation governing helically symmetric equilibria with pressure anisotropy and incompressible flow of arbitrary direction is derived. This equation recovers the respective axisymmetric and translationally symmetric ones as particular cases.
- For linearizing choices of the arbitrary functions contained in the GGSE a new class of exact helically symmetric equilibrium solutions is obtained.
- It is found that pressure anisotropy can act either paramagnetically or diamagnetically depending on the ratio of the scalar pressures parallel and perpendicular to the magnetic field.
- For $\vec{v}//\vec{B}$ the parallel flow induces paramagnetism. The non parallel flow associated with the electric field has a diamagnetic impact on equilibria, in this case the parallel flow enhances this diamagnetic action.
- Both the pressure anisotropy and the flow have an appreciable impact on the current density.

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