Stability criteria of Magnetohydrodynamic plasmas and their underlying Hamiltonian structure

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The "equilibrium and stability" framework

The concepts of equilibrium and of stability are often adopted in the description of a plasma configuration and of its dynamics.

It may not always be obvious how to apply these concepts operationally to conditions where the identification of an underlying equilibrium state is rather arbitrary, see e.g., the case of a fully developed turbulence.

Nevertheless they represent a very important logical framework within which one can constrain the bewildering richness of the plasma dynamics.

A meaningful definition of equilibria requires that we restrict the dynamics under study to a selected range of spatial and temporal scales. In this perspective magnetohydrodynamic equilibria can play an important role in providing a first step, even if fairly incomplete, in the investigation of the behaviour of magnetized plasma in the laboratory, in space and in the universe.

The “equilibrium and stability” framework

- *Besides being in general a useful descriptive tool, the concept of MHD equilibrium is very relevant to the study of externally constrained plasma configurations, such as most fusion plasmas in the laboratory. It can also be usefully applied to the study of space plasmas such as e.g. planetary magnetospheres, stellar and galactic discs, accretion discs*[1] *on compact astrophysical objects, etc.*

For these latter cases the role of the plasma flows is of paramount importance and thus we are led to distinguish between static equilibria (without flows) and stationary equilibria (with flows).

Note that the role of plasma flows, in particular of plasma rotation, is now fully recognized also for (toroidal) plasma configurations in the laboratory, e.g. as a possible source of improved energy transport.

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[1] see e.g. the *Magnetorotational instability* (Velikhov-Chandrasekhar instability or Balbus-Hawley instability) of lonf ago: *Stability of an ideally conducting liquid flowing between cylinders rotating in a magnetic field*, E.P. Velikhov - Sov. Phys. JETP, 1959

An extended framework

The generalization to equilibria with flows includes configurations where the properties, e.g. the velocity vector, of a given plasma element are not constant in time.

It may thus appear natural that some of the methods[2] that are applied to study the stability of stationary equilibria can find application even in cases where no concept of equilibrium is involved (e.g. in the study of the orbital stability of the time evolution of a plasma configuration).

In fact, different definitions of stability can be given with, in general, different mathematical and physical content.

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[2]such as the so-called *time-dependent relabelling* which I will address later in this presentation

Stability of equilibrium points of a dynamical system with a finite number of degrees of freedom.

Generally spaeking, in a dynamical system stability concerns the behaviour of solutions near equilibrium points.

An equilibrium point is stable if solutions starting close to it at $t = 0$ remain close to it for all later times.
If these solutions are determined from the linearized dynamics, the equilibrium point is *linearly stable*.

Equilibria that are unstable under nonlinear dynamics, yet stable under linear dynamics, are said to be *nonlinearly unstable*[3].

A linear system is *spectrally stable* if, assuming a time behaviour of the form $\exp(\gamma t)$ and solving for γ, there are no solutions with $Re\gamma > 0$
Linear stability implies spectral stability but the converse is not true.

[3] Equilibria can be linearly unstable and nonlinearly stable

Stability of equilibrium points of a dynamical system with an infinite number of degrees of freedom.

In a dynamical system with an infinite number of degrees of freedom similar definitions apply with "equilibrium configuration" taking the place of "equilibrium point" and functional derivatives that of partial derivatives.

Global quantities such as, e.g., the total energy of a plasma configuration will involve functionals in the form of space integrals of local functions such as the plasma energy density.

Mathematical results that can be rigorously proven for a system with a finite number of degrees of freedom turn out to be useful also in the infinite number case: the well known MHD stability $\delta \mathscr{W}$ *method is in essence the infinite-degree-of-freedom version of Lagrange's theorem (1788), while for Hamiltonian systems that are not of the separable form another old theorem, Dirichlet's theorem (1846), gives a sufficient condition for stability.*

Functional Differentiation

First variation of function:

$$\delta f(z;\delta z) = \sum_{i=1}^{n} \frac{\partial f(z)}{\partial z_i}\,\delta z_i =: \nabla f \cdot \delta z\,, \qquad f(z) = f(z_1, z_2, \ldots, z_n)\,.$$

First variation of functional:

$$\delta F[u;\delta u] = \frac{d}{d\epsilon} F[u + \epsilon\,\delta u]\Big|_{\epsilon=0} = \int_{x_0}^{x_1} \delta u\,\frac{\delta F}{\delta u(x)}\,dx =: \left\langle \frac{\delta F}{\delta u}, \delta u \right\rangle\,.$$

dot product	$\cdot$	$\Longleftrightarrow$	scalar product	$<,>$
index	i	$\Longleftrightarrow$	integration variable	x
gradient	$\frac{\partial f(z)}{\partial z_i}$	$\Longleftrightarrow$	functional derivative	$\frac{\delta F[u]}{\delta u(x)}$

Ideal MHD stability of static plasma equilibria

Stability criteria can be obtained by constructing quadratic forms starting from the linearized MHD equations. This approach requires an explicit proof that the linear operator from which the quadratic form is constructed is self-adjoint over the linear space of functions inside which solutions of the linearized equations are searched for.

This is for example the method adopted for the case of static equilibria in the famous article *I. B. Bernstein, E. A. Frieman, M. D. Kruskal, R. M. Kulsrud, Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 244, 17 (1958),*

An energy principle for hydromagnetic stability problems

By I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. M. Kulsrud

Project Matterhorn, Princeton University

(*Communicated by S. Chandrasekhar, F.R.S.—Received* 18 *April* 1957—*Revised* 26 *August* 1957)

$$\omega^2 = \frac{\delta W\{\xi,\xi\}}{K\{\xi,\xi\}}$$

Ideal MHD stability of static plasma equilibria

Conversely, stability criteria can be simply obtained as a consequence of energy conservation.

- The full, i.e., nonlinear, ideal MHD equations, as I will mention later, are Hamiltonian and thus possess a conserved energy.

From the full Hamiltonian functional a conserved quadratic functional can be derived in the linear limit. This procedure ensures automatically that the linearized force operator is self-adjoint.

This is essentially the approach adopted by W. A. Newcomb, "Lagrangian and Hamiltonian methods in magnetohydrodynamics," Nuclear Fusion Supplement, 2, 451 (1962).

NUCLEAR FUSION: 1962 SUPPLEMENT, PART 2

LAGRANGIAN AND HAMILTONIAN METHODS IN MAGNETOHYDRODYNAMICS*

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Ideal MHD stability of plasma equilibria

In the case of static equilibria these two methods lead to linear stability criteria that are both sufficient and necessary.

An extension of the δW method to plasma with flows was provided by *E. Frieman M. Rotenberg, Rev. Mod. Phys., 32 , 898 (1960)* by exploiting the concept of “equilibrium trajectory", by constructing quadratic forms in the displacement ξ with respect to this trajectory from which they can derive sufficient stability conditions.

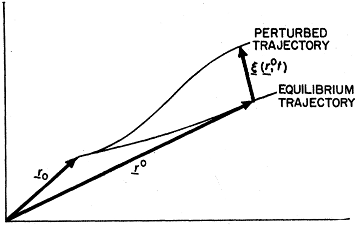

FIG. 1. Definitions of $\mathbf{r}^0$ and $\xi(\mathbf{r}^0,t)$. The vector $\mathbf{r}_0$ represents the original position of the fluid elements. *Note.* In Figs. 1 and 2, vectors are indicated by a bar beneath the letter.

The position vector $\mathbf{r}$ of a fluid element which at $t=0$ was at $\mathbf{r}_0$ is given by

$$\mathbf{r}=\mathbf{r}^0+\xi(\mathbf{r}^0,t), \qquad (12)$$

where $\mathbf{r}^0$ describes the equilibrium trajectory and $\xi(\mathbf{r}^0,t)$ describes the displacement from equilibrium. We choose ξ to be a function of $\mathbf{r}^0,t$ rather than a function of $\mathbf{r}_0,t$ so that the equilibrium quantities are time independent and solutions of the form $e^{i\omega t}$ are permitted.

Ideal MHD stability of plasma equilibria

In the rest of my presentation I will discuss the linear stability of ideal MHD equilibria with flows.

The approach I will use is based on the Hamiltonian nature of the ideal MHD equations.

Both a Lagrangian and an Eulerian formulation will be considered and, as a consequence of the different constraints that will be imposed on the perturbations, different sufficient stability conditions will be derived.

A general "inclusion order" between the different sufficient conditions obtained will be discussed.

I will mainly refer to the following string of articles

T. Andreussi, P. J. Morrison, F. Pegoraro,
Phys. Plasmas 20, 092104 (2013),
Phys. Plasmas 22, 039903 (2015),
Phys. Plasmas 23, 102112 (2016).

MHD stability of plasmas with flows

- In order to study the stability of magnetohydrodynamic (MHD) plasma equilibria with stationary flows requires an approach that generalizes the $\delta\mathscr{W}$ approach that is used for static configurations.
An obvious difficulty is mentioned by E. Frieman and M. Rotenberg: *the presence of a velocity field in the equilibrium state may lead to the phenomenon of overstability. The manifestation of this in the mathematical formalism is the appearance of non-Hermitian operators.*"

- The generalization that overcomes this difficulty is best performed by looking at the functional $\delta\mathscr{W}$ not as a quadratic form derived from the linearized MHD equations, but as the second order variation of the Hamiltonian functional $\mathscr{H}$ that describes the full dynamics of a dissipationless MHD plasma.
In this approach the Hermitian property follows automatically.

Eulerian and Lagrangian variables

- As in the case of a standard fluid the MHD dynamics can be described either in Eulerian or in Lagrangian variables.
- Lagrangian variables describe the dynamics of a plasma fluid element whereas Eulerian variables describe the evolution in time of the plasma quantities at a fixed spatial position. For an extensive presentation of the Hamiltonian formulation of the MHD plasma dynamics in Lagrangian and in Eulerian coordinates see Morrison[4].
- The second order variation of the plasma Hamiltonian can be computed either in Lagrangian or in Eulerian variables.

In the presence of stationary equilibrium flows these two procedures follow somewhat different paths.

The aim of this presentation is to illustrate these differences and to exemplify them in the simple case of a rotating pinch configuration.

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[4] P. J. Morrison, Rev. Mod. Phys., **70** , 467 (1998).

Hamiltonian of an MHD plasma in Eulerian variables

- The MHD Hamiltonian in Eulerian variables takes the form

$$H = \int d\mathbf{x} \left[\frac{\rho}{2} |\mathbf{v}|^2 + \rho\, U(s,\rho) + \frac{|\mathbf{B}|^2}{8\pi} \right],$$

where $\rho(x,t)$ is the density, $\mathbf{v}(x.t)$ the fluid velocity, $U = U(s,\rho)$ is the internal energy per unit mass, $s(x,t)$ the entropy per unit mass and $\mathbf{B}(x.t)$ the magnetic field.
The pressure is given by $p = \rho^2 \partial U / \partial \rho$ and the temperature by $T = \partial U / \partial s$. A closure condition for $U(s,\rho)$ is assumed.

- The variables $Z = \rho, \mathbf{v}, s, \mathbf{B}$ in terms of which the Hamiltonian above is expressed are not canonical and their equations of motion, i.e., the standard equations of ideal MHD, are obtained by defining generalized *noncanonical* Poisson brackets such that

$$\frac{\partial Z}{\partial t} = \{Z, H\}_Z \, .$$

Non Canonical coordinates - finite dimensions

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad [A,B] = \frac{\partial A}{\partial z^i} J^{ij}(z)\frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $[A,B] = -[B,A]\,,$

Jacobi identity $\longrightarrow$ $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

Non Canonical coordinates: free rigid body

$$\dot{\ell}_1 = \ell_2 \ell_3 \left(\frac{1}{I_3} - \frac{1}{I_2} \right),$$

$$\dot{\ell}_2 = \ell_3 \ell_1 \left(\frac{1}{I_1} - \frac{1}{I_3} \right),$$

$$\dot{\ell}_3 = \ell_1 \ell_2 \left(\frac{1}{I_2} - \frac{1}{I_1} \right),$$

$$H = \tfrac{1}{2} \sum_{i=1}^{3} \frac{\ell_i^2}{I_i}. \qquad \dot{\ell}_i = [\ell_i, H],$$

$$[f,g] = -\epsilon_{ijk} \ell_k \frac{\partial f}{\partial \ell_i} \frac{\partial g}{\partial \ell_j}.$$

$$C = \tfrac{1}{2} \sum_{i=1}^{3} \ell_i^2,$$

which satisfies

$$[C,f] = 0, \quad \forall \; f.$$

A dot means time derivative

Noncanonical Poisson brackets: Casimirs

• Contrary to the canonical Poisson brackets that involve canonical variables, the noncanonical Poisson brackets are degenerate.
This degeneracy gives rise to Casimir invariants[5], i.e. to special functionals C that satisfy $\{C,F\}=0$ for all functionals F.
The general form of noncanonical Poisson brackets is given by

$$\{F,G\} = \int dx \frac{\delta F}{\delta Z} \cdot \mathbb{J} \cdot \frac{\delta G}{\delta Z}.$$

where F and G are two functionals and $\mathbb{J}$ is an anti-selfadjoint operator that must satisfy the Jacobi identity

$$\{F,\{G,K\}\} + \{G,\{K,F\}\} + \{K,\{F,G\}\} = 0.$$

The Casimir invariance implies that *the system evolution is restricted to subdomains (foliations) of the space of Eulerian variables Z*

[5] Magnetic helicity and cross helicity belong to such a class of invariants

Noncanonical Poisson brackets for MHD

$$\{F,G\}_Z = \int dx \frac{\delta F}{\delta Z} \cdot \mathbb{J} \cdot \frac{\delta G}{\delta Z}, \qquad Z = \rho, \mathbf{v}, s, \mathbf{B},$$

$$\begin{aligned}\{F,G\}_Z = -\int_V \Big\{ & F_\rho \nabla \cdot G_\mathbf{v} - G_\rho \nabla \cdot F_\mathbf{v} \\ & + \frac{\nabla \times \mathbf{v}}{\rho} \cdot (G_\mathbf{v} \times F_\mathbf{v}) + \frac{\nabla s}{\rho} \cdot (F_s G_\mathbf{v} - G_s F_\mathbf{v}) \\ & + \mathbf{B} \cdot \left[\left(\frac{1}{\rho} F_\mathbf{v} \cdot \nabla \right) G_\mathbf{B} - \left(\frac{1}{\rho} G_\mathbf{v} \cdot \nabla \right) F_\mathbf{B} \right] \\ & + \mathbf{B} \cdot \left[\left(\nabla \frac{1}{\rho} F_\mathbf{v} \right) \cdot G_\mathbf{B} - \left(\nabla \frac{1}{\rho} G_\mathbf{v} \right) \cdot F_\mathbf{B} \right] \Big\} d^3 r,\end{aligned}$$

where F and G are two generic functionals and subscripts indicate functional derivatives.

Morrison P J, Greene J M, Phys. Rev. Lett. 45 790 (1980) & Phys. Rev. Lett. 48 569 (1982). They are obtained from the canonical Lagrangian brackets using the transformation that maps Lagrangian into Eulerian variables

Noncanonical MHD Poisson brackets: Eulerian variables

Using

$$\frac{\partial Z}{\partial t} = \{Z, H\}_Z .$$

with

$$H = \int dx \left[\frac{\rho}{2} |\mathbf{v}|^2 + \rho\, U(s,\rho) + \frac{|\mathbf{B}|^2}{8\pi} \right] ,$$

and the brackets defined above we recover the MHD equations

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) ,$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(\frac{|\mathbf{v}|^2}{2} + U + \frac{p}{\rho} \right) - (\nabla \times \mathbf{v}) \times \mathbf{v} + T \nabla s + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B},$$

$$\frac{\partial s}{\partial t} = -\mathbf{v} \cdot \nabla s,$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{B} \times \mathbf{v}) .$$

MHD Hamiltonian in Lagrangian variables

- Let $\mathbf{q}$ be the Lagrangian variable that determines the position of a fluid element and suppose that $\mathbf{q}$ has a canonical conjugate π.

Both are labelled by a continuum variable $\mathbf{a}$, i.e., the dynamical variables of the Hamiltonian description are the pair $\mathbf{q}(\mathbf{a},t),\pi(\mathbf{a},t)$.

It is common to assume that the fluid element described by $\mathbf{q}$ is labelled by its initial condition, $\mathbf{q}(\mathbf{0},\mathbf{t})=\mathbf{a}$, but this is not necessary.

The map from the Lagrangian variables $(\mathbf{q},\pi)$ to the Eulerian variables Z includes the mass, entropy and magnetic flux conservation laws and is given by

$$\rho(\mathbf{x},t)=\left.\frac{\rho_0(\mathbf{a})}{J(\mathbf{a},t)}\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)}, \qquad s(\mathbf{x},t)=\left.s_0(\mathbf{a})\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)},$$

$$v_i(\mathbf{x},t)=\left.\frac{\pi_i(\mathbf{a},t)}{\rho_0(\mathbf{a})}\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)}, \qquad B^i(\mathbf{x},t)=\left.\frac{\partial q^i(\mathbf{a},t)}{\partial a_j}\frac{B_{0j}(\mathbf{a})}{J(\mathbf{a},t)}\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)},$$

where $J=\left|\partial q^i/\partial a^j\right|$ and $_0$ indicates that these functions are attributes of the Lagrangian fluid elements and thus depend on the label $\mathbf{a}$.

MHD Hamiltonian in Lagrangian variables

- The Hamiltonian $H[\mathbf{q},\pi]$ is

$$H[\mathbf{q},\pi]=\int d\mathbf{a}\left[\frac{\pi_i\pi^i}{2\rho_0}+\rho_0 U\left(s_0,\rho_0/J\right)+\frac{\partial q_i}{\partial a^k}\frac{\partial q^i}{\partial a^\ell}\frac{B_0^k B_0^\ell}{8\pi J}\right].$$

and the equations of motion are

$$\dot{\pi}_i=\{\pi_i,H\}=-\frac{\delta H}{\delta q^i}\quad\text{and}\quad\dot{q}^i=\{q^i,H\}=\frac{\delta H}{\delta \pi_i},$$

where ' $\cdot$ ' means derivative with respect to t at fixed label $\mathbf{a}$ and the Poisson bracket $\{\cdot,\cdot\}$ is canonical and given by

$$\{F,G\}=\int d\mathbf{a}\left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta \pi_i}-\frac{\delta G}{\delta q^i}\frac{\delta F}{\delta \pi_i}\right).$$

Eulerian equilibria. The energy Casimir functional

• In the Hamiltonian context equilibrium configurations are extremal points of the MHD Hamiltonian using the (known) Casimir invariants as constraints[6]. Different choices of the Casimir invariants lead to different equilibria. We consider the energy-Casimir functional

$$\mathfrak{F} = H + \Sigma_i C_i$$

and calculate its first variation.
In general it is not easy to find explicit expressions for the Casimir invariants in such a way that sufficiently general families of equilibria that include plasma flows can be described. Thus the energy Casimir method is generally applied to the search of configurations that are assumed to have geometrical symmetries.

See T. Andreussi, P. J. Morrison, and F. Pegoraro, Plasma Phys. Control. Fusion 52, 055001(2010), & Phys. Plasmas 19, 2102 (2012), & Phys. Plasmas 20, 092104 (2013), & Phys. Plasmas 22, 039903 (2015), & Phys. Plasmas 23, 102112 (2016)

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[6] In order to avoid the trivial null extremum

Eulerian equilibria. The energy Casimir functional

For an axisymmetric and translationally invariant configuration the first variation leads to the generalized 1-D Grad-Shafranov equation

$$\frac{1}{4\pi r}\frac{d}{dr}\left[\left(1-\frac{4\pi\mathscr{F}^2}{\rho}\right)r\frac{d\psi}{dr}\right]=\rho T\mathscr{S}'-\rho\,\mathscr{J}'-B_z\mathscr{H}'$$
$$-\rho v_z\mathscr{G}'-\left(v_\phi B_\phi+v_zB_z\right)\mathscr{F}',$$

where now a prime denotes differentiation with respect to the flux function ψ (here $B_\phi=\nabla\psi\times\nabla z$) and specific equilibrium solutions are defined by the choice of the Casimir functions $\mathscr{F}$, $\mathscr{H}$, $\mathscr{J}$, $\mathscr{G}$ and $\mathscr{S}$ as functions of ψ

$$\mathscr{F}B_\phi = \rho v_\phi\,,\qquad \mathscr{F}B_z+\rho\mathscr{G}=\rho v_z\,,$$
$$\mathscr{H}+\mathscr{F}v_z = \frac{B_z}{4\pi}\,,\qquad \mathscr{J}+v_z\mathscr{G}=v_z^2/2+v_\phi^2/2+c_s^2\ln(\rho/\rho_0)\,.$$

and all terms must be expressed in terms of ψ.

Dynamically accessible equilibria

Dynamically accessible variations[7] (DA) allow us to bypass the difficulty of having to find the explicit expression of the Casimirs. DA restricts the variations to be those generated by the noncanonical Poisson brackets. This ensures that kinematical constraints are satisfied. The first order DA variations are:

$$\delta\rho_{\rm da} = \nabla\cdot(\rho\mathbf{g}_1)\,, \quad \delta\mathbf{v}_{\rm da} = \nabla g_3 + s\nabla g_2 + (\nabla\times\mathbf{v})\times\mathbf{g}_1 + \mathbf{B}\times(\nabla\times\mathbf{g}_4)/\rho$$
$$\delta s_{\rm da} = \mathbf{g}_1\cdot\nabla s\,, \quad \delta\mathbf{B}_{\rm da} = \nabla\times(\mathbf{B}\times\mathbf{g}_1)\,,$$

with $\mathbf{g}_1$, g_2, g_3, and $\mathbf{g}_4$ arbitrary. The variation of the Hamiltonian gives

$$\delta H_{\rm da} = \int d\mathbf{x}\Big[\mathbf{g}_1\cdot\left(\rho\mathbf{v}\times(\nabla\times\mathbf{v}) - \rho\nabla|\mathbf{v}|^2/2 - \rho\nabla h + \rho T\nabla s + \mathbf{j}\times\mathbf{B}\right)$$
$$- g_2\nabla\cdot(\rho s\mathbf{v}) - g_3\nabla\cdot(\rho\mathbf{v}) + \mathbf{g}_4\cdot\nabla\times(\mathbf{v}\times\mathbf{B})\Big] = 0\,,$$

The vanishing of the terms multiplying the independent quantities $\mathbf{g}_1$, g_2, g_3, and $\mathbf{g}_4$ gives the Eulerian equilibrium equations.

7 P. J. Morrison and D. Pfirsch, Phys. Rev. A 40, 3898 (1989)

Non static Lagrangian equilibria

Eulerian equilibria with flows are not Lagrangian equilibria. To treat equilibria that are not static we use a *time dependent relabelling transformation*[8] $\mathbf{a} = \mathfrak{A}(\mathbf{b},t)$, with the inverse $\mathbf{b} = \mathfrak{B}(\mathbf{a},t)$, which gives rise to the new dynamical variables and non separable Hamiltonian

$$\Pi(\mathbf{b},t) = \mathfrak{J}\,\pi(\mathbf{a},t)\,, \qquad \mathbf{Q}(\mathbf{b},t) = \mathbf{q}(\mathbf{a},t)\,,$$

$$\tilde{H}[\mathbf{Q},\Pi] = H - \int d\mathbf{b}\,\Pi\cdot(\mathbf{V}\cdot\nabla_b\mathbf{Q}) = K + H_f + W$$

$$= \int d\mathbf{b}\Big[\frac{\Pi_i\Pi^i}{2\tilde{\rho}_0} - \Pi_i V^j\frac{\partial Q^i}{\partial b^j} + \tilde{\rho}_0\,U\left(\tilde{s}_0,\tilde{\rho}_0/\tilde{J}\right) + \frac{\partial Q_i}{\partial b^k}\frac{\partial Q^i}{\partial b^\ell}\frac{\tilde{B}_0^k\tilde{B}_0^\ell}{8\pi\tilde{J}}\Big]$$

K is the kinetic energy, H_f is due to the relabelling,
W is the sum of the internal and magnetic field energies,
$\mathbf{V}(\mathbf{b},t) = \dot{\mathfrak{B}}\circ\mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{B}(\mathbf{b},t),t)$, is the velocity of the label,
$\nabla_b = \partial/\partial\mathbf{b}$, $\mathfrak{J} = \det(\partial a^i/\partial b^j)$, $\tilde{J} = \det(\partial Q^i/\partial b^j) = J\,\mathfrak{J}$,
$\tilde{\rho}_0/\tilde{J} = \rho_0/J$, $\tilde{s}_0(\mathbf{b},t) = s_0(\mathfrak{A}(\mathbf{b},t))$, $B^i(\mathbf{x},t) = [\partial Q^i/\partial b_j]\,[\tilde{B}_{0j}/\tilde{J}]|_{\mathbf{b}=\mathbf{Q}^{-1}(\mathbf{x},t)}$

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[8] T. Andreussi, P. J. Morrison, and F. Pegoraro, Phys. Plasmas 22, 039903 (2015)

Non static Lagrangian equilibria: relabelling

The extremization of Hamiltonians give equilibrium equations.
For the Hamiltonian $H[\mathbf{q},\pi]$ this gives static equilibria,
from the Hamiltonian $\tilde{H}[\mathbf{Q},\Pi]$ in relabelled variables one obtains stationary equilibria.
Relabelling allows us to express stationary equilibria in terms of Lagrangian variables, which would ordinarily be time dependent, as time-independent orbits with moving labels.
The equilibrium equations are (index $_e$)

$$0 = \partial_t \mathbf{Q}_e = \Pi_e / \tilde{\rho}_0 - \mathbf{V}_e \cdot \nabla_b \mathbf{Q}_e \qquad 0 = \partial_t \Pi_e = -\nabla_b \cdot (\mathbf{V}_e \otimes \Pi_e) + \mathbf{F}_e \,,$$

where $\mathbf{F}_e$ comes from the W part of the Hamiltonian.

Using $\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathfrak{B}_e(\mathbf{a},t)$ and $\mathbf{V}(\mathbf{b},t) = \mathbf{v}_e(\mathbf{b})$, where $\mathbf{v}_e(\mathbf{b})$ denotes an Eulerian equilibrium state and setting $\mathbf{b} = \mathbf{x}$ we recover the usual stationary equilibrium equation $\nabla \cdot (\rho_e \mathbf{v}_e \mathbf{v}_e) = \mathbf{F}_e$, where $\rho_e(\mathbf{x})$ is the usual equilibrium density.

Energy Casimir stability: translational symmetry

For MHD equilibria that satisfy $\delta\mathfrak{F} = 0$ a sufficient condition for stability follows if the second variation $\delta^2\mathfrak{F}$ can be shown to be positive definite.

For perturbations invariant along z, $\delta^2\mathfrak{F}$ can be put into the form

$$\delta^2\mathfrak{F}[Z_e;\delta Z_s] = \int d\mathbf{x}\left[a_1\,|\delta\mathbf{S}|^2 + a_2\,(\delta Q)^2 + a_3(\delta R_z)^2 + a_4\,|\delta\mathbf{R}_\perp|^2 + a_5\,(\delta\psi)^2\right],$$

where $(\delta\mathbf{S},\delta\mathbf{R},\delta Q,\delta\psi)$ are linear combinations of $(\delta\mathbf{v},\delta\mathbf{B},\delta\rho,\delta\psi)$.

The coefficients a_i depend on space through the equilibrium density, Alfvén and sound velocity $c_a^2 = B^2/(4\pi\rho)$ and $c_s^2 = \partial p/\partial\rho$ and the poloidal Alfvén Mach number $M^2 = 4\pi\mathscr{F}^2/\rho$.

Energy Casimir stability: translational symmetry

Extremizing over all variables except $\delta\psi$ and back substituting

$$\delta^2\mathfrak{F}[Z_e;\delta\psi] = \int d\mathbf{x}\left[b_1\,|\nabla\delta\psi|^2 + b_2\,(\delta\psi)^2 + b_3\left|\mathbf{e}_\psi\times\nabla\delta\psi\right|^2\right],$$

with $\mathbf{e}_\psi = \nabla\psi/|\nabla\psi|$

$$b_1 = \frac{1-\mathscr{M}^2}{4\pi}\,\frac{c_s^2-\mathscr{M}^2\left(c_s^2+c_a^2\right)}{c_s^2-M^2\left(c_s^2+c_a^2\right)+\frac{\mathscr{M}^4}{4\pi\rho}|\nabla\psi|^2},$$

$$b_2 = \nabla\cdot\left[\frac{\partial}{\partial\psi}\left(\frac{\mathscr{M}^2}{4\pi}\right)\nabla\psi\right] - \frac{\partial^2}{\partial\psi^2}\left(p+\frac{B_z^2}{8\pi}+\frac{2}{4\pi}|\nabla\psi|^2\right),$$

$$b_3 = \frac{1-\mathscr{M}^2}{4\pi} - b_1\,.$$

and $\mathscr{M}^2 = 4\pi\mathscr{F}^2/\rho < 1$ has been assumed.

Lagrangian stability

Expand $\mathbf{Q} = \mathbf{Q}_e(\mathbf{b},t) + \eta(\mathbf{b},t)\,, \quad \Pi = \Pi_e(\mathbf{b},t) + \pi_\eta(\mathbf{b},t)\,,$ and calculate the second variation of the Hamiltonian in terms of the relabelled canonically conjugate variables $(\eta\,,\pi_\eta)$

$$\delta^2 H_{\mathrm{la}}\left[Z_e;\eta\,,\pi_\eta\right] = \frac{1}{2}\int d\mathbf{x}\left[\frac{1}{\rho_e}\left|\pi_\eta - \rho_e \mathbf{v}_e \cdot \nabla\eta\right|^2 + \eta\cdot\mathfrak{V}_e\cdot\eta\right],$$

which depends on the time independent equilibrium quantities $Z_e = (\rho_e, s_e, \mathbf{v}_e, \mathbf{B}_e)$, the operator $\mathfrak{V}_e$ has no explicit time dependence.

$$\delta^2 W_{\mathrm{la}}\left[Z_e;\eta\right] = \frac{1}{2}\int d\mathbf{x} \quad \eta\cdot\mathfrak{V}_e\cdot\eta =$$

$$\frac{1}{2}\int d\mathbf{x}\left[\rho_e\left(\mathbf{v}_e\cdot\nabla\mathbf{v}_e\right)\cdot\left(\eta\cdot\nabla\eta\right) - \rho_e\left|\mathbf{v}_e\cdot\nabla\eta\right|^2\right] + \delta^2 W\left[Z_e;\eta\right],$$

is identical to the functional obtained by Frieman and Rotenberg.

Lagrangian stability

The Hamilton equations for π_η and η give

$$\rho_e \left(\partial^2 \eta / \partial t^2\right) + 2\rho_e \mathbf{v}_e \cdot \nabla \left(\partial \eta / \partial t\right) = \mathbf{F}_e,$$

with

$$\mathbf{F}_e(\eta) = \nabla \cdot (\rho_e \eta \mathbf{v}_e \cdot \nabla \mathbf{v}_e - \rho_e \mathbf{v}_e \mathbf{v}_e \cdot \nabla \eta) +$$

$$\nabla \left[\rho_e (\partial p_e / \partial \rho_e) \nabla \cdot \eta + \eta \cdot \nabla p_e\right] + \left[\mathbf{B}_e \cdot \nabla \delta \mathbf{B} + \delta \mathbf{B} \cdot \nabla \mathbf{B}_e - \nabla \left(\mathbf{B}_e \cdot \delta \mathbf{B}\right)\right] / (4\pi).$$

Due to the arbitrariness of π_η which does not contribute to $\delta^2 W_{\mathrm{la}}$, the quadratic term $\left|\pi_\eta - \rho_e \mathbf{v}_e \cdot \nabla \eta\right|^2$ in the integrand can be put equal to zero and a sufficient condition for stability is given by $\delta^2 W_{\mathrm{la}} > 0$ for any perturbation η.

Lagrangian stability

The term $\delta^2 W$ can be written in the standard way[9]

$$\delta^2 W\left[Z_e;\eta\right]=\frac{1}{2}\int d\mathbf{x}\left[\rho_e\frac{\partial p_e}{\partial\rho_e}\left(\nabla\cdot\eta\right)^2+\left(\nabla\cdot\eta\right)\left(\nabla p_e\cdot\eta\right)+\frac{|\delta\mathbf{B}|^2}{4\pi}+\mathbf{J}_e\times\eta\cdot\delta\mathbf{B}\right],$$

where $4\pi\mathbf{J}_e=\nabla\times\mathbf{B}_e$ is the equilibrium current and $\delta\mathbf{B}=\nabla\times\left(\eta\times\mathbf{B}_e\right)$.

• The first order Eulerian perturbations induced by the Lagrangian variation written in terms of the displacement η:

$$\begin{aligned}
&\delta\rho_{\rm la}=-\nabla\cdot\left(\rho_e\eta\right), \qquad \delta s_{\rm la}=-\eta\cdot\nabla s_e\\
&\delta\mathbf{v}_{\rm la}=\pi_\eta/\rho_e-\eta\cdot\nabla\mathbf{v}_e=\partial\eta/\partial t+\mathbf{v}_e\cdot\nabla\eta-\eta\cdot\nabla\mathbf{v}_e\\
&\delta\mathbf{B}_{\rm la}=-\nabla\times\left(\mathbf{B}_e\times\eta\right)
\end{aligned}$$

where $\delta s_{\rm la}$ can be replaced by the pressure perturbation, $\delta p_{\rm la}=-\gamma p_e\nabla\cdot\eta-\eta\cdot\nabla p_e$, that is often used.

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9 I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London, Ser. A 244, 17 (1958)

Dynamical accessible stability

Dynamically accessible stability is assessed by expanding the Hamiltonian expressed in Eulerian variables to second order using the dynamically accessible constraints to this order:

$$\delta^2 H_{\rm da}[Z_e;\mathbf{g}] = \int d\mathbf{x}\ \rho \left|\delta \mathbf{v}_{\rm da} - \mathbf{g}_1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{g}_1\right|^2 + \delta^2 W_{\rm la}[\mathbf{g}_1]\,.$$

If $\delta \mathbf{v}_{\rm da} = \nabla g_3 + s\nabla g_2 + (\nabla \times \mathbf{v}) \times \mathbf{g}_1 + \mathbf{B} \times (\nabla \times \mathbf{g}_4)/\rho$ were independent and arbitrary we could use it to nullify the first term. Then setting $\mathbf{g}_1 = -\eta$, we would see that dynamically accessible stability is identical to Lagrangian stability.

However in general there is not sufficient freedom in the generating functions to cancel the positive definite first term[10] [*solvability condition*].

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[10] see also E. Hameiri, Phys. Plasmas 10, 2643 (2003), Phys. Plasmas 11, 3423 (2004)

Comparison between the three different criteria

Because different constraints are imposed, stability conditions for dissipationless fluids and magnetofluids take different forms when derived within the Lagrangian, Eulerian (energy-Casimir), or dynamical accessible frameworks.

We obtained three quadratic energy expressions which can be written in terms of the Eulerian perturbation variables

$$\mathfrak{P} = \{\delta\rho, \delta\mathbf{v}, \delta s, \delta\mathbf{B}\}\,.$$

Different perturbations are associated with the three expressions and, we recall, can be written as

$$\begin{cases} \delta\rho_{\rm la} & = -\nabla\cdot(\rho\eta) \\ \delta\mathbf{v}_{\rm la} & = \frac{\partial\eta}{\partial t} + \mathbf{v}\cdot\nabla\eta - \eta\cdot\nabla\mathbf{v} \\ \delta s_{\rm la} & = -\eta\cdot\nabla s \\ \delta\mathbf{B}_{\rm la} & = -\nabla\times(\mathbf{B}\times\eta) \end{cases} \qquad \begin{cases} \delta\rho_{\rm ec} \\ \delta\mathbf{v}_{ec} \\ \delta s_{\rm ec} \\ \delta\mathbf{B}_{ec} \end{cases} \qquad \begin{cases} \delta\rho_{\rm da} & = -\nabla\cdot(\rho\mathbf{g}_1) \\ \delta\mathbf{v}_{\rm da} & = \mathbf{X} + \mathbf{v}\cdot\nabla\mathbf{g}_1 - \mathbf{g}_1\cdot\nabla\mathbf{v} \\ \delta s_{\rm da} & = -\mathbf{g}_1\cdot\nabla s \\ \delta\mathbf{B}_{\rm da} & = -\nabla\times(\mathbf{B}\times\mathbf{g}_1) \end{cases}$$

where $\mathbf{X} = 2(\mathbf{v}\cdot\nabla)\mathbf{g}_1 + \mathbf{v}\times(\nabla\times\mathbf{g}_1) + s\nabla g_2 + \nabla g_3 + \frac{1}{\rho}\mathbf{B}\times(\nabla\times\mathbf{g}_4)$

Comparison between the three different criteria

In the case of the Lagrangian energy, the set of perturbations $\mathfrak{P}_{\rm la}$ are constrained, while for the energy-Casimir expression the perturbations $\mathfrak{P}_{\rm ec}$ are entirely unconstrained provided they satisfy the translation symmetry we have assumed.
The dynamically accessible perturbations are constrained.
Thus the following inclusion applies

$$\mathfrak{P}_{\rm da} \subset \mathfrak{P}_{\rm la} \subset \mathfrak{P}_{\rm ec} \,,$$

which leads to the conclusion

$$\mathfrak{stab}_{ec} \Rightarrow \mathfrak{stab}_{\rm la} \Rightarrow \mathfrak{stab}_{\rm da} \,,$$

Dynamically accessible stability is the most limited because its perturbations are the most constrained, while energy-Casimir stability is the most general, when it exists, for its perturbations are not constrained at all.

Explicit comparison in the case of a rigid rotating isothermal configuration

We use cylindrical coordinates (r, ϕ, z) and consider plasma equilibrium configurations where all equilibrium quantities depend only on the radial coordinate r:

$$\mathbf{B} = B_z(r)\hat{\mathbf{z}} + B_\phi(r)\hat{\phi}, \quad \mathbf{v} = v_z(r)\hat{\mathbf{z}} + v_\phi(r)\hat{\phi},$$

$$\rho = \rho(r), \quad s = s(r), \quad B_\phi = \hat{\phi}\cdot\nabla\psi\times\hat{\mathbf{z}} = -\frac{d\psi(r)}{dr}.$$

Generalized Grad-Shafranov equation for the flux function $\psi(r)$

$$\frac{1}{r}\frac{d}{dr}\left(\frac{1-\mathscr{M}^2}{4\pi} rB_\phi\right) - \frac{1}{\psi_r}\frac{d}{dr}\left(p + \frac{B_z^2}{8\pi}\right) + \frac{d}{dr}\left(\frac{\mathscr{M}^2}{4\pi}B_\phi\right) = 0,$$

where

$$\mathscr{M}(r) = \left[4\pi\rho(r)v_\phi^2(r)/B_\phi^2(r)\right]^{1/2}$$

is the poloidal Alfvèn Mach number, $v_z(r)$ does not appear in GGS and will be set equal to zero.

Explicit comparison in the case of a rigidly rotating, isothermal, uniform current configuration

Define a dimensionless radius r and use dimensionless units.
Set $B_z(r) = B_z$, $B_\phi(r) = B_0 r$, and $v_\phi(r) = \Omega r$ with B_z, B_0, Ω constants.
Treat GGS as an equation for $p(r)$. Since the plasma is isothermal the relationship between $p(r)$ and $\rho(r)$ is linear. One obtains a one-parameter family of equilibria with $w = \Omega r_0 / c_s$ $(w^2/2 < 1)$

$$\hat{p}(r) = \frac{2}{w^2}\left[1 - \left(1 - \frac{w^2}{2}\right)\exp\left(\frac{w^2 r^2}{2}\right)\right],$$

c_s sound velocity, $\hat{p}(0) = 1$, $\hat{p}(\bar{r}) = 0$ for $\bar{r}^2 = -(2/w^2)\ln(1 - w^2/2)$.
For $w \to 0$ it reduces to the standard parabolic pinch with $\bar{r} = 1$ and $\hat{p}(r) = 1 - r^2$, while for $w^2 \to 2$ we have $\bar{r} = \infty$ and $\hat{p}(r) \equiv 1$.
A uniform B_z field does not alter these equilibrium configurations but affects their stability.

Explicit comparison in the case of a rigidly rotating, isothermal, uniform current configuration

We performed[11] an analytical comparison of the stability boundaries in the w, $\hat{b} = B_z/B_0$ plane for translationally invariant perturbations illustrating the different steps in the procedure including the derivation of the equilibrium from the first variation of the Hamiltonian in the three different formulations and the explicit implementation of the time dependent relabelling.

• The Lagrangian and the dynamically accessible approaches lead to equivalent conditions.
The constraints obeyed by the dynamically accessible perturbations in the presence of flows lead to an additional stabilizing term that cannot be made to vanish for azimuthally symmetric perturbations. This term does not modify the stability analysis since azimuthally symmetric perturbations are stable even within the Lagrangian framework. For more general equilibria this need not be the case.

[11] T. Andreussi, P.J. Morrison, F. Pegoraro, Phys. Plasmas, 23, 102112 (2016).

Explicit comparison in the case of a rigidly rotating, isothermal, uniform current configuration

The minimization of $\delta^2 W_{\mathrm{la}}$ leads to the study of the positivity of a 3×3 matrix[12] function of the equilibrium quantities for $|m| = 1$ perturbations.
A necessary and sufficient condition for the positivity of this matrix is provided by the Sylvester criterion which yields $w^2 < 1/2$ for $B_z = 0$ and $w^2 B_z^2 < 1$ for $B_z \neq 0$ and $w^2 \to 0$ and $B_z^2/B_0 < 1/3$, for $w^2 \to 1/2^-$.
A partial minimization procedure with respect to η_ϕ (to η_z and η_ϕ for $B_z \neq 0$) leads to less restrictive conditions: $w^2 \lesssim 0.62$ for $B_z = 0$ and $w^2 \lesssim 0.46$ choosing, e.g., $B_z/B_0 = 1$.
Even less restrictive conditions could be found by solving the Euler-Lagrange equation for η_r obtained via variation of the resulting "reduced" $\delta^2 \tilde{W}_{\mathrm{la}}$ subject to the constraint of $\int r dr |r\eta_r|^2$.
Such a procedure leads to an eigenvalue equation that can be searched for the lowest eigenvalue.

[12] A 4×4 matrix for $B_z \neq 0$ as η_z is no longer decoupled

Explicit comparison in the case of a rigidly rotating, isothermal, uniform current configuration

Extremization of the energy-Casimir functional over all variables except $\delta\psi$ leads to sufficient stability bounds on w^2 that, similarly to the Lagrangian case, become stricter as B_z^2 increases.

These bounds are in general more restrictive than those found within the Lagrangian framework, as shown, e.g., by considering again $B_z^2 = 1$*, in which case we find* $w^2 \lesssim 0.31$*.*

Sharper stability conditions could be obtained by solving the Euler-Lagrange equation associated with this “reduced ” energy-Casimir functional subject to a normalization constraint on $\delta\psi$.

Conclusions

The methods described in this presentation for the three approaches are of general utility – they apply to all important plasma models, kinetic as well as fluid, when dissipation is neglected. In fact, the approaches were compared[13] for the Vlasov and guiding-center kinetic equations including a dynamically accessibly calculation in this kinetic context.

Given the large amount of recent progress on extended magnetofluid models a great many stability calculations like the ones described here are possible.

Finally note that the time dependent relabelling does not require that the configuration be stationary in Eulerian variables and thus can be applied to the time evolution of any ideal MHD configuration in order to study its orbital stability.

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[13] P. J. Morrison and D. Pfirsch, Phys. Rev. A 40, 3898 (1989) & Phys. Fluids B 2, 1105 (1990).

Thanks for your attention

Energy Casimir - second variation- helical symmetry

$$\delta^2 \mathfrak{F} = \int d^3x \, [a_1|\delta \mathbf{S}|^2 + a_2(\delta Q)^2 + a_3|\delta \mathbf{R}|^2 + a_4(\delta\psi)^2],$$

where

$$a_1 = \frac{1}{\rho}, \quad a_2 = \frac{\rho}{(c_s^2 - \mathrm{M}^2 c_a^2)},$$

$$a_3 = \frac{4\pi(c_s^2 - \mathrm{M}^2 c_a^2)}{(c_s^2 + c_a^2)(\mathrm{M}_c^2 - \mathrm{M}^2)},$$

$$a_4 = -\Upsilon - a_1 \left|\frac{\delta \mathbf{S}}{\delta\psi}\right|^2_{\bar{z}_s} - a_2 \left.\frac{\delta Q}{\delta\psi}\right|^2_{\bar{z}_s} - a_3 \left|\frac{\delta \mathbf{R}}{\delta\psi}\right|^2_{\bar{z}_s}$$

$$\delta \mathbf{S} = \rho\delta\mathbf{v} + \mathbf{v}\delta\rho - \mathcal{F}\delta\mathbf{B} - \mathcal{F}'\mathbf{B}\delta\psi - \frac{\mathbf{h}}{k}(\mathcal{G}\delta\rho + \rho\mathcal{G}'\delta\psi),$$

$$\delta Q = \frac{1}{\rho}(c_s^2 - \mathrm{M}^2 c_a^2)\delta\rho - \mathcal{M}\,\delta\psi + \frac{\mathrm{M}^2}{4\pi}\frac{\mathbf{B}}{\rho}\delta\mathbf{B},$$

$$\delta \mathbf{R} = \frac{(c_s^2 + c_a^2)(\mathrm{M}_c^2 - \mathrm{M}^2)}{4\pi(c_s^2 - \mathrm{M}^2 c_a^2)}\delta\mathbf{B} - \left[\frac{2\mathcal{F}\mathcal{F}'}{\rho}\mathbf{B} + \mathcal{N}\mathbf{h} + \frac{\mathrm{M}^2\mathcal{M}}{(c_s^2 - \mathrm{M}^2 c_a^2)}\frac{\mathbf{B}}{4\pi}\right]\delta\psi,$$

Circulation Integral

Consider the variation of the circulation integral $\Gamma = \oint_c \mathbf{v} \cdot \mathbf{dx}$ on a fixed closed contour c for an equilibrium with $\mathbf{v}_e \equiv 0$ and $\mathbf{B}_e \neq 0$. A general Lagrangian variation $\delta \mathbf{v}_{\mathrm{la}}$ can generate any amount of circulation. For a dynamically accessible variation,

$$\delta\Gamma = \oint_c \delta \mathbf{v}_{\mathrm{da}} \cdot \mathbf{dx} = \oint_c s \nabla g_2 \cdot \mathbf{dx} + \oint_c (\nabla \times \mathbf{g}_4) \cdot (\mathbf{dx} \times \mathbf{B})/\rho$$

If c is a closed magnetic field line $\mathbf{dx} \parallel \mathbf{B}$ and $\mathbf{B} \cdot \nabla s = 0$ along c

$$\delta\Gamma_B = \oint_c s \nabla g_2 \cdot \mathbf{dx} = \oint_c (\nabla(s g_2) - g_2 \nabla s) \cdot \mathbf{dx} = -\oint_c g_2 \nabla s \cdot \mathbf{dx} = 0 \, .$$

If instead c is taken at constant s, and $\mathbf{B} \cdot \nabla s \neq 0$ along c,

$$\delta\Gamma_s = \oint_c (\nabla \times \mathbf{g}_4) \cdot (\mathbf{dx} \times \mathbf{B})/\rho \neq 0 \qquad (1)$$

In silico studies

A number of numerical magnetohydrodynamic (MHD) codes have been developed for modeling plasma flows in astrophysics.
Some of the most well-known codes are ZEUS (Stone and Norman, 1992a), FLASH (Fryxell, 2000), PLUTO (Mignone et al., 2007), and ATHENA (Stone et al., 2008; Skinner and Ostriker, 2010).
A number of different numerical algorithms have also been developed for the numerical integration of the MHD equations including different approaches for the spatial and temporal approximations (Brio and Wu, 1988; Cockburn et al., 1989; Dai and Woodward, 1994a; 1994b; Ryu et al., 1995; Balsara and Spicer, 1999; Gurski, 2004; Ustyugov, 2009).
The linear MHD stability code MINERVA investigates the toroidal rotation effect on the stability of ideal MHD modes in tokamak plasmas. This code solves the Frieman-Rotenberg equations.

Non canonical Poisson brackets for helical symmetry

$$F_{B_h} = F_{\mathbf{B}}\cdot\mathbf{h}, \quad F_\psi = \nabla\cdot(F_{\mathbf{B}}\times k\mathbf{h}), \quad \text{and}$$
$$F_{\mathbf{M}} = F_{M_h}\mathbf{h} + F_{\mathbf{M}_\perp}. \qquad (16)$$

In term of the variables $\mathcal{Z}_S := (\rho, \mathbf{M}_\perp, M_h, \sigma, \psi, B_h)$, the Poisson bracket of Eq. (3) transforms into the "symmetric" MHD bracket given by

$$\begin{aligned}
\{F,G\}_{\mathcal{SYM}} = -\int_V \{ & \rho(F_{\mathbf{M}_\perp}\cdot\nabla G_\rho - G_{\mathbf{M}_\perp}\cdot\nabla F_\rho) \\
& + M_h[F_{\mathbf{M}_\perp}\cdot\nabla(kG_{M_h}) - G_{\mathbf{M}_\perp}\cdot\nabla(kF_{M_h})]/k \\
& + (k^2[l]\sin 2\alpha)M_h\mathbf{h}\cdot(F_{\mathbf{M}_\perp}\times G_{\mathbf{M}_\perp}) + \mathbf{M}_\perp \\
& \cdot[(F_{\mathbf{M}_\perp}\cdot\nabla)G_{\mathbf{M}_\perp} - (G_{\mathbf{M}_\perp}\cdot\nabla)F_{\mathbf{M}_\perp}] \\
& + \sigma(F_{\mathbf{M}_\perp}\cdot\nabla G_\sigma - G_{\mathbf{M}_\perp}\cdot\nabla F_\sigma) \\
& + kB_h[F_{\mathbf{M}_\perp}\cdot\nabla(G_{B_h}/k) - G_{\mathbf{M}_\perp}\cdot\nabla(F_{B_h}/k)] \\
& + \psi(F_{\mathbf{M}_\perp}\cdot\nabla G_\psi - G_{\mathbf{M}_\perp}\cdot\nabla F_\psi) \\
& - \psi(F_\psi\nabla\cdot G_{\mathbf{M}_\perp} - G_\psi\nabla\cdot F_{\mathbf{M}_\perp}) \\
& - (k^3[l]\sin 2\alpha)\nabla\psi\cdot(F_{B_h}G_{\mathbf{M}_\perp} - G_{B_h}F_{\mathbf{M}_\perp}) \\
& + \psi([G_{B_h}/k, kF_{M_h}] - [F_{B_h}/k, kG_{M_h}])\}d^3r,
\end{aligned} \qquad (17)$$

where $[F,G] := (\nabla F\times\nabla G)\cdot k\mathbf{h}$. Because this calculation is

The Poisson bracket of (3) can be rewritten in terms of any complete set of variables—switching from one set to another amounts to a change of coordinates. A convenient form of the MHD Poisson bracket is obtained by using, instead of the variables $\mathbf{v}$ and s, the density variables $\mathbf{M} = \rho\mathbf{v}$ and $\sigma = \rho s$. We let $\mathcal{Z} := (\rho, \mathbf{M}, \sigma, \mathbf{B})$ denote the new set. To transform from Z to $\mathcal{Z}$, we use the functional chain rule identities,

$$F_\rho|_{\mathbf{v},s} = F_\rho|_{\mathbf{M},\sigma} + \mathbf{v}\cdot F_{\mathbf{M}} + sF_\sigma, \quad F_{\mathbf{v}} = \rho F_{\mathbf{M}}, \quad F_s = \rho F_\sigma, \qquad (10)$$

with $F_{\mathbf{B}}$ unchanged, to transform the Poisson bracket of (3) into

$$\begin{aligned}
\{F,G\}_{\mathcal{Z}} = -\int_V \{ & \rho(F_{\mathbf{M}}\cdot\nabla G_\rho - G_{\mathbf{M}}\cdot\nabla F_\rho) \\
& + \mathbf{M}\cdot[(F_{\mathbf{M}}\cdot\nabla)G_{\mathbf{M}} - (G_{\mathbf{M}}\cdot\nabla)F_{\mathbf{M}}] \\
& + \sigma(F_{\mathbf{M}}\cdot\nabla G_\sigma - G_{\mathbf{M}}\cdot\nabla F_\sigma) \\
& + \mathbf{B}\cdot[(F_{\mathbf{M}}\cdot\nabla)G_{\mathbf{B}} - (G_{\mathbf{M}}\cdot\nabla)F_{\mathbf{B}}] \\
& + \mathbf{B}\cdot(\nabla F_{\mathbf{M}}\cdot G_{\mathbf{B}} - \nabla G_{\mathbf{M}}\cdot F_{\mathbf{B}})\}d^3r.
\end{aligned} \qquad (11)$$

The bracket of (11) is the Lie-Poisson bracket (see Ref. 22), i.e., a bracket linear in each variable, obtained in Ref. 16.

Hamiltonian symmetries

Systems with symmetry possess other invariants, constants of motion that commute with the particular Hamiltonian (unlike Casimir invariants, which commute with all Hamiltonians). Extremization of the energy with these constants held fixed, which can be achieved by using Lagrange multipliers, yields relative equilibria, i.e., equilibria in frames of reference generated by the invariants. For example, extremization of $H+\lambda\cdot P$, where P is the momentum, gives a state that is uniformly translating at the velocity λ. Alternatively, one can interpret the extremal points obtained from extremizing the Hamiltonian at fixed invariants as being the equilibrium of interest observed in a different frame of reference. If there exists any frame of reference in which the energy functional is definite, then both linear and, for finite systems, nonlinear instability are precluded. Further, one can analyze the energy with the variations restricted to lie within the surfaces defined by the invariants,

In general, if any combination of known invariants implies the existence of a family of compact invariant sets about an equilibrium, then t that equilibrium is nonlinearly stable.

G-operator

$$\partial_t^2 \xi + 2(\mathbf{v}\cdot\nabla)\partial_t \xi = \mathscr{F}[\xi] = \mathscr{G}[\xi] - (\mathbf{v}\cdot\nabla)^2 \xi.$$

Stability of ideal MHD configurations. I. Realizing the generality of the $\mathscr{G}$ operator

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A field theoretical approach, applied to the time-reversible system described by the ideal magnetohydrodynamic (MHD) equations, exposes the full generality of MHD spectral theory. MHD spectral theory, which classified waves and instabilities of static or stationary, usually axisymmetric or translationally symmetric configurations, actually governs the stability of flowing, (self-)gravitating, single fluid descriptions of nonlinear, time-dependent idealized plasmas, and this *at any time during their nonlinear evolution*. At the core of this theory is a self-adjoint operator $\mathscr{G}$, discovered by Frieman and Rotenberg [Rev. Mod. Phys. **32**, 898 (1960)] in its application to stationary (i.e., time-independent) plasma states. This Frieman-Rotenberg operator dictates the acceleration identified by a Lagrangian displacement field $\boldsymbol{\xi}$, which connects two ideal MHD states in four-dimensional space-time that share initial conditions for density, entropy, and magnetic field. The governing equation reads $\frac{d^2\boldsymbol{\xi}}{dt^2} = \mathscr{G}[\boldsymbol{\xi}]$, as first noted by Cotsaftis and Newcomb [Nucl. Fusion, Suppl. Part **2**, 447 and 451 (1962)]. The time derivatives at left are to be taken in the Lagrangian way, i.e., moving with the flow $\mathbf{v}$. Physically realizable displacements must have finite energy, corresponding to being square integrable in the Hilbert space of displacements equipped with an inner product rule, for which the $\mathscr{G}$ operator is self-adjoint. The acceleration in the left-hand side features the Doppler-Coriolis operator $\mathbf{v}\cdot\nabla$, which is known to become an antisymmetric operator when restricting attention to stationary equilibria. Here, we present all derivations needed to get to these insights and connect results throughout the literature. A first illustration elucidates what can happen when self-gravity is incorporated and presents aspects that have been overlooked even in simple uniform media. Ideal MHD flows, as well as Euler flows, have essentially $6+1$ wave types, where the 6 wave modes are organized through the essential spectrum of the $\mathscr{G}$ operator. These 6 modes are actually three pairs of modes, in which the Alfvén pair (a shear wave pair in hydro) sits comfortably at the middle. Each pair of modes consists of a leftgoing wave and a rightgoing wave, or equivalently stated, with one type traveling from past to future (forward) and the other type that goes from future to past (backward). The Alfvén pair is special, in its left-right categorization, while there is full degeneracy for the slow and fast pairs when reversing time and mirroring space. The Alfvén pair group speed diagram leads to the familiar Elsässer variables.

Shear stabilization?

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Stability of localized modes in rotating tokamak plasmas

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Abstract
The ideal magnetohydrodynamic stability is investigated of localized interchange modes in a large-aspect ratio tokamak plasma. The resulting stability criterion includes the effects of toroidal rotation and rotation shear and contains various well-known limiting cases. The analysis allows for a general adiabatic index, resulting in a stabilizing contribution from the convective effect. A further stabilizing effect from rotation exists when the angular frequency squared decreases radially more rapidly than the density. Flow shear, however, also decreases the stabilizing effect of magnetic shear through the Kelvin–Helmholtz mechanism. Numerical simulations reveal the merits and limitations of the performed local analysis.

A sufficient condition for the ideal instability of shear flow with parallel magnetic field

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A simple sufficient condition is given for the linear ideal instability of plane parallel equilibria with antisymmetric shear flow and symmetric or antisymmetric magnetic field. Application of this condition shows that plane Couette flow, which is stable in the absence of a magnetic field, can be driven unstable by a symmetric magnetic field. Also, although strong magnetic shear can stabilize shear flow with a hyperbolic tangent profile, there exists a range of magnetic shear that causes destabilization.

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Abstract
The present understanding of edge pedestal structure is reviewed. Pedestal plasma strongly affects fusion power and divertor heat load, and as such, characterization of the pedestal structure has significantly progressed. In high-confinement mode (H-mode) plasmas, the pedestal component plays the role of a boundary condition in determining the core heat transport through profile stiffness. On the other hand, a higher global poloidal beta or Shafranov shift improves the stability of the plasma edge in the low magnetic field side particularly at high triangularity. Toroidal rotation also influences the edge stability boundary. While toroidal flow stabilizes high-n ballooning modes, it destabilizes low-n kink/peeling modes. On the basis of this background,

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ORIGINAL PAPER

Stabilization in the ZaP Flow Z-Pinch

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Additional references

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Variational principles for equilibrium states with plasma flow

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A new constant of the motion is utilized to formulate a variational principle for plasma equilibria with general flow fields. Two additional variational principles are derived from the original one. None of these formulations leads to a stability criterion if the velocity is not parallel to the magnetic field since the functionals used, the first of which being the energy, do not possess in this case a minimum but only stationary points. It is shown that other stability criteria already reported in the literature also suffer from the same deficiency. It is suggested that the lack of a minimum is due to the presence of ballooning modes.
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A Lagrangian perspective on the stability of ideal MHD equilibria with flow

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We take a careful look at two approaches to deriving stability criteria for ideal MHD equilibria. One is based on a tedious analysis of the linearized equations of motion, while the other examines the second variation of the MHD Hamiltonian computed with proper variational constraints. For equilibria without flow, the two approaches are known to be fully consistent. However, for equilibria with flow, the stability criterion obtained from the constrained variation approach was claimed to be stronger than that derived using the linearized equations of motion. We show this claim is incorrect by deriving and comparing both criteria within the same framework. It turns out that the criterion obtained from the constrained variation approach has stricter requirements on the initial perturbations than the other. Such requirements naturally emerge in our new treatment of the constrained variation approach using the Euler-Poincaré structure of ideal MHD, which is more direct and simple than the previous derivation from the Poisson perspective.