Fractional Transport in Strongly Turbulent Plasmas

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Introduction I

- Particle transport in weakly turbulent environments ($\delta B/B << 1$) has been discussed extensively with the use of the Fokker-Planck (FP) equation, mostly in combination with the quasi-linear (QL) approximation

Strong turbulence is though also important and abundant

Recent research on the development of strong magnetic turbulence ($\delta B/B \approx 1$) has shown the importance of two scenarios:

1. Extended current filaments (CF) or multiple interacting CFs develop on fast time scales into a strongly turbulent environment, fragmented into a collection of small scale CFs.
2. Propagating Alfvén waves reinforce reconnection at existing CF and new CF are formed.

Introduction II

In this context, we address two open questions:

1. Is the FP equation still valid in strongly turbulent environments ?
2. How to model transport when the FP approach is not valid anymore ?

- In the following
  1. we consider a large scale environment of strong turbulence
  2. and we analyze statistically the energization of particles in this environment, focusing on the high energy part (tail) of the energy distribution.
  3. we develop an appropriate transport model
- Applications: Solar flares, Earth's magnetosphere, accretion disks, jets, ..., may-be the plasma edge in tokamaks ?

The MHD turbulent environment I

- We consider a strongly turbulent environment as it naturally results from the nonlinear evolution of the MHD equations
- We do not set up a specific reconnection geometry
- 3D, nonlinear, resistive, compressible and normalized MHD equations

$$\partial_t \rho = -\nabla \cdot \mathbf{p} \tag{1}$$

$$\partial_t \mathbf{p} = -\nabla \cdot (\mathbf{pu} - \mathbf{BB}) - \nabla P - \nabla B^2/2 \tag{2}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} \tag{3}$$

$$\partial_t (S\rho) = -\nabla \cdot [S\rho \mathbf{u}] \tag{4}$$

  with ρ the density, $\mathbf{p}$ the momentum density, $\mathbf{u} = \mathbf{p}/\rho$, P the thermal pressure, $\mathbf{B}$ the magnetic field, $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{J}$ the electric field, $\mathbf{J} = \nabla \times \mathbf{B}$ the current density, η the resistivity, $S = P/\rho^\Gamma$ the entropy, and $\Gamma = 5/3$ the adiabatic index.
- The MHD equations are solved numerically with the pseudo-spectral method combined a the strong-stability-preserving Runge Kutta scheme of
  - Cartesian coordinates
  - periodic boundary conditions

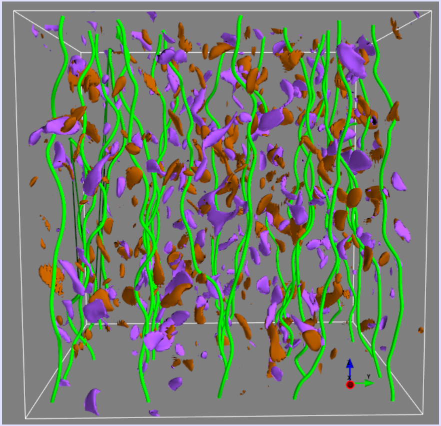
The MHD turbulent environment II

- initial conditions: superposition of Alfvén waves, with a Kolmogorov type spectrum
- constant background magnetic field B_0 in the z-direction.

- The mean value of the initial magnetic perturbation is $< b >= 0.6B_0$, its standard deviation is $0.3B_0$, so that we indeed consider strong turbulence.
- For the MHD turbulent environment to build, we let the MHD equations evolve until the largest velocity component starts to exceed twice the Alvfèn speed.
- The magnetic Reynolds number at final time is $< |\mathbf{u}| > l/\eta = 3.5 \times 10^3$
- The test-particle are tracked in a fixed snapshot of the MHD evolution
- Also, we take into account anomalous resistivity effects by increasing the resistivity to $\eta_{an} = 1000\eta$ locally when the current density $J = |\mathbf{J}|$ exceeds a threshold J_{cr}.

Iso-contours of the supercritical current density component J_z (positive in brown, negative in violet), magnetic field lines (green):

clear fragmentation into a large number of small-scale coherent structures



Test-particle simulations I

- The relativistic guiding center equations (without collisions) are used for the evolution of the position $\mathbf{r}$ and the parallel component $u_{||}$ of the relativistic 4-velocity of the particles (X. Tao et al., PoP **14**, 092107 (2007))

$$\frac{d\mathbf{r}}{dt} = \frac{1}{B^*_{||}}\left[\frac{u_{||}}{\gamma}\mathbf{B}^* + \hat{\mathbf{b}} \times \left(\frac{\mu}{q\gamma}\nabla B - \mathbf{E}^*\right)\right] \quad (5)$$

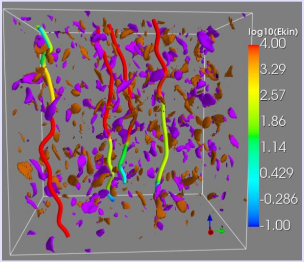
$$\frac{du_{||}}{dt} = -\frac{q}{m_0 B^*_{||}}\mathbf{B}^* \cdot \left(\frac{\mu}{q\gamma}\nabla B - \mathbf{E}^*\right) \quad (6)$$

where $\mathbf{B}^* = \mathbf{B} + \frac{m_0}{q}u_{||}\nabla \times \hat{\mathbf{b}}$, $\mathbf{E}^* = \mathbf{E} - \frac{m_0}{q}u_{||}\frac{\partial \hat{\mathbf{b}}}{\partial t}$, $\mu = \frac{m_0 u_\perp^2}{2B}$ is the magnetic moment, $\gamma = \sqrt{1 + \frac{u^2}{c^2}}$, $B = |\mathbf{B}|$, $\hat{\mathbf{b}} = \mathbf{B}/B$, $u_\perp$ is the perpendicular component of the relativistic 4-velocity, and q, m_0 are the particle charge and rest-mass, respectively.

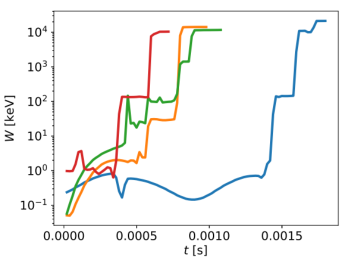
- The test-particles we consider throughout are electrons. Initially, all particles are located at random positions, they obey a Maxwellian distribution with temperature $T = 100\,\mathrm{eV}$. The simulation box is open, the particles can escape from it when they reach any of its boundaries.

- The acceleration process, is very efficient, and we consider a final time of 0.002 s (7×10^5 gyration periods), at which the asymptotic state has already been reached.

4 orbits of energetic particles (reaching 10 MeV), colored according to the logarithm of their kinetic energy in keV



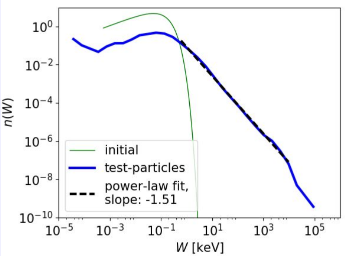
The energy evolution of the same 4 energetic particles



The particles mostly gain energy in a number of sudden jumps in energy, the energization process thus is localized,
and there is multiple energization at different current filaments

Test-particle simulations III

the energy distribution at final time (blue):
clear power law tail,
power-law index -1.51



Question 1

Can the test-particle results be reproduced as a solution of the FP equation ?

- For simplification, we consider the FP equation only in energy space

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial W}\left[Fn - \frac{\partial[Dn]}{\partial W}\right] = -\frac{n}{t_{esc}}, \tag{7}$$

n: the distribution function, W: kinetic energy, t_{esc}: the escape time.
D is the energy diffusion coefficient,

$$D(W,t) = \frac{\left\langle (W(t+\Delta t) - W(t))^2 \right\rangle_W}{2\Delta t}, \tag{8}$$

F is the energy convection coefficient,

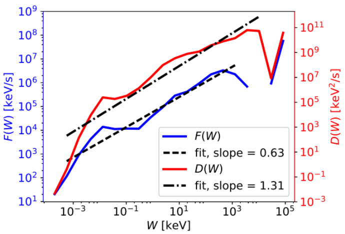
$$F(W,t) = \frac{\langle W(t+\Delta t) - W(t)\rangle_W}{\Delta t}, \tag{9}$$

with Δt a small time-interval.
$\langle ...\rangle_W$ denotes the conditional average that $W(t) = W$

- For the estimate of the coefficients F, D from the simulation data: we monitor the particle energy at a number of fixed times separated by Δt, the conditional averaging is done through binned statistics
  - divide the energies of the particles at time t into a number of logarithmically equi-spaced bins and perform the requested averages separately for the particles in each bin.

The estimates of $F(W)$ and $D(W)$ at $t = 0.002\,\mathrm{s}$ as function of the energy:

—> power-law shape, indices $a_F = 0.63$ and $a_D = 1.31$.

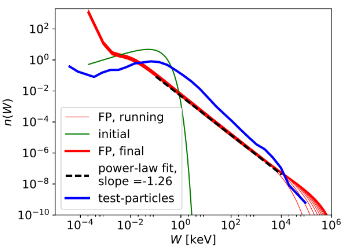


- Verification of the estimates of F and D: insert F and D, into the FP equation and solve it numerically in $[0, \infty)$ (pseudospectral method, based on rational Chebyshev polynomials)
  - escape time estimate $t_{esc} = 0.004\,\mathrm{s}$ (assuming the number of particles staying in the box to decay exponentially)

The solution of the FP equation up to final time 0.002 s:

—> clear power-law tail,

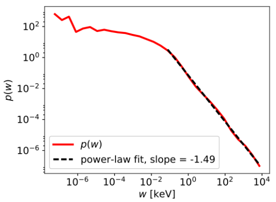
—> much flatter though than the test-particle simulations.



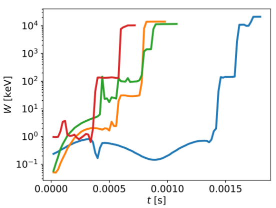
- In Vlahos et al., ApJ **827**, L3, (2016) we have shown that the above procedure can be successful:
  Why does it fail here ?

estimates of F and D are based on the sample of energy increments
$w_j := W_j(t + \Delta t) - W_j(t)$
(with j the particle index)

The distribution of increments has a power law tail (index -1.49)



—> occasionally very large jumps in energy space: Levy flights



- energy increments with a power-law tail imply:
  1. The estimates of F as a mean value and D as a variance theoretically are infinite, and thus in practice they are very problematic
  2. The prerequisites for deriving a FP equation are not fulfilled (see below)

Question 2

How to model transport when the FP approach is not valid anymore ?

- General description of transport in energy space: Chapman-Kolmogorov equation

$$\begin{aligned} n(W,t) &= \int dw \int_0^t d\tau\, n(W-w, t-\tau)\, q_w(w)\, q_\tau(\tau) \\ &\quad + n(W,0)\int_t^\infty q_\tau(\tau) d\tau \end{aligned} \tag{10}$$

  expresses a conservation law, and can be interpreted as a Continuous Time Random Walk.

- q_w: probability density for a particle to make a random walk step w in energy,
  q_τ: probability density for this step to be performed in a time interval τ
- When both q_w and q_τ have finite mean and variance (i.e. only small increments) (as e.g. for Gaussians), then the FP equation can be derived from Eq. (10) through Taylor-expansions

Fractional transport equation (FTE) II

- Here, we do not make the assumption of small increments
- distribution of increments, expressed in Fourier (k) and Laplace space (s):
  1. distribution of energy increments: symmetric stable Levy distributions
     $\hat{q}_w(k) = \exp(-a|k|^\alpha)$, with $0 < \alpha \leq 2$,
     which exhibit a power-law tail in energy-space, $q_w(w) \sim 1/w^{1+\alpha}$ for $\alpha < 2$ and w large,
     and for $\alpha = 2$ they are Gaussian distributions
  2. waiting time distribution: one sided stable Levy distributions,
     $\tilde{q}_\tau = \exp(-bs^\beta)$, with $b > 0$ and $0 < \beta \leq 1$,
     which have a power-law tail, $q_\tau \sim 1/\tau^{1+\beta}$ for $\beta < 1$ and τ large,
     and for $\beta = 1$ they equal $q_\tau(\tau) = \delta(\tau - b)$
- In order to derive a meso-scopic transport equation, we consider the fluid-limit: w, τ are large, and thus k, s are small, so that the distributions of increments can be approximated as
  $\hat{q}_w \approx 1 - a|k|^\alpha$
  $\tilde{q}_\tau \approx 1 - bs^\beta$.

Fractional transport equation (FTE) III

- Chapman Kolmogorov equation —> make Fourier Laplace transform —> apply convolution theorems —> insert distributions of increments in the fluid limit:

$$bs^{\beta}\tilde{\hat{n}}(k,s) - bs^{\beta-1}\hat{n}(k,0) = -a|k|^{\alpha}\tilde{\hat{n}}(k,s) \tag{11}$$

which can be written as a fractional transport equation (FTE)

$$bD_t^{\beta} n = aD_{|W|}^{\alpha} n \tag{12}$$

with D_t^{β} the Caputo fractional derivative of order β, defined in Laplace space as

$$\mathcal{L}\left(D_t^{\beta} n\right) = s^{\beta}\tilde{n}(W,s) - s^{\beta-1}n(W,0) \tag{13}$$

and $D_{|W|}^{\alpha}$ the symmetric Riesz fractional derivative of order α, defined in Fourier space as

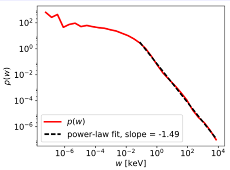
$$\mathcal{F}\left(D_{|W|}^{\alpha} n\right) = -|k|^{\alpha}\hat{n}(k,t) \tag{14}$$

- We need to estimate two parameter sets, α, a and β, b

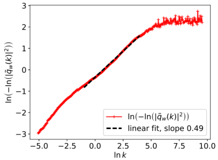
Fractional transport equation (FTE) IV

- the order of the fractional derivatives (α, β) is given by the index of the power-law tail of the distribution of increments, if any
  - otherwise, if the mean and variance of the increments are finite, then the classical FP equation is appropriate.

the distribution of energy increments $p_w(w)$ has a power-law tail, its index z yields $\alpha = -z - 1 = 0.49$.



- As second method to determine α and also a, we use the characteristic function approach:
  $\alpha = 0.49$ (as before) and $a = 0.36$



- "Waiting times": We have considered energy increments over a fixed time interval Δt,
  —> we use 'observation/sampling times', not 'waiting times'
  —> "waiting time" distribution $p_\tau(\tau) = \delta(t - \Delta t)$,
  —> it follows that $\beta = 1$ and $b = \Delta t$.
  - This approach seems unavoidable if the test-particle data are given in the form of time-series, where there is no direct information on the waiting times between scattering events.
- Thus, we consider the fractional transport equation to have a first order derivative in time-direction and a fractional derivative in energy direction,

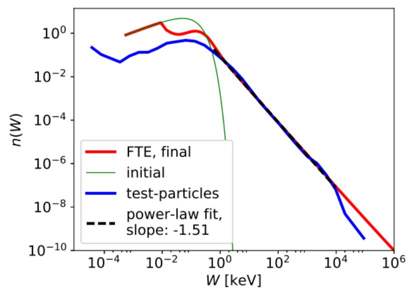
$$\partial_t n = (a/b) D^{\alpha}_{|W|} n - n/t_{esc}, \tag{15}$$

where we also have added an escape term.

Fractional transport equation (FTE) VI

- numerical solution of the FTE:
  Grünwald-Letnikov definition of fractional derivatives (e.g. (Kilbas et al.(2006))), in the matrix formulation of (Podlubny et al.(2009), Podlubny et al.(2013)):
  same non equi-distant grid-points in $[0, \infty)$ as above for the FP equation

Solution of the FTE at $t = 0.002\,\mathrm{s}$:
the FTE reproduces very well the power-law tail from the test-particle simulations in its entire extent



Conclusion I

We posed two questions:

1. Is the FP equation still valid in strongly turbulent environments ?
   Answer: No !
2. How to model transport when the FP approach is not valid anymore ?
   Answer: With a kind of fractional transport equation (work is still needed)

- statistical analysis of the distribution of energy increments:
  —> allows deciding whether a FP or a FTE is appropriate
  —> in the FP case the estimate of the transport coefficients is based on it
  —> In the FTE case, the form of the FTE and its parameters (the order of the fractional derivative etc), are directly inferred from the simulation data (and thus they are not universal or unique).
- simplifying assumption:
  instead of 'waiting times' we used 'observation/sampling times'
  —> did not affect the success of the FTE approach
- We made no effort to model the low energy part of the distribution
- published in Phys. Rev. Lett. **119**, 045101 (2017)

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