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Integrable perturbed magnetic fields in toroidal geometry: An exact analytical flux surface label for large aspect ratio

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An analytical description of magnetic islands is presented for the typical case of a single perturbation mode introduced to tokamak plasma equilibrium in the large aspect ratio approximation. Following the Hamiltonian structure directly in terms of toroidal coordinates, the well known integrability of this system is exploited, laying out a precise and practical way for determining the island topology features, as required in various applications, through an analytical and exact flux surface label. [http://dx.doi.org/10.1063/1.4885082]

One important topic of fusion research is the appearance of magnetic islands, e.g., in the form of neoclassical tearing modes (NTMs), and in particular, their role in plasma confinement. Their theoretical investigation, besides direct numerical integration of magnetic field lines, sometimes requires an analytical description, as well. The latter can be provided by a magnetic surface quantity, which may be quite useful in various modelling approaches, such as test-particle simulations, ray tracing, and wave propagation. For magnetic perturbations consisting of an infinite series of modes, such quantities can only exist locally, though. Therefore, the general method considered[1–3] follows a Taylor expansion in the neighborhood of the resonant surface, in order to approximate the island chain corresponding to a specific mode, leading to an expression very similar to the well known pendulum Hamiltonian.

However, when testing the previously described flux surface label against field line tracing, certain discrepancies are apparent, depending on the problem. Thus, the topology of the actual magnetic field under investigation and the magnetic surface labeling used to investigate it would be inconsistent. On the other hand, in electron-cyclotron resonance methods, for example, often one particular mode is considered right from the start. However, simplified these perturbations are, they are typically used, when studying magnetic island effects on wave propagation, absorption, current drive, etc., for NTM integration.[4,5] In this case, the above general method is no longer necessary, since the perturbed magnetic field is still integrable.

In particular, when one perturbation mode is taken into account in the field line Hamiltonian, no series expansion is required, but only analyzing the perturbed magnetic surfaces. Nonetheless, the latter are characterized by an effective Hamiltonian, which is not separable and may, in general, be quite complicated, far from typical mechanical systems. Thus, we present how to determine the perturbed topology and all its characteristics for these cases in a straightforward and simple manner. In doing so, we take on a geometric approach, utilizing the flux surface label obtained to recover Poincare surfaces of section for the magnetic field lines by projecting the magnetic surfaces in a poloidal cross section. The technique is not bound to the particular form of the Hamiltonian system, and could be applied in similar cases.

In addition—and although we use the large aspect ratio approximation for the equilibrium magnetic field—for the purpose of higher precision, the actual toroidal geometry is adopted instead of the approximate cylindrical one generally used.[6–9] Within the Hamiltonian formulation, emphasis is given on the noncanonical nature of the toroidal coordinates as well as the use of action-angle variables for expressing the induced mode. The latter is essential for truly providing the magnetic field with helical symmetry without the appearance of satellite islands,[10,11] and thus focusing on the effects of one single island structure.

The magnetic fields under consideration are the ones applied in toroidal configurations of plasma confinement devices, such as tokamaks. Therefore, we introduce a (right-handed) toroidal coordinate system $\boldsymbol{x} = (r, \theta, \varphi)$, where r is the minor radius of the torus, θ is the poloidal angle measured counterclockwise from the outer edge, and φ is the toroidal angle measured clockwise from the y-axis. The transformation to Cartesian coordinates is $x = R\sin\varphi$, $y = R\cos\varphi$, and $z = r\sin\theta$, where $R = R_0 + r\cos\theta$, R_0 being the major radius of the torus. We denote the unit base as $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$ and the covariant one with $(e_r, e_\theta, e_\varphi)$.

The background equilibrium of a tokamak plasma is very often approximated by an axisymmetric magnetic field $\boldsymbol{B}(r, \theta)$. Thus, we can equivalently start off with a vector potential $\boldsymbol{A}(r, \theta)$, being, too, independent of φ,[12] meaning

$$\boldsymbol{B} = \frac{1}{\sqrt{g}}\left[\frac{\partial A_\varphi}{\partial \theta}\, e_r - \frac{\partial A_\varphi}{\partial r}\, e_\theta + \left(\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}\right) e_\varphi\right], \quad (1)$$

where $g = (rR)^2$ is the determinant of the metric tensor defined by the toroidal coordinates, and A_r, A_θ, A_φ are the covariant components of $\boldsymbol{A}$.

The dynamics of a general magnetic field $\boldsymbol{B}$ in three-dimensional Euclidean space, is described by the set of equations $\boldsymbol{x}'(\tau) = \boldsymbol{B}(\boldsymbol{x})$, where τ is related to the line element of the magnetic field lines and $\boldsymbol{B} = (B^r, B^\theta, B^\varphi)$ is expressed using contravariant components. The Hamiltonian structure of $\boldsymbol{B}$[13–18] can be obtained by the usual technique of treating one of the coordinates, say φ, as the new independent variable, arriving at

$$\frac{d\theta}{d\varphi} = \frac{B^\theta}{B^\varphi}, \quad \frac{dr}{d\varphi} = \frac{B^r}{B^\varphi}, \tag{2}$$

as long as $B^\varphi \neq 0$. In the case of axisymmetry, i.e., in light of (1), the above system can be easily casted into Hamiltonian form, by choosing the Hamiltonian function as $H_0(r,\theta) = -A_\varphi(r,\theta)$, while the symplectic structure as $\omega = \sqrt{g}B^\varphi\, dr \wedge d\theta$. In other words, system (2) can be written as a two-dimensional Hamiltonian system in non-canonical variables[3]

$$\frac{d\theta}{d\varphi} = \frac{1}{\sqrt{g}B^\varphi}\frac{\partial H_0}{\partial r}, \quad \frac{dr}{d\varphi} = -\frac{1}{\sqrt{g}B^\varphi}\frac{\partial H_0}{\partial \theta}. \tag{3}$$

Due to the axisymmetry of $\boldsymbol{B}$, i.e., the independence of the toroidal angle, carried over to the Hamiltonian, this system is autonomous and therefore integrable. Thus, action-angle variables, ψ and ϑ, can be constructed, in terms of which the above system takes the form[19]

$$\frac{d\vartheta}{d\varphi} = H_0'(\psi), \qquad \frac{d\psi}{d\varphi} = 0. \tag{4}$$

Its solutions, lying on surfaces that are topologically equivalent to a torus, are then simply $\psi = $ const. and $\vartheta = w(\psi)\varphi + \vartheta_0$, where $w(\psi) = H_0'(\psi)$ and ϑ_0 some constant. In fusion literature, ϑ is commonly known as the intrinsic poloidal angle, while the function $w(\psi)$ as the winding number. Its inverse, denoted by $q(\psi)$, is the safety factor, expressing the number of turns of magnetic field lines around φ per one turn along ϑ.

The above system gives a simplified picture of actual experiments, for, in real tokamak plasmas, ubiquitous MHD instabilities introduce small perturbations to the equilibrium magnetic field. These can be modelled by considering a perturbed Hamiltonian function[20]

$$H(\psi,\vartheta,\varphi) = H_0(\psi) + \epsilon H_1(\psi,\vartheta,\varphi), \tag{5}$$

in terms of the action-angle variables ψ, ϑ and the "time" φ, where ϵ is the perturbation strength. In order for H_1 to be single-valued, it has to be a 2π-periodic function of ϑ and φ. Thus, it can always be represented as a Fourier series, $H_1(\psi,\vartheta,\varphi) = \sum_m \sum_n h_{mn}(\psi)\cos(m\vartheta - n\varphi)$.

Here, we consider the case of simply one perturbation mode, i.e., one particular term in the above sum, denoted as (m_0, n_0)

$$H_1 = h_{m_0 n_0}(\psi)\cos(m_0\vartheta - n_0\varphi). \tag{6}$$

These types of perturbations lead to systems that are also integrable,[2] owing now to a helical symmetry in terms of ϑ and φ. The integrability in this case can be shown in many ways, the easiest of which is probably by replacing ϑ with the new variable $\xi = m_0\vartheta - n_0\varphi$, for which the equations for the magnetic field lines

$$\begin{aligned} \frac{d\xi}{d\varphi} &= m_0\frac{d\vartheta}{d\varphi} - n_0 = m_0\frac{\partial H}{\partial \psi} - n_0 = \frac{\partial \tilde{H}}{\partial \psi} \\ \frac{d\psi}{d\varphi} &= -\frac{\partial H}{\partial \vartheta} = -m_0\frac{\partial H}{\partial \xi} = -\frac{\partial \tilde{H}}{\partial \xi} \end{aligned} \tag{7}$$

are casted again in Hamiltonian form,[21] using $\tilde{H}(\psi,\xi) = m_0 H(\psi,\xi) - n_0\psi$ as the new, *effective* Hamiltonian. The latter is independent of φ, and therefore system (7) is integrable, meaning the magnetic field lines lie on the surfaces $\tilde{H} = $ const.

So, the invariant surfaces $\psi = $ const. of the unperturbed system are now replaced by $\tilde{H}(\psi,\xi) = $ const. A small perturbation, however, affects mostly the so-called resonant surfaces located at ψ_s. These are the rational surfaces of the unperturbed system, for which $q(\psi_s) = m_0/n_0$.

As previously shown, the function $\tilde{H}$ characterizes the magnetic surfaces and, thus, can serve as a flux surface label. The latter can be useful in many ways, one of which is the analytical construction of Poincare plots.

In the case of magnetic field lines, where the role of "time" is played by the toroidal angle, Poincare surfaces are simply obtained by poloidal cross sections of the torus, that is, $\varphi = $ const. Since magnetic field lines lie on the surface $\tilde{H}(\psi,\xi) = $ const., their intersection with a poloidal cross section $\varphi = \varphi_p$ is the projection of the magnetic surfaces on the $\psi\vartheta$-plane, accordingly described by the equation $\Omega(\psi,\vartheta) = \tilde{H}(\psi,\vartheta,\varphi_p) = $ const. Therefore, a contour plot of the function Ω would simply yield the desired Poincare surface of section. The critical points of Ω correspond then to the equilibrium points of the Poincare map, representing periodic orbits of the system. The maxima or minima give rise to centers, the o-points, while the saddle points accordingly to saddles, the x-points.

In conclusion, when only one particular resonance (m_0, n_0) is under investigation, Poincare section at any given $\varphi = \varphi_p$ can be constructed by the contour plot of the function

$$\Omega(\psi,\vartheta) = m_0 H(\psi,\vartheta,\varphi_p) - n_0\psi. \tag{8}$$

Both o- and x-points can be determined from the condition $\nabla\Omega = 0$, which, for $h_{m_0 n_0}(\psi) \neq 0$ at least in a neighborhood around ψ_s, trivially results in

$$m_0\vartheta - n_0\varphi_p = k\pi, \tag{9}$$

$$w(\psi) + (-1)^k \epsilon h'_{m_0 n_0}(\psi) = \frac{n_0}{m_0}, \tag{10}$$

where k is any integer. From the first equation, we can find the $2m_0$ angles ϑ_i in the interval $[0, 2\pi)$ for different values of k, while from the second one the two actions ψ_i depending on whether k is even or odd. To determine which case corresponds to the o- or x-point, we turn to the Hessian matrix of Ω, calculated at (ψ_i, ϑ_i). The eigenvalues of this matrix, whenever (9)–(10) hold, are

$$\begin{aligned} \lambda_1 &= (-1)^{k+1}\epsilon m_0^3 h_{m_0 n_0}(\psi_i), \\ \lambda_2 &= m_0\left[w'(\psi_i) + (-1)^k \epsilon h''_{m_0 n_0}(\psi_i)\right]. \end{aligned} \tag{11}$$

If $\lambda_1\lambda_2 > 0$, we have an o-point, while in the opposite case, $\lambda_1\lambda_2 < 0$, we have an x-point. Finally, the equation for the separatrix is $\Omega(\psi,\vartheta) = \Omega_x$, where $\Omega_x = \Omega(\psi_x, \vartheta_x)$ for (ψ_x, ϑ_x) any x-point.

In many cases, e.g., for a strictly monotonous profile of the safety factor and, consequently, of the winding number, λ_2 is defined mostly by the first term, since ϵ is relatively small. Therefore, the product $\lambda_1\lambda_2$ has the same sign as $(-1)^{k+1}h_{m_0n_0}(\psi_i)w'(\psi_i)$ has. The changing sign of the latter for successive values of k, causes the interchange of o- and x-points, resulting in the island chain formation we typically see. In light of (9), the number of islands is then equal to the poloidal mode number m_0. From Eq. (10), on the other hand, it is evident that o- and x-points are ϵ-close to the unperturbed resonant surface. Taylor expansion around ψ_s suggests that the deviation $\delta=\psi_i-\psi_s$ to a first order approximation is $\delta=(-1)^{k+1}\epsilon h_{m_0n_0}(\psi_s)/w'(\psi_s)$, also found in Ref. 7.

Before proceeding with a concrete and detailed example, a special class of axisymmetric systems to begin with is considered, which is widely used in applications. These are unperturbed magnetic fields with the simplified assumption of vanishing radial component, $B^r=0$. The latter reflects an equilibrium for large aspect ratio, meaning we do not take into account the poloidal current density, nor the Shafranov shift, though we retain the toroidal geometry, opposite to the cylindrical often used.

First of all, we have $dr/d\tau=0$. Therefore, the invariant surfaces of the unperturbed system are simply defined by $r=\text{const.}$, meaning the magnetic field lines lie on the surface of a torus of radius r. In terms of the Hamiltonian description (3), this also implies that H_0 is a function of r alone, and, consequently, in combination with (2) that it represents the normalized poloidal magnetic flux

$$H_0(r)=\int_0^r\sqrt{g}B^\theta\,dr=\frac{1}{2\pi}\int_0^{2\pi}\int_0^r\hat{B}_\theta(r,\theta)Rdrd\varphi. \tag{12}$$

On the other hand, the construction of action-angle variables in the general case of an axisymmetric system would first require finding canonical coordinates for system (3). When $B^r=0$, we may skip this step. Actually, the transformation from (r,θ) to (ψ,ϑ) can be made via the relation $\sqrt{g}B^\varphi\,dr\wedge d\theta=d\psi\wedge d\vartheta$. Since r is an integral of motion, ψ has to be a function of r alone. Consequently, the previous condition can be simplified, yielding

$$\frac{d\psi}{dr}\frac{\partial\vartheta}{\partial\theta}=r\hat{B}_\varphi. \tag{13}$$

Thus, we can begin with a given function $\psi(r)$ and then construct $\vartheta(r,\theta)$ or vice versa. A typical choice, widely used in the literature, is expressing the action ψ through the normalized toroidal magnetic flux

$$\psi(r)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^r\hat{B}_\varphi(r,\theta)rdrd\theta, \tag{14}$$

while considering ϑ to be a 2π-periodic function of θ.

Another consequence of the zero radial magnetic component is an alternative expression of the winding number directly in terms of r instead of ψ. For when $B^r=0$, then

$$w(r)=\frac{d\vartheta}{d\varphi}=\frac{\partial\vartheta}{\partial\theta}\frac{d\theta}{d\varphi}=\frac{R\hat{B}_\theta}{r\hat{B}_\varphi}\frac{\partial\vartheta}{\partial\theta}. \tag{15}$$

Inverting this relation and taking into account that $\vartheta(r,\theta+2\pi)=\vartheta(r,\theta)+2\pi$, we recover the formula

$$q(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{r\hat{B}_\varphi}{R\hat{B}_\theta}d\theta, \tag{16}$$

in Refs. 20 and 22. Equivalently, once ψ is fixed, w and consequently q can be calculated more simply using

$$w(r)=\frac{dH_0}{d\psi}=\frac{1}{\psi'(r)}\frac{dH_0}{dr}=\frac{R\hat{B}_\theta}{\psi'(r)}. \tag{17}$$

At this point, we should comment that although action-angle variables have been used in the analysis so far, this is by no means a restriction. Yet in the general case of an axisymmetric system the transformation back to the original coordinates r and θ would be quite complicated. In the case of vanishing radial magnetic component, however, the latter task would be a simple one. Actually, the unperturbed Hamiltonian and the safety factor have already been expressed naturally in terms of r. The determination of the o- and x-points for the perturbed system could also follow. Equation (10) can be solved with respect to r still independently from (9), and then replace its solutions r_i in (9) to find θ_i. From Eq. (15), we also deduce that when $\partial\vartheta/\partial\theta\to 1$, then the number of turns around φ per one turn along either ϑ or θ is the same.

So, in fact, all the above conclusions allow us to draw one more. Since r is an integral of motion (for the unperturbed system) just like the action ψ, as long as the intrinsic poloidal angle behaves like the geometrical one, the island topology, realised in the abstract $\psi-\vartheta$ space, is carried over to the geometric $r-\theta$ plane intact. On this ground, $B^r=0$ allows us to switch easily from the action-angle variables, appealing in theory, to the actual toroidal coordinates, as desired in practice, and study the dynamics of the magnetic field lines therefrom.

A typical axisymmetric model for the background equilibrium that falls into the previous category is the so-called standard magnetic field, introduced by Balescu[23]

$$\boldsymbol{B}_0(r,\theta)=\frac{B_0}{R}\left(rw_c(r)\,\hat{e}_\theta+R_0\,\hat{e}_\varphi\right), \tag{18}$$

where B_0 is a constant, expressing the toroidal field on the magnetic axis, while w_c is the winding number in the approximation of cylindrical geometry, i.e., for $r/R_0\to 0$. The Hamiltonian equations (3) for the standard magnetic field are

$$\frac{d\theta}{d\varphi}=\frac{R_0+r\cos\theta}{B_0R_0\,r}\frac{dH_0}{dr},\qquad\frac{dr}{d\varphi}=0, \tag{19}$$

where the Hamiltonian function can be deduced directly from the poloidal flux (12), yielding

$$H_0(r) = B_0 \int r w_c(r)\, dr. \tag{20}$$

Choosing ψ as the toroidal flux (14) and then using (13), we end up with the following action-angle variables:[20]

$$\psi = B_0 R_0 \left(R_0 - \sqrt{R_0^2 - r^2} \right), \tag{21}$$

$$\vartheta = 2 \arctan \left(\sqrt{\frac{R_0 - r}{R_0 + r}} \tan \frac{\theta}{2} \right). \tag{22}$$

So, proceeding in terms of the toroidal coordinates, from Eq. (17) the actual winding number w with respect to the approximate one w_c is

$$w(r) = \frac{\sqrt{R_0^2 - r^2}}{R_0}\, w_c(r). \tag{23}$$

From (10), the radial position r_i of o- and x-points, for any mode (m_0, n_0) we choose to perturb the system with, can be found through

$$w(r_i) + (-1)^k \epsilon h'_{m_0 n_0}(r_i) \frac{\sqrt{R_0^2 - r_i^2}}{B_0 R_0 r_i} = \frac{n_0}{m_0}, \tag{24}$$

and then the corresponding poloidal angle at any given cross section $\varphi = \varphi_p$ would be

$$\theta_i = 2 \arctan \left(\sqrt{\frac{R_0 + r_i}{R_0 - r_i}} \tan \frac{\vartheta_i}{2} \right), \tag{25}$$

where $\vartheta_i = (n_0 \varphi_p + k\pi)/m_0$ from (9).

Finally, it is also worth noting that the value ϑ_0 of the intrinsic poloidal angle at $\varphi = 0$, introduced just after Eqs. (4) of the unperturbed system, may serve as a rotation parameter for the islands, without affecting whatsoever the unperturbed system. Actually, it should reappear as an integration constant, meaning ϑ could be defined as in (22) minus ϑ_0. The resulting replacement $\vartheta \to \vartheta - \vartheta_0$ enters only in (9) and, consequently, (25), changing the poloidal position θ_i of the o- and x-points to some other angle $\theta_i + \theta_0$, leaving, in general, any other dependence on θ unchanged. Thus, we can rotate the island topology exactly where we want to, without rotating the whole system. This is a common requirement in experiments, such as NTM integration with electron cyclotron current drive, where electromagnetic waves are launched, targeting the o-points. This way of rotating the islands is independent of the specific Hamiltonian or action-angle variables.

Let us demonstrate the previous methods for constructing Poincare plots with an example of a (3,2) resonance mode for the standard magnetic field with ITER-like parameters, $B_0 = 5.51$ T and $R_0 = 6.2$ m. Following Ref. 24, with some assumptions on the density and temperature profiles, we consider

$$w_c(r) = \frac{1}{4} \left(2 - \frac{r^2}{a^2} \right) \left(2 - 2\frac{r^2}{a^2} + \frac{r^4}{a^4} \right), \tag{26}$$

where $a = 1.9$ m is the maximum value of r. Thus, from (20), the Hamiltonian of the unperturbed system is

$$H_0(r) = \frac{B_0 r^2}{2} \left(1 - \frac{3r^2}{4a^2} + \frac{r^4}{3a^4} - \frac{r^6}{16a^6} \right), \tag{27}$$

while the (3,2) resonant surface is located at $r_s = 96.5743$ cm. We assume a perturbation of the form[9]

$$h_{32}(\psi(r)) = \frac{r}{3} \left(1 + \frac{r - r_s}{b} \right), \tag{28}$$

expressed directly in terms of r instead of ψ, where $b = 12$. The associated Hamiltonian system describes the field lines of the magnetic field $\boldsymbol{B}(r,\theta,\varphi) = \boldsymbol{B}_0(r,\theta) + \epsilon \boldsymbol{B}_1(r,\theta,\varphi)$, where

$$\boldsymbol{B}_1 = \frac{3\sqrt{R_0^2 - r^2}}{rR^2} h_{32}(r) \sin(3\vartheta - 2\varphi) \hat{e}_r + \frac{1}{R} \left[h'_{32}(r) \cos(3\vartheta - 2\varphi) + \frac{3R_0 \sin\theta}{R\sqrt{R_0^2 - r^2}} h_{32}(r) \sin(3\vartheta - 2\varphi) \right] \hat{e}_\theta, \tag{29}$$

for h_{32} given in (28) and ϑ in (22). Notice that for $r/R_0 \to 0$ the last term of the poloidal component would be neglected. The flux surface label (8) for this magnetic field is

$$\Omega(r,\theta) = 3H_0(r) - 2\psi(r) - 3\epsilon h_{32}(r) \cos(3\vartheta(r,\theta)), \tag{30}$$

at the poloidal cross section $\varphi_p = \pi/2$, that is, $y = 0$, where H_0, h_{32} are replaced from (27) and (28) and ψ, ϑ from (21) and (22), respectively.

The contour plot of (30) is shown in Figure 1 for $\epsilon = 0.005$. The same Poincare section was drawn using numerical integration of the magnetic field lines, expressed in Cartesian coordinates, with a 4th-order adaptive step-size Runge-Kutta scheme. Comparison of the two plots reveals actually no difference, they look identical. Figure 2 shows that the points from the numerical Poincare map lie exactly on the contour lines of the analytical flux surface label Ω.

For the calculation of the o- and x-points, we notice first that for $0 \le r \le a$ the perturbation h_{32} is positive, the winding number w is decreasing monotonously and the toroidal flux ψ is increasing instead. Thus, from the relations (11), assuming $\lambda_2 \approx 3w'(r)/\psi'(r)$, we immediately deduce that when k is odd, the critical point of the function Ω would be a saddle point, indicating an x-point, while when k is even it would be a maximum, indicating an o-point. So (24) and (25) yield six (real) solutions altogether, three o-points at $r_o = 96.6336$ cm,

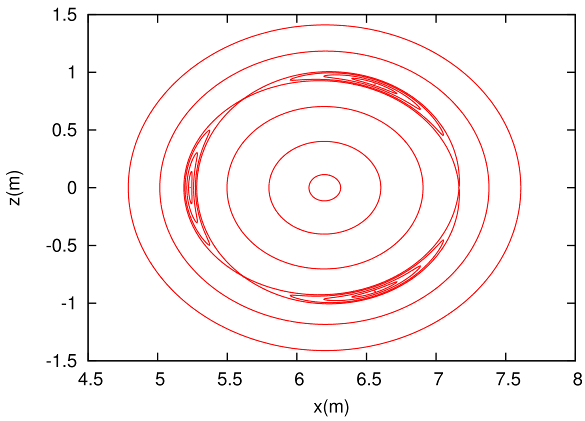

FIG. 1. Contour plot of flux surface label Ω (30).

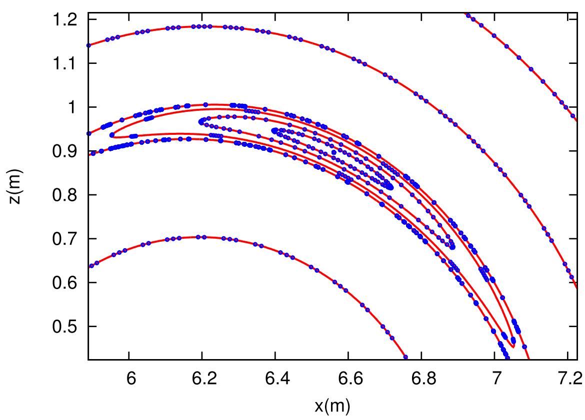

FIG. 2. Comparison of the analytical flux surface label (red lines) and the numerically determined Poincare plot (blue points), zoomed in the island region.

$\theta_o = 68.0854°$, $180°$, and $291.915°$ and three x-points at $r_x = 96.5149\,\mathrm{cm}$, $\theta_x = 0°$, $127.468°$, and $232.532°$, respectively. These values, also visualised in Figure 1, are in absolute agreement with the ones from the field line tracing.

Finally, it is also worth noting that in the cylindrical approximation, often adopted in the large aspect ratio limit, we recover the usual considerations $\psi \approx B_0 r^2/2$, $\vartheta \approx \theta$ (as previously requested), $w \approx w_c$ and accordingly $\boldsymbol{B}_1(r,\theta,\varphi) \approx 3h_{32}(r)\sin(3\theta - 2\varphi)/(rR)\hat{e}_r + h'_{32}(r)\cos(3\theta - 2\varphi)/R\hat{e}_\theta$, reflecting a (3,2) mode in terms of θ and φ instead of ϑ and φ. The deviations, however, from the above treatment are not negligible, as, for example, the corresponding intrinsic poloidal angle values $\vartheta_o = 60°$, $180°$, and $300°$ and $\vartheta_x = 0°$, $120°$, and $240°$ indicate.

In this work, we have presented a simple analytical way for determining the island topology of the magnetic field, when a single perturbation mode is introduced to the plasma equilibrium. Poincare sections of field lines have been constructed by contour plotting a flux surface label that is consistent with the magnetic field, and the positions of o- and x-points, as well as the separatrix, have been given analytically. The method addressed follows the nontrivial Hamiltonian nature of the magnetic field lines in terms of their toroidal structure.

The integrability of this kind of systems is widely known in the context of Hamiltonian mechanics, yet, at times, neglected in applications such as these. And though quite often employed in the cylindrical approximation, to the authors' knowledge, it has not been fully utilized in the actual toroidal geometry of tokamaks. The technique described here, requiring no assumptions on the particular form of the integrable Hamiltonian, is quite general and could be applied elsewhere.

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[1]R. B. White, *The Theory of Toroidally Confined Plasmas*, 2nd ed. (Imperial College Press, 2001).
[2]R. D. Hazeltine and J. D. Meiss, *Plasma Confinement*, 2nd ed. (Dover Publications, Inc., 2003).
[3]W. Horton and Y.-H. Ichikawa, *Chaos and Structures in Nonlinear Plasmas* (World Scientific, 1996).
[4]H. Isliker, I. Chatziantonaki, C. Tsironis, and L. Vlahos, Plasma Phys. Controlled Fusion **54**, 095005 (2012).
[5]I. Chatziantonaki, C. Tsironis, H. Isliker, and L. Vlahos, Plasma Phys. Controlled Fusion **55**, 115012 (2013).
[6]P. H. Rutherford, Phys. Fluids **16**, 1903 (1973).
[7]R. B. White, D. A. Monticello, M. N. Rosenbluth, and B. V. Waddell, Phys. Fluids **20**, 800 (1977).
[8]R. Fitzpatrick, Phys. Plasmas **2**, 825 (1995).
[9]C. Ren, J. D. Callen, T. A. Gianakon, C. C. Hegna, Z. Chang, E. D. Fredrickson, K. M. McGuire, G. Taylor, and M. C. Zarnstorff, Phys. Plasmas **5**, 450 (1998).
[10]L. H. A. Monteiro, V. Okano, M. Y. Kucinski, and I. L. Caldas, Phys. Lett. A **193**, 89 (1994).
[11]R. L. Viana, Chaos, Solitons Fractals **11**, 765 (2000).
[12]N. Kallinikos and E. Meletlidou, J. Phys. A: Math. Theor. **46**, 305202 (2013).
[13]A. H. Boozer, Phys. Fluids **26**, 1288 (1983).
[14]J. R. Cary and R. G. Littlejohn, Ann. Phys. **151**, 1 (1983).
[15]K. Elsässer, Plasma Phys. Controlled Fusion **28**, 1743 (1986).
[16]M. S. Janaki and G. Ghosh, J. Phys. A: Math. Gen. **20**, 3679 (1987).
[17]H. R. Lewis, Phys. Fluids **2**, 2551 (1990).
[18]P. J. Morrison, Phys. Plasmas **7**, 2279 (2000).
[19]Here and in the following, a function either in terms of r and θ or ψ and ϑ will be denoted by the same symbol, without indicating the same functional dependency, though.
[20]S. Abdullaev, *Construction of Mappings for Hamiltonian Systems and Their Applications* (Springer, 2006).
[21]L. Turner, J. Math. Phys. **26**, 991 (1985).
[22]J. A. Wesson, *Tokamaks*, International Series of Monographs on Physics Vol. 118, 3rd ed. (Oxford University Press, 2004).
[23]R. Balescu, *Transport Processes in Plasmas*, Neoclassical Transport Vol. II (North-Holland, 1988).
[24]J. H. Misguich, J. D. Reuss, D. Constantinescu, D. Steinbrecher, M. Vlad, F. Spineanu, B. Weyssow, and R. Balescu, Ann. Phys. **28**, 1 (2003).