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A TEST FOR STATIONARITY: FINDING PARTS IN TIME SERIES APT FOR CORRELATION DIMENSION ESTIMATES

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We propose a method to identify stationary phases in time series. Stationarity is a necessary condition for many concepts in dynamical systems theory, e.g. deterministic chaos. Therefore, testing for stationarity should necessarily be the first step in any data analysis. Above all, this testing is highly important whenever one deals with systems for which stationarity is not guaranteed by the data acquisition procedure: if only short and unique time series are accessible and if the experimental situation is not or only restrictedly controllable, as for instance in astronomy, economy, or medicine.

The proposed stationarity test is easily workable and easy to implement in the form of a systematically searching loop. It singles out the parts of a time series which are a reasonable input to a dimension estimate algorithm. Thereby, it can ascertain finite correlation dimensions which are *not* indicative of deterministic behavior; this kind of dimensions can occur in *stochastic* processes which are nonstationary, e.g. self-affine.

1. Introduction

Dimension estimate is a widely used method in dynamical systems theory to ascertain whether a process is deterministic or stochastic, and it is the most popular one for *short* time series (see, for example, Mayer-Kress [1986]). In order to categorize a process by the notion of dimension the phase space of a system is reconstructed [Takens, 1981] and the limit set spanned by the trajectories, the so-called attractor, is considered. The path of a recurrent stochastic system spans a subvolume of dimension equal to the embedding dimension, after long enough time. A quasiperiodic system moves on a torus of finite dimension. And finally, a deterministic but irregular, so-called chaotic system — i.e. a system with a great sensitivity to initial conditions — yields a usually intricate set of finite dimension. The latter two processes are distinguishable via the power-spectrum.

Not every measured time series, however, is suited for a dimension estimate: it has to be stationary, since stationarity is an important prerequisite to deterministic chaos. Theiler [1986, 1991] noted that analyzing nonstationary data can mimic finite correlation dimensions — so that a finite correlation dimension does not imply stationarity. One known example of stochastic processes with finite correlation dimension is fractional Brownian motion [Osborne & Provenzale, 1989]. It is, however, not stationary. Definitely, a separate test of stationarity is needed.

In several branches of science like astronomy, geophysics, economy, or medicine, investigating stationarity is highly important from a practical point of view, too: The data are not gained in an experiment where the boundary conditions and, by that, the exterior parameters are technically fixed and the measurements are naturally stationary. Controlling the boundary conditions is not — or only partially — possible. Burst phenomena in stellar coronae for instance have a start and an end phase, so that, as a whole, they are not stationary. The question is whether there are parts of a time series which can be considered stationary. To know whether a system is in a *stationary* state or not (e.g. in a transient phase) is physically meaningful by itself. And, as mentioned, the stationary parts can then be enquired for finite dimensions.

The aim of this paper is to introduce a reliable stationarity test which preselects the parts of a time series that are a reasonable input to dimension estimate algorithms. After exposing a typical example of data from astronomy in Sec. 2, where a stationarity test is highly useful, we propose a stationarity test in Sec. 3. Section 4 is dedicated to its application to experimental and some numerically generated data, among which is the interesting case of fractional Brownian motion. We show that the stationarity test is easily workable, even for short time series.

2. Example: Possible Nonstationarity in Short Data Sets from Astronomy

The application of dimension estimates where we meet the fundamental problem of stationarity is the investigation of the dynamics of eruptive phenomena in the solar and stellar atmospheres (so-called flares), thin plasmas (the particle density is about 10^8 cm^{-3}) whose coherent emissions we observe by radio telescopes. There are many time series with a good signal to noise ratio. Güdel & Benz [1988] report typical examples. Different kinds of coronal bursts occur in the course of a flare: decimetric pulsations, type I–V events, and millisecond spikes. An example of a type I event is given in Fig. 1. Investigations of pulsations concerning possible chaos were done by Kurths *et al.* [1991] who found that this type can be characterized by a low-dimensional attractor. Millisecond spikes were investigated by

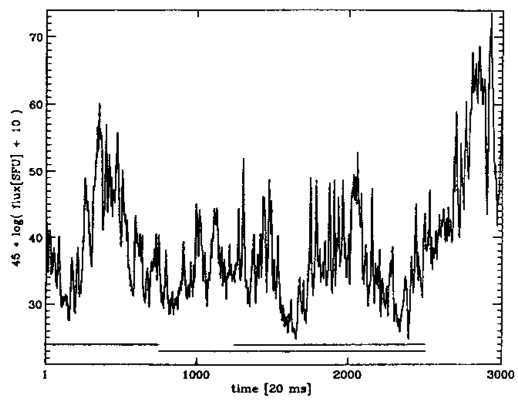

Fig. 1. The time profile of a measurement of a type I event (82/04/23, 09:08:15) (the largest stationary sections are marked by horizontal bars).

Isliker [1992]. An analysis of other burst types (type I, II and III events, again millisecond spikes, and stellar flares) is in preparation.

Looking at Fig. 1, the question is: what parts of the time series should be analyzed — the entire time series? Or else: how to select subsections? The answer is given by the stationarity test in the next section.

We have to be able to ascertain the stationarity of a process in a given interval mainly under the condition that only a small amount of data is available: as the phenomena under scrutiny have a limited duration (typically 20–200 sec), the time series are rather short, often near the limit of the dimension estimate being applicable, with 500 to some thousand points typically (an adequately chosen resolution is implied). (For conditions on the minimum number N of points in phase space see, for instance, Isliker [1992], Brandstater & Swinney [1987], Ruelle [1990], Abraham *et al.* [1986], Kurths & Herzel [1987], and Atmanspacher *et al.* [1988]).

3. Detecting Nonstationarity

Stationarity in the strong sense is the property that all statistical quantities of a process are independent of absolute time; they are at most a function of relative times. Fractional Brownian motion for instance is not stationary: its variance is a function of its duration.

In order to investigate stationarity a time series is usually divided into several parts and statistical properties of each part are compared. This approach can be inconclusive if statistical quantities are considered *which contain too little information* about the process. This often occurs for the various techniques which test for the stationarity of the second-order properties of a time series (*cf.* Priestley [1991]). Look at the following examples:

(a) Using simply the variances of the parts of a time series may be misleading: for instance the variance of the nonstationary fractional Brownian motion depends only on its duration t (it is proportional to t^{1/D_H}, with D_H a constant (see Mandelbrot [1982]). Parts of a time series of this nonstationary process will thus have equal variances if they are of equal length.

(b) Though it is a necessary condition for stationarity that the power spectrum does not change, comparing the power spectra of singular parts is deceptive: the nonstationary self-affine process constructed by Osborne & Provenzale [1989] (see next section) will by construction have the same power spectrum in every part of a time series!

The physical invariant measure ρ gives a complete picture of a stationary process. It is a statistical description of a system in state space, a probability density, measuring how frequently the different parts of state space are visited, loosely speaking. It reflects the dynamics of a system by taking into account that some parts of the state space are more frequently visited than others. This density contains the information of all statistical moments, which are calculated from it.

The invariant measure ρ is operationally defined in the n-dimensional state space as the time average of Dirac δ-distributions along a trajectory $\mathbf{x}(t)$,

$$\rho := \lim_{T\to\infty} \frac{1}{T} \int_0^T \delta_{\mathbf{x}(t)}\, dt\,, \tag{1}$$

[Eckmann & Ruelle, 1985]. If the system is assumed to be ergodic, then space averages, with ρ as weight, indeed equal time averages:

$$\int_{\text{state space}} f(\mathbf{x})\rho(d\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\mathbf{x}(t))dt\,, \tag{2}$$

for any function $f(\mathbf{x})$ in this state space.

As only one coordinate of state space is accessible (say x_1), the projection $\rho(dx_1)$ of $\rho(d\mathbf{x})$ onto this coordinate x_1 is considered:

$$\rho(dx_1) := \int \rho(dx_2 dx_3 \cdots dx_n)\,. \tag{3}$$

We calculate the measure empirically by dividing the x_1-axis (which is the amplitude or flux in our data) into intervals $[x_1^{(k)}, x_1^{(k+1)}]$, $k = 1, \ldots, K$, counting the measured points falling into these intervals, which yields a frequency n_k:

$$\begin{aligned} n_k &:= \text{number}[x_1^{(k)} \le x_1 < x_1^{(k+1)}] \\ &\approx \sum_{x_1} \int_{x_1^{(k)}}^{x_1^{(k+1)}} \delta(x - x_1)dx \\ &= \sum_{x_1} X_{[x_1^{(k)}, x_1^{(k+1)}]}(x_1)\,. \end{aligned} \tag{4}$$

The sums are over the flux values x_1 and $X_{[a,b]}(x)$ is the characteristic function of the interval $[a, b]$ (i.e. $X_{[a,b]}(x) = 1$ if $x \in [a, b]$, else 0).

In order to check the stationarity of a given time series we estimate the invariant measure for the entire series ($n_k^{(\text{all})}$), which yields a reference density $p_k^{(\text{all})}$,

$$p_k^{(\text{all})} = \frac{n_k^{(\text{all})}}{\sum_k n_k^{(\text{all})}}\,. \tag{5}$$

To see how much a given subpart of the time series, e.g. the first half of it, deviates from this reference population, the invariant measure for this first half is estimated ($n_k^{(\text{half})}$). The natural way to compare the two probability distributions is the χ^2-test. The test quantity is defined as

$$\chi^2 := \sum_k \frac{(n_k^{(\text{half})} - Zp_k^{(\text{all})})^2}{Zp_k^{(\text{all})}}\,, \tag{6}$$

with Z the number of points in the first half of the time series: $Z = \sum_k n_k^{(\text{half})}$. This quantity can be expected to have a χ^2-distribution, the degrees of freedom are the number of intervals minus one, $K - 1$. If $Zp_k^{(\text{all})} < 5$, the interval k and a neighboring one, e.g. $k - 1$, have to be merged.

The number of intervals K has to be chosen with care for it influences the outcome of the test. If it is chosen too small, most of the details of the dynamics are missed, and hardly any data set will be stationary. With increasing K, the relative errors of the estimated $p_k^{(\text{all})}$'s increase. The optimum choice is an intermediate value. This is reflected in how the test depends on K: the test is negative ($\equiv$ not acknowledging stationarity) for small K's, no matter whether the data are stationary. Increasing K beyond a particular threshold, the test gets positive if the data are stationary, and it remains positive if K is further increased. In the case of nonstationary data the test remains negative with increasing K.

Experiments with data point out that for peaky data of length N, with $N \approx 1000$–2000, a good and intermediate value of K is $K \approx 90$. It is beyond the mentioned threshold and results in typically 30–60 effectively used intervals (concatenation of neighboring intervals in order to fulfill the requirement $Zp_k^{(\text{all})} \geq 5$). A better way to do the binning is to use equiprobable bins in which case one can start with about 50 bins. Their number remains unchanged, for it is comfortably below the maximum $K_{\max}$, derived from the condition $Zp_k^{(\text{all})} \equiv Z(1/K) \geq 5$.

With a 95% significance level we demand that the invariant measure in the first half of the section does not differ from the one in the entire section — the invariant measure remaining unchanged in time implies stationarity.

Stationary regimes of a time series $\{X_{t_i}\}_{i=1}^{N}$ can now systematically be searched for: first, the entire time series $\{X_{t_i}\}_{i=1}^{N}$ is tested, i.e. the invariant density ρ estimated from $\{X_{t_i}\}_{i=1}^{N}$ has to be compared to the one estimated from $\{X_{t_i}\}_{i=1}^{N/2}$. Then, subsections $\{X_{t_i}\}_{i=i_j}^{n_j}$ of the time series of smaller size are tested, i.e. the invariant density ρ estimated from $\{X_{t_i}\}_{i=i_j}^{n_j}$ has to be compared to the one estimated from $\{X_{t_i}\}_{i=i_j}^{(i_j+n_j)/2}$. One can look upon analyzing the data of a subsection as analyzing data in a window put onto the time series. In these terms, the best way to proceed is to decrease a window's length, to shift it through the data, to decrease its length again and so on, down to a minimum length of interest.

Remark 1. The invariant measure ρ is the basis of correlation dimension calculation — with the usual definition of correlation dimension: At a first step, reconstruct the phase space from the time series by the time delay technique [Takens, 1981; Packard *et al.*, 1980]. The resulting vectors $\boldsymbol{\xi}_i$ span a set in phase space. Using the invariant measure ρ, the correlation dimension of this set is determined by the correlation integral [Grassberger & Procaccia, 1983a, 1983b]

$$C_d^{(2)}(\varepsilon) := \int_A \int_A d\boldsymbol{\xi} d\boldsymbol{v}\, \rho(\boldsymbol{\xi})\rho(\boldsymbol{v})\Theta(\varepsilon - |\boldsymbol{\xi} - \boldsymbol{v}|)$$
$$\approx \lim_{N\to\infty} \frac{2}{(N-W)(N-W-1)} \times \sum_{i+W<j}^{N} \Theta(\varepsilon - |\boldsymbol{\xi}_i - \boldsymbol{\xi}_j|)\,, \qquad (7)$$

with the Heaviside function Θ and "$|\,.\,|$" denoting any vector norm, in the sense that the correlation dimension $D^{(2)}$ is defined as the scaling property of this correlation integral: $C_d^{(2)}(\varepsilon) \sim \varepsilon^{D^{(2)}}$, for $\varepsilon \to 0$. The decorrelation parameter $W \geq 0$ has been introduced by Theiler [1986]. (Details on the technical questions around the evaluation of correlation dimensions can be taken from Atmanspacher *et al.* [1988].) *Therefore, the invariant measure ρ is exactly the quantity which must not change in time in order for a dimension estimate to be reasonable.*

Remark 2. A frequently used means, in time series analysis, to arrive at a time series which is more likely to be stationary is to derivate the original time series. Derivation, however, increases noise and decreases signal so seriously that a dimension estimate becomes too critical, in our cases.

4. Exemplary Application: Identifying Stationarity

Figure 2(a) shows the measure $\rho(dx_1)$ of the type I in Fig. 1, based on the entire time series (solid line) together with the measure based on the first half of the same time series (broken line). The coincidence is bad, there are systematic deviations between the two densities: the χ^2-test refuses to accept that the data are stationary. According to our way of proceeding, a window of smaller and smaller size is shifted through the data, performing the stationarity test on the window each time: the entire window is compared to the first half of it. The largest stationary section found in the upper type I is 750–2500, marked by one of the horizontal bars in Fig. 1. The fairly coinciding measures $\rho(dx_1)$ are given in Fig. 2(b), suggesting that this event is stationary, which is confirmed by the χ^2-test on the usual 95% significance level. Some other subsections of the event are also stationary, the largest two are marked in Fig. 1 (see also Table 1).

Similar coincidences are obtained for (Table 1 gives a list of the particular stationary regions)

— some subsections of other bursts;
— a coordinate of a trajectory of the Lorenz equations in a chaotic regime (2048 points);
— an auto-regressive process of order 1 — this is a representative of the general class of the so-called auto-regressive processes of order p (AR-p processes), which actually are the general form of linear stochastic processes [Kurths & Herzel 1987]:

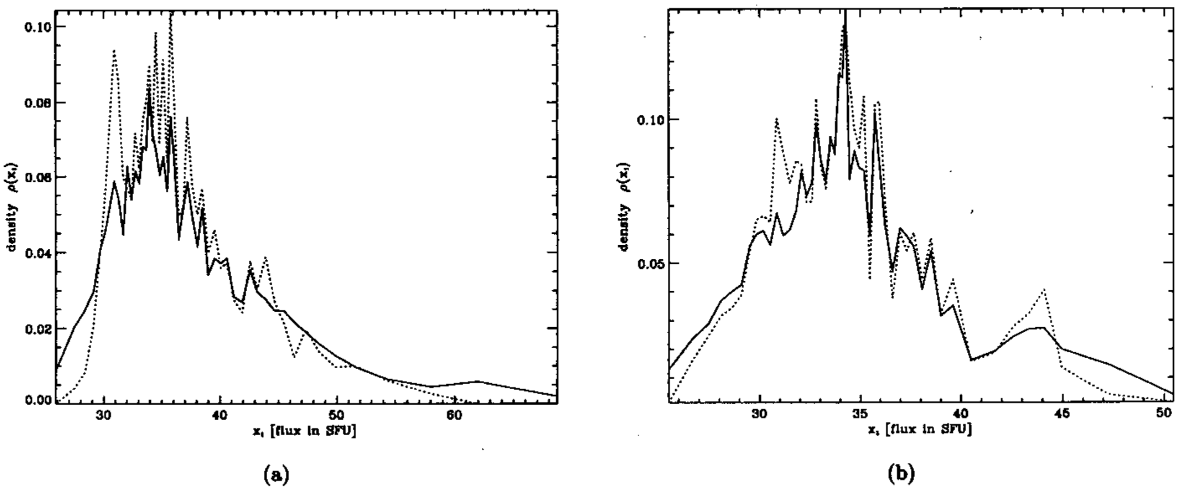

Fig. 2. The invariant densities $\rho(x_1)$, based on an entire part of a time series (solid line) and based on the first half of the same part, respectively (broken line). We used 50 equiprobable bins on the x_1-axis. (For a better visualization we plot the midpoints of the intervals against the density instead of the histograms). The data sets are parts of the type I event on 82/04/23, 09:08:15 (cf. Fig. 1): (a) the entire time series, 1–3000; (b) the part 750–2500 out of the same event (marked by a horizontal bar in Fig. 1).

Table 1. Parameters estimated for both observational data and samples of typical models: A measured time series (type I event), a coordinate of the Lorenz model, an auto-regressive process of order 1 [Eq. (8)] and a self-affine process [Eqs. (10) and (11)]. We list the result of the stationarity test, using a 95% significance level, and the correlation dimension $D^{(2)}$ [Eq. (7)].

Data				Stationarity:	Corr'dim.
Date	Start	Points	Subsection	χ^2-test	$D^{(2)}$
82/4/23	09:08:15	3000	1–750	stat.	div.
			750–2500	stat.	div.
			1250–2500	stat.	div.
Lorenz-model		2048	1–2048	stat.	2.06
AR-1 process		2048	1–2048	stat.	no plateau
Self-affine process		2048	1–2048	not stat.	3.0 ± 0.2

$$X(t_i) = \sum_{k=1}^{p} a_k X(t_{i-k}) + N_{t_i}, \quad i = 1, \ldots, M, \tag{8}$$

where N_{t_i} is identically and independently distributed (white noise). We chose $p = 1$, $a_1 = 0.95$, the noise spreads over the interval $[-0.25; 0.25]$, and $M = 2048$.

No coincidence is yielded by

— fractional Brownian motion, which is a self-affine random path. For $\{X(t_i)\}_{i=1}^{M}$, a time series measured at temporal resolution $\tau = t_{i+1} - t_i$, self-affinity of the increments is defined by the property

$$\langle |X(t_i + \lambda\,\Delta t) - X(t_i)| \rangle = \lambda^H \langle |X(t_i + \Delta t) - X(t_i)| \rangle, \tag{9}$$

where "$\langle . \rangle$" is the average over all points, and H is called the characteristic scaling exponent. Self-affinity means that a scaling exponent H does exist independent of the shift in time λ [Mandelbrot, 1982]. The example of such a random process investigated by Osborne & Provenzale [1989] starts from a power-law decay of its power spectrum $P(\omega_k)$,

$$P(\omega_k) = C\omega_k^{-\alpha}, \tag{10}$$

with spectral index α, and constructs a stochastic time series by

$$X(t_i) := \sum_{k=1}^{M/2} \zeta_k \cos(\omega_k t_i + \varphi_k), \quad i = 1, \ldots, M, \tag{11}$$

where $\omega_k := k\,\Delta\omega$ $(k = 1, \ldots, M/2)$, with $\Delta\omega = 2\pi/M\,\Delta t$, and $\zeta_k := \sqrt{P(\omega_k)\,\Delta\omega}$. The φ_k are chosen at random. Osborne & Provenzale [1989] justify their assertion that such time series are one component of a self-similar random path by testing it with the defining property of self-affinity [Eq. (9)]. They carry out a second self-affinity test by measuring the length of the curves with different rulers. The results are the same: these time series have a self-affine structure. For self-affine processes, the variance σ is time dependent:

$$\sigma \sim t^H, \tag{12}$$

with t being the considered duration of the process. The invariant measures $\rho(dx_1)$ obtained for this process (with $\alpha = 1.75$ and $M = 2048$) exhibit a large discrepancy, the χ^2-test states that the two distributions significantly change (Table 1) — without problems, the process is identified to be not stationary.

Just as an aside: Physical ideas on probable processes in the solar corona suggest that the eruptive phenomena we investigate can sometimes be superimposed upon a varying background. In these cases it is reasonable to subtract a minimum envelope whose characteristic time is much greater than the characteristic time of the process (the autocorrelation time). Whether such a background is present or not has to be inferred from the individual measured spectra.

The proposed test is a quantitative instrument to judge stationarity. It shows that the solar data investigated are sometimes in a stationary phase. The test renders it possible to select stationary phases if it is implemented in the way of a systematic search algorithm.

5. Conclusion

The proposed stationarity test fills in the somewhat neglected gap of data analysis: ascertaining stationarity. Basing on the invariant measure, a description of the dynamical behavior of a system, it fits naturally into the context of dimension estimate. Stationarity is a *statistical* property, and the test formulates a necessary condition for stationarity by means of a *statistical* test, the χ^2-test; if it rejects some data, then with the confidence of the significance level the data are asserted not to be stationary.

The test allows us to overcome some difficulties in interpreting correlation dimensions in time series: Self-affine stochastic processes are easily recognized by it. The situation does not really change when long time series are accessible — then for instance the Lyapunov exponents could be calculated. Stationarity of the time series, however, still remains a prerequisite.

And above all, the test is a well suited tool for handling *short and unique experimental* time series. Applicable without problems, it is a practical help in the selection of the stationary phases in an *unknown* process: whenever the stationarity of a time series is not guaranteed by fixing the boundary conditions in an experiment, the stationarity test, implemented as a loop, allows us to search efficiently for subsections which can be considered stationary. On these intervals dimensions can now be estimated, a class of misinterpretations being thus excluded, above all if correlation integrals are evaluated with a careful choice of the decorrelation parameter ['W' in Eq. (7)].

Applying the stationarity test and a dimension estimate algorithm yields a detailed picture of a system's behavior in the course of time: it provides a list of the nonstationary sections, the stationary ones, and inside the latter the deterministic chaotic ones, all in their sequential order.

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