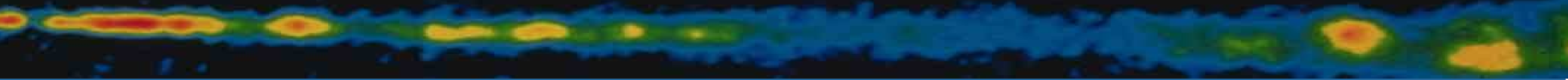

Gravitational wave propagation in astrophysical plasmas

An MHD approach

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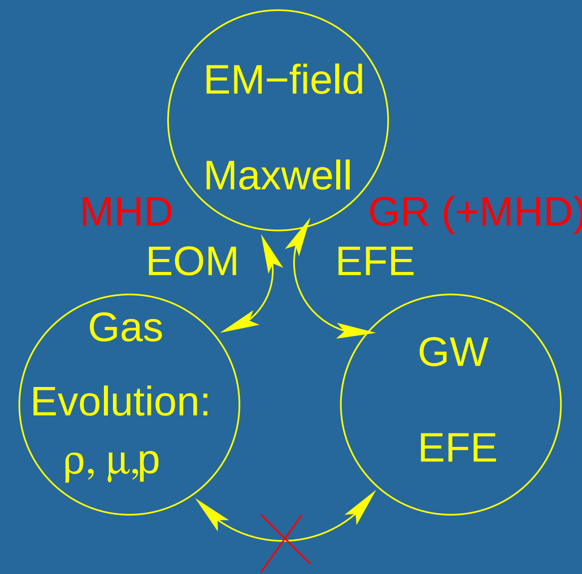
Outline

- The MHD approximation & assumptions,
- Derivation of General Relativistic MHD:
  - ⋆ $3+1$ split & proper reference frames,
  - ⋆ Covariant Maxwell $\Rightarrow$ EM fields,
  - ⋆ Thermodynamics $\Rightarrow$ gas,
  - ⋆ Cons. laws for matter, energy & momentum,
  - ⋆ Set-up: Linearized MHD,
  - ⋆ Note on the coupling GW $\leftrightarrow$ MHD.
- Wave solutions and their properties:
  - ⋆ MHD Wave equation,
  - ⋆ Alfvén, slow and fast magneto-acoustic modes,
  - ⋆ Damping of the GW.
- [end tutorial, start qualitative & quantitative results & astrophysical applications]



The MHD approximation

Ideal Magnetohydrodynamics Approximation

- In kinetic plasma theory, distribution function of 7 independent variables $\boldsymbol{r}$, $\boldsymbol{v}$, t,
- Velocity space effects removed (moments of Boltzmann) to reduce complexity,
- Assumptions to get rid of remaining 2-fluid variables and close set of eqns.

$$\Downarrow$$

$\star$ In MHD model, plasma treated as continuum conducting fluids

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⋆ In MHD model, plasma treated as continuum conducting fluids

⋆ Macroscopic variables defined as linear combination of 2-fluid vars:

$$\rho \equiv n_e m_e + n_i m_i \qquad \textit{total mass density}$$

$$\tau \equiv -e(n_e - Z n_i) \qquad \textit{charge density}$$

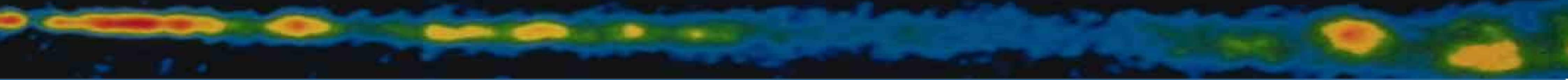
$$\boldsymbol{v} \equiv \frac{1}{\rho}(n_e m_e \boldsymbol{u}_e + n_i m_i \boldsymbol{u}_i) \qquad \textit{center of mass velocity}$$

$$\boldsymbol{j} \equiv -e(n_e \boldsymbol{u}_e - Z n_i \boldsymbol{u}_i) \qquad \textit{current density}$$

$$p \equiv p_e + p_i \qquad \textit{total pressure}$$

Ideal MHD Assumptions

- Typical length scales $\gg$ typical internal scales (gyro radius, collision mean free path / Debye length),
- Charge neutrality $|n_e - Zn_i| \ll n_e$,
- Small relative velocity $|\boldsymbol{u}_e - \boldsymbol{u}_i| \ll v$,
- Negligible viscosity,
- Negligible heat flow,
- Neglect electron skin depth $c/\omega_{pe} \to 0$,
- Negligible resistivity $\boldsymbol{R}_e$ / infinite conductivity $\Rightarrow$ vanishing electric field in rest frame.

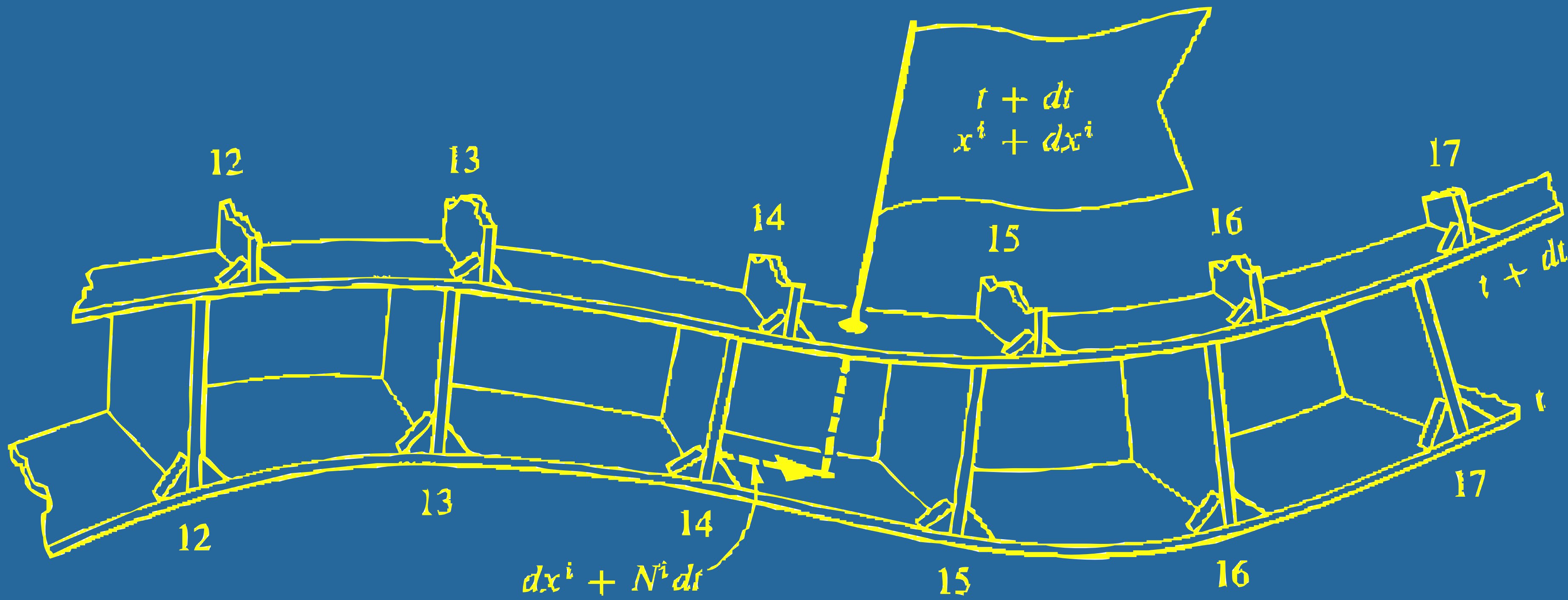


$3 + 1$ split

&

proper reference frames

$3 + 1$ Space-time split



- Timelike observer with 4-velocity u^μ perceives u^μ as time axis ($u^\mu u_\mu = -1$).
- $3D$ hypersurfaces orthogonal to time axis are snapshots of space.
- 'Time projection operator': $U^\mu_{\ \nu} \equiv -u^\mu u_\nu$,
- 'Space projection operator': $H^\mu_{\ \nu} \equiv \delta^\mu_{\ \nu} + u^\mu u_\nu$.

Coordinate vs Non-coordinate Frames

$$\Gamma_{\mu\beta\gamma} = \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) + \frac{1}{2}(c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu})$$

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- Holonomic basis $=$ coordinate,
- $c_{\mu\beta\gamma} = 0$,
- 'Christoffel symbols',
- Twisting, turning, expansion, contraction of basis vectors by *directional derivatives of metric*:

$$g_{\beta\gamma,\mu} = \frac{\partial g_{\beta\gamma}}{\partial x^{\mu}}$$

- If $g_{\beta\gamma,\mu} = 0$, Lorentz frame, FFO.

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- If $g_{\beta\gamma,\mu} = 0$, Lorentz frame, FFO.

- Anholonomic = non-coordinate,
- In tidal field of GW:
- Basis vectors do not commute:

$$\begin{aligned} [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] &= \nabla_\alpha \boldsymbol{e}_\beta - \nabla_\beta \boldsymbol{e}_\alpha \\ &= c_{\alpha\beta}{}^\gamma \boldsymbol{e}_\gamma, \end{aligned}$$

- ‘Ricci rotation coefficients’.

[illustrationI] [illustrationII] [continue]

Illustration I: Proper reference frame of accelerated observer

- Acceleration $\boldsymbol{a}$ and rotation ω of basis,
- If $\omega = 0 \Rightarrow$ Fermi-Walker (gyroscope) transport,
- If also $\boldsymbol{a} = 0$ FFO, geodesic motion, parallel transport,
- Coordinate frame with connection coeff:

$$
\begin{aligned}
\Gamma^{0}{}_{00} &= \Gamma_{000} = 0 \\
\Gamma^{0}{}_{j0} &= -\Gamma_{0j0} = +\Gamma_{j00} \\
&= \Gamma^{j}{}_{00} = \boxed{a^{j}} \\
\Gamma^{j}{}_{k0} &= \Gamma_{jk0} = \boxed{-\omega^{i}\epsilon_{0ijk}}
\end{aligned}
$$

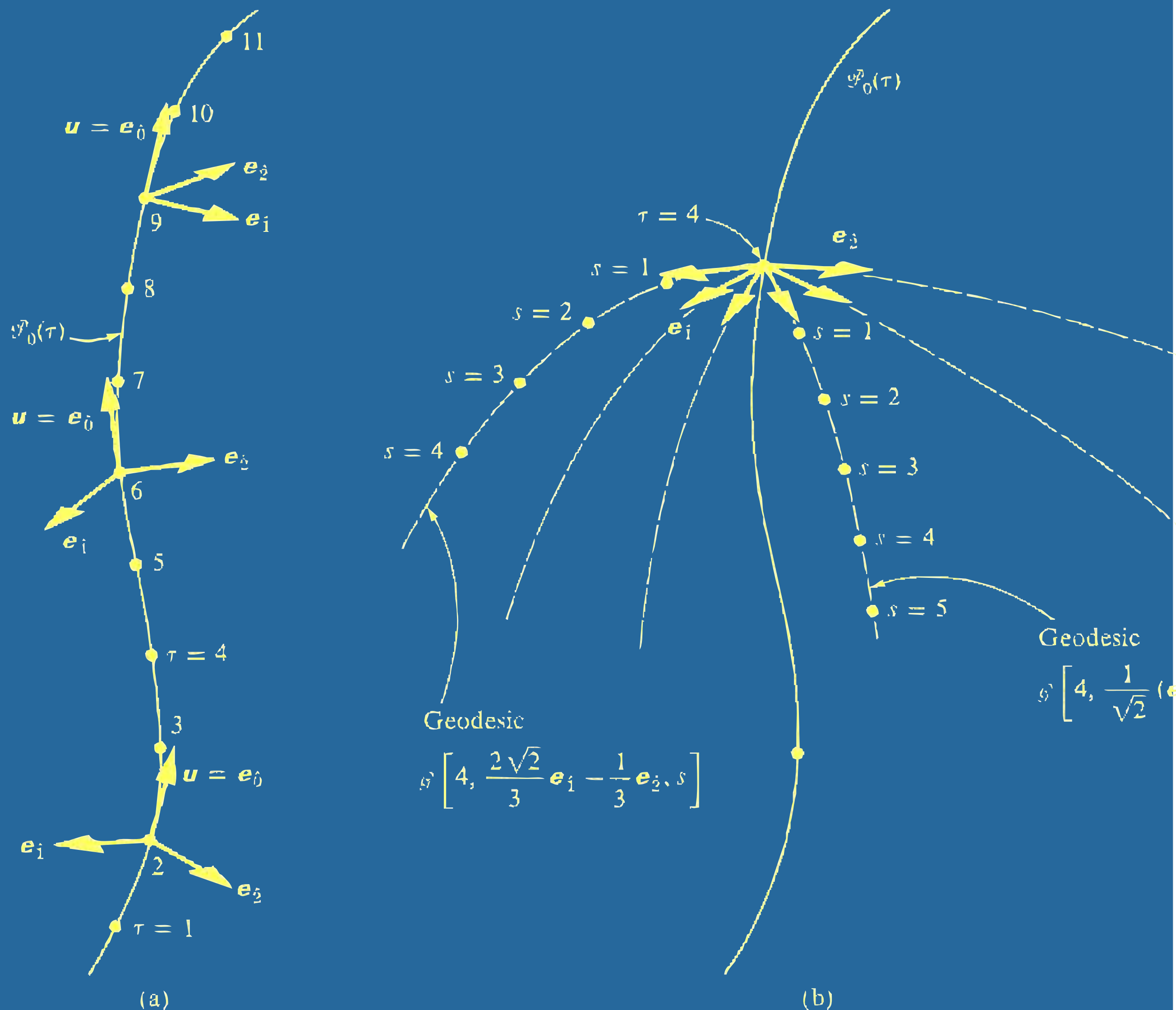
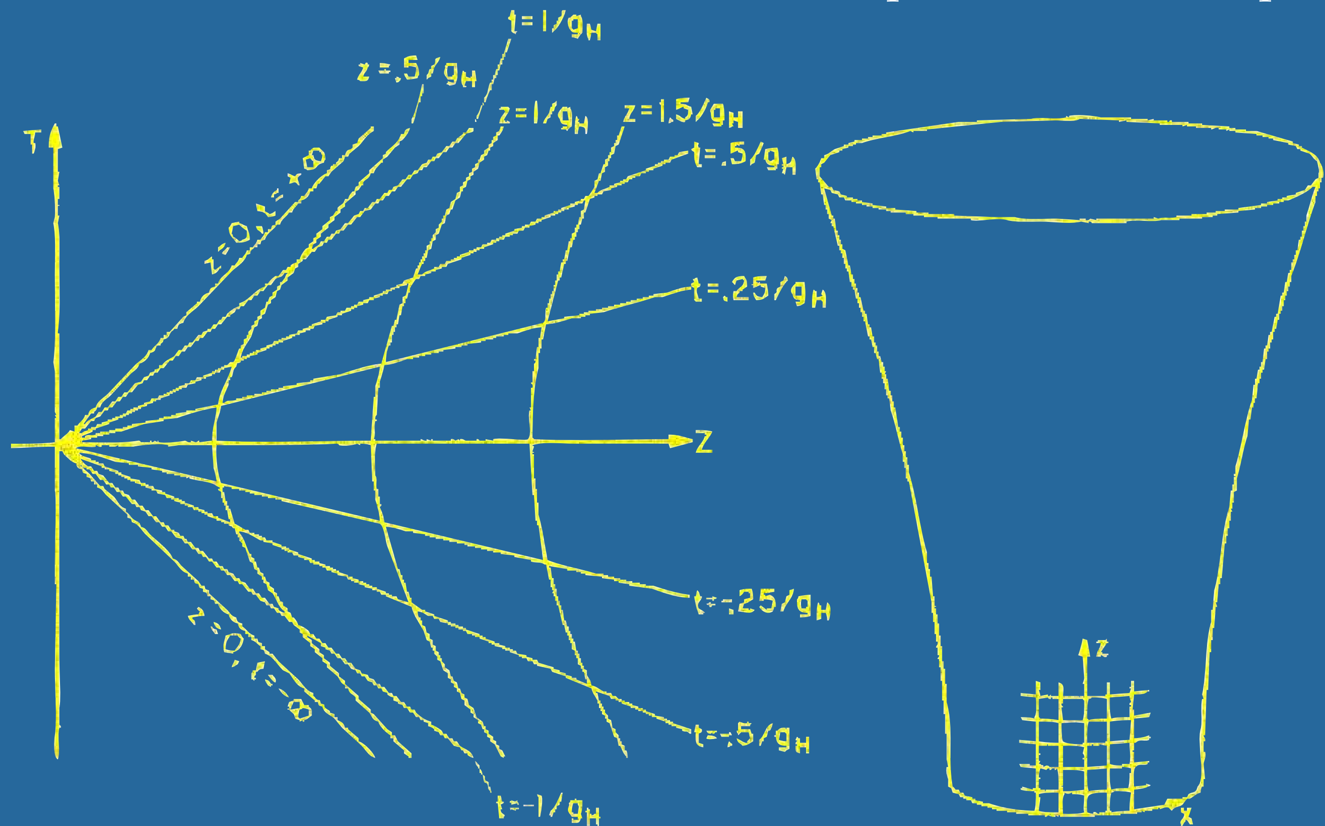


Illustration II: Rindler coords of FIDO near BH horizon

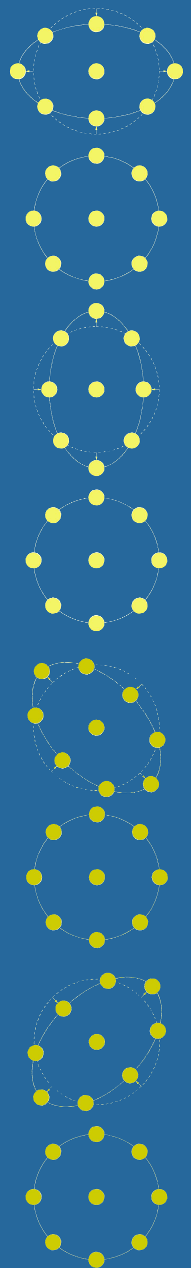
Close to horizon, coords transf: $x = 2M\left(\theta - \frac{\pi}{2}\right)$, $y = 2M\phi$, $z = 4M\sqrt{1-\frac{2M}{r}}$ turns Schwarzschild into Rindler geometry $=$ Minkowski:

$$\begin{aligned} ds^2 &= -\left(\frac{z}{4M}\right)^2 dt^2 + (dx^2 + dy^2 + dz^2)\left\{1 + \mathcal{O}\left[\left(\frac{z}{4M}\right)^2, \left(\frac{x}{4M}\right)^2\right]\right\} \\ &= -dT^2 + dX^2 + dY^2 + dZ^2 \quad \left(T = z\sinh\frac{Mt}{4},\ Z = z\cosh\frac{Mt}{4},\ X = x,\ Y = y\right). \end{aligned}$$



(a) (b)

Non-coordinate frame in GW tidal field



Metric tuned to TTGW: $$g^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + h_+(z,t) & h_\times(z,t) & 0 \\ 0 & h_\times(z,t) & 1 - h_+(z,t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observer metric: $g_{(\mu\nu)} = \eta_{\mu\nu}$ w. r. t. orthonormal basis vectors:

$$\boldsymbol{e}_0 = \left(\frac{\partial}{\partial t}, 0, 0, 0\right), \quad \boldsymbol{e}_1 = \left(0, \left[1 - \frac{h_+}{2}\right]\frac{\partial}{\partial x}, -\frac{h_\times}{2}\frac{\partial}{\partial y}, 0\right),$$

$$\boldsymbol{e}_3 = \left(0, 0, 0, \frac{\partial}{\partial z}\right), \quad \boldsymbol{e}_2 = \left(0, -\frac{h_\times}{2}\frac{\partial}{\partial x}, \left[1 + \frac{h_+}{2}\right]\frac{\partial}{\partial y}, 0\right)$$

Covariant derivatives: $\nabla_a T_{bc} = \boldsymbol{e}_a T_{bc} - \Gamma^d{}_{ba} T_{dc} - \Gamma^d{}_{ca} T_{bd}$, with:

$$-\Gamma_{[01]1} = \Gamma_{[02]2} = \frac{1}{2}\frac{\partial h_+}{\partial t}, \quad -\Gamma_{[31]1} = \Gamma_{[32]2} = \frac{1}{2}\frac{\partial h_+}{\partial z},$$

$$-\Gamma_{[01]2} = \Gamma_{[20]1} = \frac{1}{2}\frac{\partial h_\times}{\partial t}, \quad -\Gamma_{[32]1} = \Gamma_{[13]2} = \frac{1}{2}\frac{\partial h_\times}{\partial z}.$$

Covariant derivation of MHD

$3+1$ Maxwell

Covariant electromagnetic field tensor split up in space & time components:

$$F_{\mu\nu} = (U^{\alpha}_{\ \mu} U^{\beta}_{\ \nu} + H^{\alpha}_{\ \mu} H^{\beta}_{\ \nu}) F_{\alpha\beta} = u_\mu E_\nu - E_\mu u_\nu + \epsilon_{\mu\nu\alpha} B^\alpha$$

$$\mathcal{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

- 3D comoving volume element $\epsilon_{\alpha\beta\gamma} \equiv \epsilon_{\alpha\beta\gamma\delta} u^\delta$ and $\epsilon_{0123} = \sqrt{|\det g|} = 1$
- $B^\mu \equiv \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta}$: magnetic field in comoving frame.

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- $E_\mu \equiv F_{\mu\nu} u^\nu$: com. electric field $\stackrel{\mathrm{MHD}}{\Rightarrow}$ vanishes $\Rightarrow E_\mu = 0$ in every frame.
- In different frame: $\boldsymbol{E} = F_{\mu 0} u^0 = \epsilon_{\mu 0 \alpha\beta} u^\beta B^\alpha = -\boldsymbol{v} \times \boldsymbol{B} \Rightarrow$ ‘Ohm’s law’.

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Maxwell (current density $j^{\mu} = [\tau, \boldsymbol{j}]$):

$$\nabla_\nu F^{\mu\nu} = \boxed{\epsilon^{\mu\nu\alpha} \nabla_\nu B_\alpha = 4\pi j^{\mu}} \quad \Leftarrow \quad \textit{Ampère \& Gauss}$$

$$\nabla_\nu \mathcal{F}^{\mu\nu} = \frac{1}{4} \boxed{\nabla_\nu (B^{\mu} u^{\nu} - u^{\mu} B^{\nu}) = 0} \quad \Leftarrow \quad \textit{Faraday \& no monopoles}$$

Thermodynamics

(a) Internal energy per unit mass U as function of p and specific volume $V = 1/\rho$:

$$dU = -pdV \ \ + dQ$$

Thermodynamics

(a) Internal energy per unit mass U as function of p and specific volume $V = 1/\rho$:

$$\boxed{dU = -pdV} + dQ$$

No heat-flow in ideal MHD, $dQ = 0$.

(b) Adiabatic gas law: $p = K\rho^\gamma$ ($\frac{4}{3} \leq \gamma \leq \frac{5}{3} \neq \Gamma$ Lorentz).

$(a+b)$ Comoving relativistic matter energy density of ideal fluid (EOS):

$$\rho(c^2 + U) = \boxed{\mu = \rho c^2 + \frac{p}{\gamma - 1}}$$

- Proper relativistic sound velocity is pressure change at constant entropy ($w = \mu + p$ is enthalpy):

$$\left.\frac{\partial p}{\partial \mu}\right|_{\mathrm{ad}} = \boxed{c_{\mathrm{s}}^2 = \frac{\gamma p}{w}}.$$

Conservation of particles, energy & momentum

- Covariant *energy-momentum* tensor for ideal magneto-fluid:

$$T_{\mu\nu} = wu_\mu u_\nu + pg_{\mu\nu} + \frac{1}{4\pi}\left(F^{\alpha}_{\mu}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

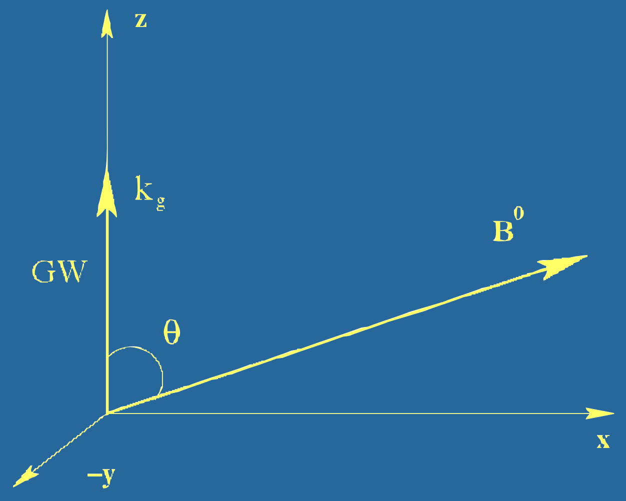
- Conservation of energy & momentum: $\nabla_\nu T^{\mu\nu} = 0$ (because $\nabla_\nu G^{\mu\nu} = 0$).
- Time ($\mu = 0$) and space $\mu = 1, 2, 3$) projections [with $u^\mu = (1, \boldsymbol{v})$]:

$$\frac{\partial\mu}{\partial t} + \nabla_{\boldsymbol{e}} \cdot (w\boldsymbol{v}) = \boldsymbol{j} \cdot \boldsymbol{E} \qquad \textit{energy}$$

$$\frac{\partial(w\boldsymbol{v})}{\partial t} + \nabla_{\boldsymbol{e}} \cdot (w\boldsymbol{v}\boldsymbol{v} + p\mathrm{I}) = \tau\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} \qquad \textit{momentum}$$

- Cons. of proper number density $n = \frac{\rho}{m_e}$: $\nabla_\mu(nu^\mu) = \frac{1}{m_e}\left[\frac{\partial\rho}{\partial t} + \nabla_{\boldsymbol{e}} \cdot (\rho\boldsymbol{v})\right] = 0$.

Set-up



- Oblique GW propagation at angle θ w. r. t. $\boldsymbol{B}^0$,
- Choose z-axis along GW propagation,
- Choose x- and y-axis such that

$$\boldsymbol{B}^0 = (B_x^0, 0, B_z^0) = |\boldsymbol{B}^0|(\sin\theta, 0, \cos\theta),$$

- In comoving frame $\boldsymbol{E}^0, \boldsymbol{v}^0, \tau^0, \boldsymbol{j}_{\mathrm{m}}^0, h^0 = 0$ & $\Gamma = 1$,
- Equilibrium characterized by $\mu^0, p^0, \rho^0 \neq 0$ and $\boldsymbol{B}^0$.

♦ To study plasma properties (stability, wave modes etc.) $\Rightarrow$ consider plasma response to small perturbations $\Rightarrow$ linearized MHD equations.

♦ [NB: Metric perturbation (GW) treated on equal footing.]

General Relativistic Magnetohydrodynamics

PART. CONS: $\frac{\partial \rho^1}{\partial t} = -\rho^0 \nabla \cdot \boldsymbol{v}^1$

ENERGY: $\frac{\partial p^1}{\partial t} = -\gamma p^0 \nabla \cdot \boldsymbol{v}^1$

NO MONOP.: $\nabla \cdot \boldsymbol{B} = 0$

E.O.M: $w^0 \frac{\partial \boldsymbol{v}^1}{\partial t} = -\nabla p^1 + \boldsymbol{j}^1_{\mathrm{m}} \times \boldsymbol{B}^0$

GAUSS $\nabla \cdot \boldsymbol{E}^1 = 4\pi\tau^1$

OHM: $\boldsymbol{E}^1 = -\boldsymbol{v}^{\mathbf{1}} \times \mathbf{B}^0$

FARADAY: $\nabla \times \boldsymbol{E}^1 + \frac{\partial \boldsymbol{B}^1}{\partial t} = -\boldsymbol{j}^1_{\mathrm{B}}$

AMPERE: $\nabla \times \boldsymbol{B}^1 - \frac{\partial \boldsymbol{E}^1}{\partial t} = 4\pi \boldsymbol{j}^1_{\mathrm{m}} + \boldsymbol{j}^1_{\mathrm{E}}$

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GW induced source terms:

$$\boldsymbol{j}^1_{\mathrm{B}} = -\frac{B^0_\perp}{2}\frac{\partial}{\partial t}\begin{pmatrix} h^1_+ \\ h^1_\times \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{j}^1_{\mathrm{E}} = -\frac{B^0_\perp}{2}\frac{\partial}{\partial z}\begin{pmatrix} h^1_\times \\ -h^1_+ \\ 0 \end{pmatrix}$$

Coupling to GW: – EFE & Maxwell

- Einstein weak Field Eqns (GW): $G_{\mu\nu} \simeq \boxed{-\frac{1}{2}\Box h_{\mu\nu} = 8\pi\delta T_{\mu\nu}}$.
- 'Poisson' eq. for gravitational field with solution (in Lorenz gauge):

$$h_{\mu\nu}(\boldsymbol{x}, t) = 4\int \frac{\delta T_{\mu\nu}(\boldsymbol{x}', t'_{\rm ret})}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'$$

- Not necessarily TT, but if $h_{\mu\nu}(z-t)$ & plane $\Rightarrow$ TT after gauge trans. $\xi(z-t)$.
- 'Throw away' all non-TT components and use $h_{ij}^{\rm TT} \sim T_{ij}^{\rm TT}$ ($i, j = 1, 2$).

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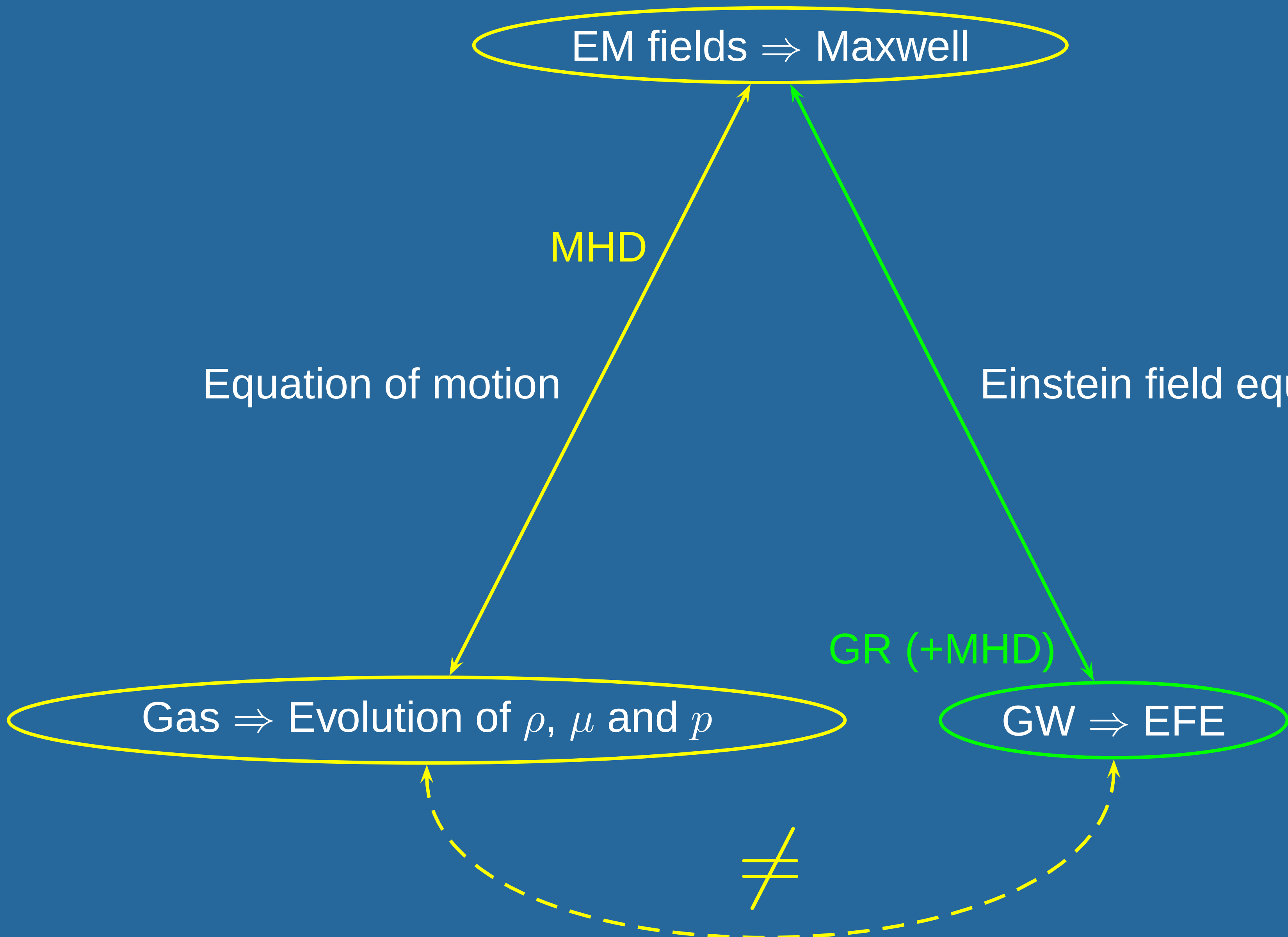
- Not necessarily TT, but if $h_{\mu\nu}(z-t)$ & plane $\Rightarrow$ TT after gauge trans. $\xi(z-t)$.
- 'Throw away' all non-TT components and use $h_{ij}^{\text{TT}} \sim T_{ij}^{\text{TT}}$ $(i, j = 1, 2)$.
- No coupling to ideal fluid: $T_{xy}^{\text{TT}} = T_{yx}^{\text{TT}} = \mathcal{O}[v^1 \propto h]^2$ & $T_{xx}^{\text{TT}} = T_{yy}^{\text{TT}}$ gauge.
- Back-reaction on GW through EFE:

$$\begin{aligned} \Box h_+ &= -8\pi(\delta T_{xx} - \delta T_{yy}) = 4B_x^0 B_x^1, \\ \Box h_\times &= -8\pi(\delta T_{xy} + \delta T_{yx}) = 4B_x^0 B_y^1. \end{aligned}$$

- Cons. of stress-energy independent of GW: $\nabla_\nu T_{\text{EM}}^{\mu\nu} = -F^{\mu\nu} j_\nu = 0$.
- GW couples to EM field through covariant derivatives in Maxwell.

Schematic Overview



Summary

18 Equations

(1) Cons. of number density,

(1) Cons. of energy density,

(1) Gauss's law,

(1) No monopoles (contraint),

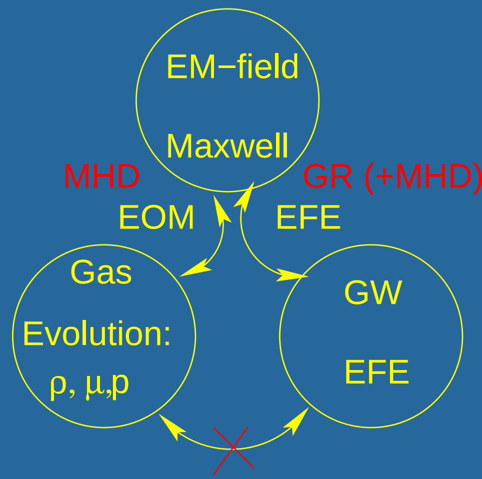
(3) Faraday's law,

(3) Ampère's law,

(3) Ohm's law,

(3) Cons. of momentum (or EOM),

(2) Einstein field equations.



18 Variables

(1) number density $\frac{\rho}{m_e}$,

(1) energy density μ,

(1) charge density τ,

(1) pressure p,

(3) current density $\boldsymbol{j}$,

(3) magnetic field $\boldsymbol{B}$,

(3) electric field $\boldsymbol{E}$,

(3) velocity $\boldsymbol{v}$,

(2) GW h_+ and $h_\times$.

MHD Wave Solutions

The Wave mode Zoo

Kinetic

- Thermal, unmagnetized:
  - ⋆ Transverse EM,
  - ⋆ Langmuir,
  - ⋆ Ion sound,
- Magnetoionic (cold plasma):
  - ⋆ Electron cyclo. maser & Low frequency:
    - $\|$ (electron /) ion cyclotron ($\sim$ Alfvén),
    - $\perp$ (electron /) ion cyclotron (Bernstein),
    - magnetoacoustic (joins whistler at ω_{LH}),
  - ⋆ High frequency:
    - Ordinary magnetoionic (o & whistler),
    - Extraordinary magnetoionic (x & z $\sim$ magnetized Langmuir).
- warm, inhomogeneous plasma, etc . . .

MHD

- Non-magnetic
  - ⋆ Sound
- Magnetic
  - ⋆ Alfvén
  - ⋆ Magneto-acoustic
    - slow
    - fast

Wave equation

Lorentz force couples matter to EM fields; eliminate $\boldsymbol{j}^1$ and $\boldsymbol{B}^1$ from Maxwell.

Wave equation $\Rightarrow$ $$\left[\frac{\partial^2}{\partial t^2} - u_{\rm m}^2 \nabla\nabla\cdot\right]\boldsymbol{v}^1 - \left[\boldsymbol{u}_{\rm A}\frac{\partial^2}{\partial t^2} - (\boldsymbol{u}_{\rm A}\cdot\nabla)\nabla\right](\boldsymbol{v}^1\cdot\boldsymbol{u}_{\rm A}) = (\boldsymbol{u}_{\rm A}\cdot\nabla)^2\boldsymbol{v}^1 - \boldsymbol{u}_{\rm A}(\boldsymbol{u}_{\rm A}\cdot\nabla)\nabla\cdot\boldsymbol{v}^1 +$$

GW source terms $\Rightarrow$ $$\sqrt{\frac{w_{\rm tot}}{4\pi}}\left[\nabla(\boldsymbol{j}_{\rm B}\cdot\boldsymbol{u}_{\rm A}) - \frac{\partial}{\partial t}(\boldsymbol{j}_{\rm E}\times\boldsymbol{u}_{\rm A}) - (\boldsymbol{u}_{\rm A}\cdot\nabla)\boldsymbol{j}_{\rm B}\right],$$

Defs: Total enthalpy $w_{\rm tot} \equiv w^0 + \frac{|\boldsymbol{B}^0|^2}{4\pi}$ & wave vel.: $u_{\rm A}^2 \equiv \frac{|\boldsymbol{B}^0|^2}{4\pi w_{\rm tot}}$, $u_{\rm m}^2 \equiv \frac{\gamma p^0}{w_{\rm tot}} + \frac{|\boldsymbol{B}^0|^2}{4\pi w_{\rm tot}}$.

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Wave equation $\Rightarrow$

$$\left[\frac{\partial^2}{\partial t^2} - u_{\rm m}^2 \nabla\nabla\cdot\right]\boldsymbol{v}^1 - \left[\boldsymbol{u}_{\rm A}\frac{\partial^2}{\partial t^2} - (\boldsymbol{u}_{\rm A}\cdot\nabla)\nabla\right](\boldsymbol{v}^1\cdot\boldsymbol{u}_{\rm A}) = (\boldsymbol{u}_{\rm A}\cdot\nabla)^2\boldsymbol{v}^1 - \boldsymbol{u}_{\rm A}(\boldsymbol{u}_{\rm A}\cdot\nabla)\nabla\cdot\boldsymbol{v}^1 +$$

GW source terms $\Rightarrow$

$$\sqrt{\frac{w_{\rm tot}}{4\pi}}\left[\nabla(\boldsymbol{j}_{\rm B}\cdot\boldsymbol{u}_{\rm A}) - \frac{\partial}{\partial t}(\boldsymbol{j}_{\rm E}\times\boldsymbol{u}_{\rm A}) - (\boldsymbol{u}_{\rm A}\cdot\nabla)\boldsymbol{j}_{\rm B}\right],$$

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Solve boundary value problem algebraically with Fourier $[t \rightarrow \omega]$ and Laplace $[z \rightarrow k]$.

$\Downarrow$

$D\boldsymbol{v}^1 = \boldsymbol{J}^1_{\rm GW}$; D is response tensor of plasma.

$\Downarrow$

Symmetric Hyperbolic PDE

3×3 Symmetric Matrix representation

Alfvén, Slow and Fast Magneto-acoustic Modes

$$\begin{pmatrix} \omega^2(1-u_{A\perp}^2)-k^2u_{A\|}^2 & 0 & -(\omega^2-k^2)u_{A\|}u_{A\perp} \\ 0 & \omega^2-k^2u_{A\|}^2 & 0 \\ -(\omega^2-k^2)u_{A\|}u_{A\perp} & 0 & \omega^2(1-u_{A\|}^2)-k^2(u_m^2-u_{A\|}^2) \end{pmatrix} \begin{pmatrix} v_x^1 \\ v_y^1 \\ v_z^1 \end{pmatrix} = \frac{i\omega^2 u_{A\perp}}{k-\omega} \begin{pmatrix} h_+ u_{A\|} \\ h_\times u_{A\|} \\ -h_+ u_{A\perp} \end{pmatrix}$$

Solutions for: $\Lambda(\omega,k) = (\omega^2 - k^2u_{A\|}^2)(\omega^2 - k^2u_f^2)(\omega^2 - k^2u_s^2) = 0$

$$\frac{\omega}{k_A} = \boxed{u_A = \pm k_A u_{A\|}} = \pm u_A \cos\theta,$$

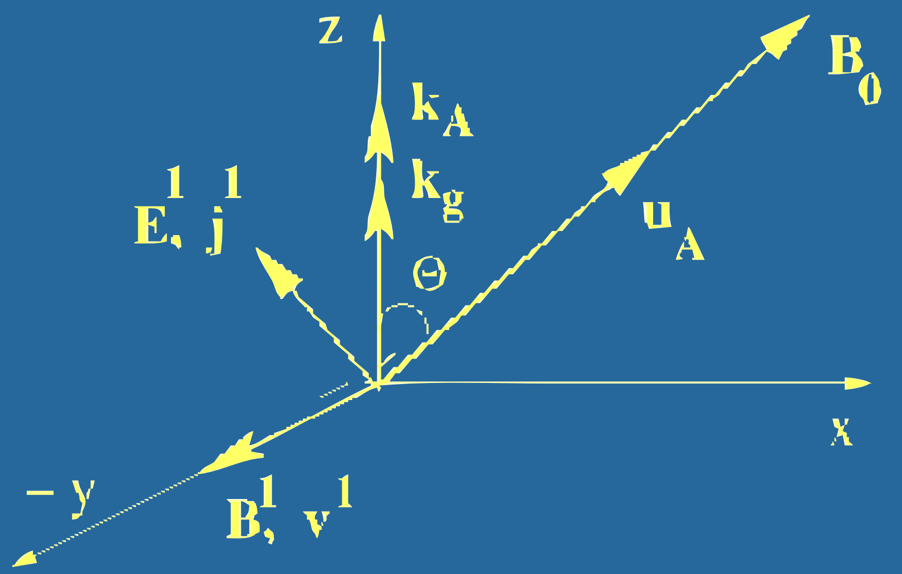
$$\frac{\omega}{k_{s,f}} = \boxed{u_{s,f} = \pm\sqrt{\tfrac{1}{2}(u_m^2 + c_s^2u_{A\|}^2)}\sqrt{1 \pm \sqrt{(1-\sigma)}}}; \quad \sigma(\theta) \equiv \frac{4c_s^2u_{A\|}^2}{(u_m^2 + c_s^2u_{A\|}^2)^2}.$$

Inhomogeneous solution: $\boldsymbol{v}^1 = D^{-1}\boldsymbol{J}_{GW}^1 = \frac{\lambda_{ij}(J_{GW}^1)_j}{\Lambda};$

Unit polarization vectors: $n_{Mi}(k)n_{Mj}^*(k) = \frac{\lambda_{ij}(\omega,k_M)}{\Lambda(\omega,k_M)}.$

Alfvén Waves

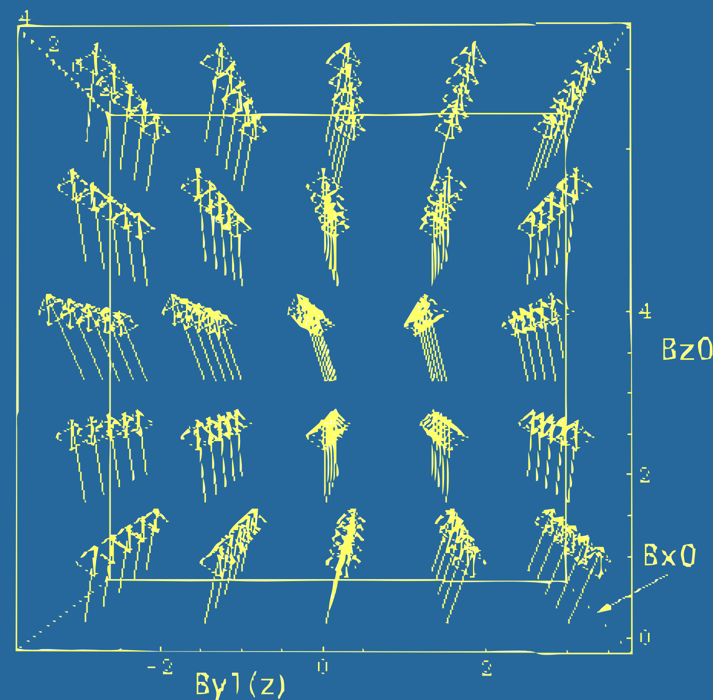
- Non-compressional shear wave,
- $\boldsymbol{E}_{||}$ component $\Rightarrow \tau^1$,
- $\frac{\boldsymbol{E}^1 \times \boldsymbol{B}^0}{|\boldsymbol{B}^0|^2}$ drift velocity along B_y^1,
- current density along $\boldsymbol{E}^1$,
- excited by $h_\times$,



- $B_y^1 \propto \frac{1}{2} h_\times B_x^0$; Equiv. to LIGO :

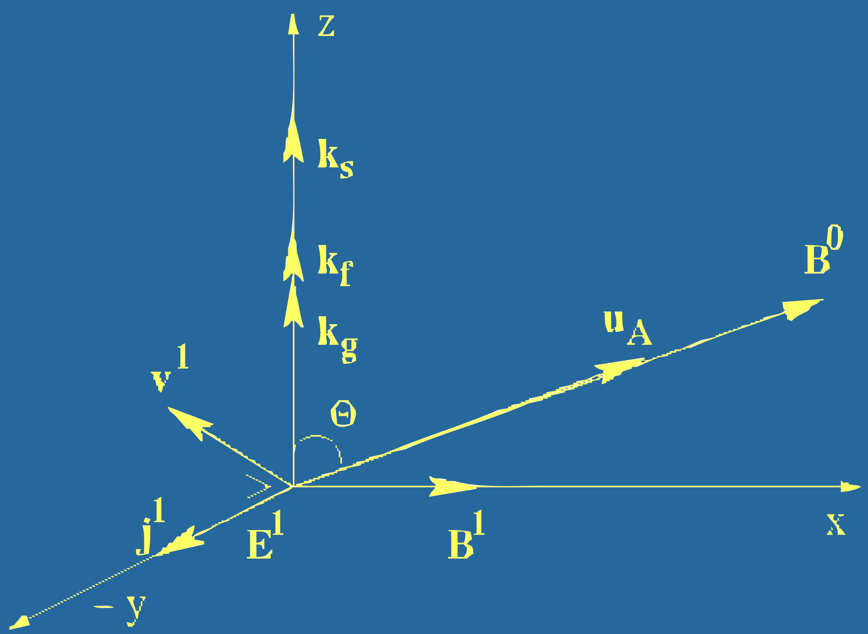
$$\boxed{y^1 = \frac{1}{2} h_\times x^0} - \frac{1}{2} h_+ y^0),$$

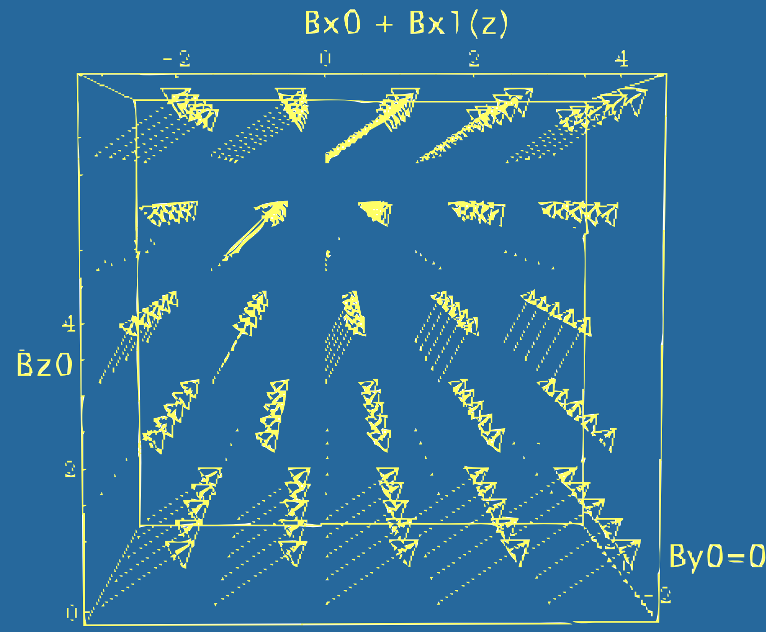
- amplitude $\propto \sin\theta$;
- coherent interaction if $\cos\theta \simeq 1$...



Magneto-acoustic Waves

- Compressional waves,
- gas & electromagnetic properties,
- perturbations in ρ, p, μ,
- drift velocity along GW (& wind),
- transverse: $\boldsymbol{E}^1 \perp \boldsymbol{B}^1 \perp \boldsymbol{k}_{\mathrm{g;s,f}}$,
- in PFD plasma ↑ vacuum EM wave,
- $B^1_x \propto \frac{1}{2} h_+ B^0_x$; equiv. to $x^1 = \frac{1}{2} h_+ x^0$,
- coherent interaction $\forall \theta$;
- amplitude $\propto \sin\theta$.





Damping of the GW

Most efficient interaction in PFD plasma ($c_{\rm s} \ll u_{\rm A}$), where:

$$B_y^1(k,\omega) = \frac{B_x^0 h_\times(k,\omega)}{2}\frac{\omega^2 + k^2 u_{{\rm A}\parallel}^2}{\omega^2 - k^2 u_{{\rm A}\parallel}^2}, \qquad B_x^1(k,\omega) \simeq \frac{B_x^0 h_+(k,\omega)}{2}\frac{\omega^2 + k^2 u_{\rm A}^2}{\omega^2 - k^2 u_{\rm A}^2}.$$

Self-consistent dispersion relations from:

$$\Box h_+ = 4 B_x^0 B_x^1 \Rightarrow \boxed{\omega^2 - k^2 = 2(B_x^0)^2 \frac{\omega^2 + k^2 u_{\rm A}^2}{\omega^2 - k^2 u_{\rm A}^2}},$$

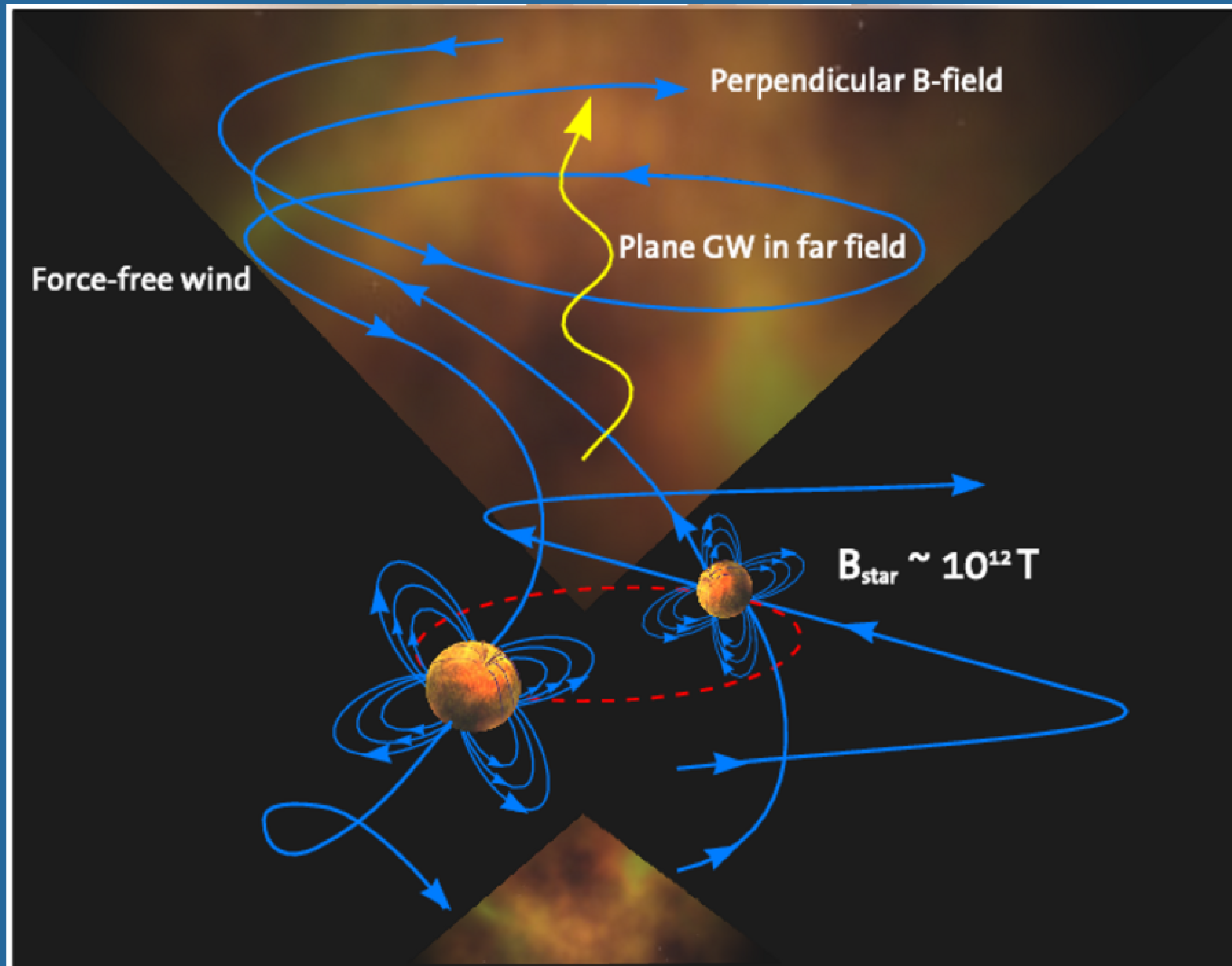
(same for $h_\times$ with $u_{\rm A} \to u_{{\rm A}\parallel}$)

modified GW sol. ($v_{\rm ph,1} v_{\rm gr,1} = 1$) & modified plasma sol. ($v_{\rm ph,2} v_{\rm gr,2} = u_{\rm A}^2$):

$$\begin{aligned} v_{\rm ph,1}^2 &\simeq 1 + \frac{2(B_x^0)^2}{\omega^2}\frac{1+u_{\rm A}^2}{1-u_{\rm A}^2}, & v_{\rm gr,1}^2 &\simeq 1 - \frac{2(B_x^0)^2}{\omega^2}\frac{1+u_{\rm A}^2}{1-u_{\rm A}^2} \\ \frac{v_{\rm ph,2}^2}{u_{\rm A}^2} &\simeq 1 - \frac{(2B_x^0)^2}{\omega^2}\frac{u_{\rm A}^2}{1-u_{\rm A}^2}, & \frac{v_{\rm gr,2}^2}{u_{\rm A}^2} &\simeq 1 + \frac{(2B_x^0)^2}{\omega^2}\frac{u_{\rm A}^2}{1-u_{\rm A}^2} \end{aligned}$$

GW acts as driver (with $\omega = kc$) when $\boxed{\frac{8\pi G}{c^2}\frac{(B_x^0)^2}{\mu_0} < \omega(\Delta k)c} = \omega^2\left(\frac{c}{u_{\rm A}} - 1\right).$

Explicit expressions & Astrophysical Applications



Separate presentation . . .