Relativistic Cosmology

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Spacetime Splitting

Introduce a timelike 4-velocity field, u_a, with

$$u_a u^a = -1\,. \tag{1}$$

This defines the family of the fundamental observers and is usually identified with the motion of the fluid.

In cosmological studies, u_a is taken as the 4-velocity that sets the CMB dipole to zero.

The spacetime splitting is completed by means of the projection tensor

$$h_{ab} = g_{ab} + u_a u_b\,, \tag{2}$$

where g_{ab} is the spacetime metric. The above projects orthogonal to u_a into the observer's instantaneous 3-D rest space.

The projector has the standard properties

$$h_{ab}u^b = 0\,, \quad h_a{}^c h_{cb} = h_{ab} \text{ and } h_a{}^a = 3\,, \tag{3}$$

and it is also the metric of the 3-space (when there is no rotation).

On using u_a and h_{ab} one obtains a unique $1+3$ spacetime splitting into "time" and "space".

Timelike and Spacelike Derivatives

We define two derivative operators:

one along the fundamental timelike direction

$$\dot{T}_{abc\cdots} = u^d \nabla_d T_{abc\cdots}\,, \tag{4}$$

which is the covariant time derivative;

and one operating on the observer's rest space

$$\mathrm{D}_d T_{abc\cdots} = h_d{}^m h_a{}^e h_b{}^f h_c{}^l \cdots \nabla_m T_{efl\cdots}\,. \tag{5}$$

Note that all the tensor indices are projected orthogonal to u_a.

Kinematics

The covariant derivative of u_a splits as

$$\begin{aligned} \nabla_b u_a &= \mathrm{D}_b u_a + \dot{u}_a u_b \\ &= \tfrac{1}{3}\Theta + \sigma_{ab} + \omega_{ab} + \dot{u}_a u_b \,, \end{aligned} \tag{6}$$

where

$$\Theta = \nabla_a u^a \,, \qquad \sigma_{ab} = \mathrm{D}_{\langle b} u_{b\rangle} \,,$$
$$\omega_{ab} = \mathrm{D}_{[b} u_{a]} \quad \text{and} \quad \dot{u}_a = u^b \nabla_b u_a \,. \tag{7}$$

are respectively, the expansion scalar, the shear, the vorticity and the 4-acceleration.

The first three quantities in the right-hand of (6) determine the relative position vector between neighbouring worldlines.

The vorticity tensor defines an associated vector, parallel to the rotation axis, by

$$\omega_a = \tfrac{1}{2}\epsilon_{abc}\omega^{bc} \,. \tag{8}$$

In cosmology we use Θ to define an average scale factor, a, by

$$\frac{\dot{a}}{a} = \frac{\Theta}{3} \,. \tag{9}$$

Matter Fields

Matter fields modify the spacetime geometry through the Einstein field equations (EFE)

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}\,, \tag{10}$$

where R_{ab} is the Ricci tensor, $R = R_a{}^a$, Λ is the cosmological constant and T_{ab} is the energy-momentum tensor of the matter ($8\pi G = 1 = c$).

Also,

$$R = 4\Lambda - T \tag{11}$$

with $T = T_a{}^a$.

Relative to u_a, the stress-energy tensor of a general imperfect fluid decomposes as

$$T_{ab} = \mu u_a u_b + p\, h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}\,, \tag{12}$$

where

$$\mu = T_{ab}u^a u^b\,, \qquad p = \tfrac{1}{3}T_{ab}h^{ab}\,,$$

$$q_a = h_a{}^b T_{bc} u^c \quad \text{and} \quad \pi_{ab} = h_{\langle a}{}^c h_{b\rangle}{}^d T_{cd}\,. \tag{13}$$

Perfect Fluids

For a perfect fluid $q_a = 0 = \pi_{ab}$, which means that

$$T_{ab} = \mu u_a u_b + p\, h_{ab}\,. \tag{14}$$

The medium is specified by the equation of state. In the case of a barotropic fluid, we have

$$p = p\,(\mu) \tag{15}$$

with

$$c_{\mathrm{s}}^2 = \frac{dp}{d\mu} \tag{16}$$

representing the adiabatic sound speed.

Matter fields traditionally satisfy the constraints

$$\mu > 0\,, \qquad \mu + p > 0\,, \quad \mu + 3p > 0$$

$$\text{and} \quad 0 \leq c_{\mathrm{s}}^2 \leq 1\,. \tag{17}$$

Spacetime Curvature

The Riemann tensor of the spacetime decomposes as

$$\begin{aligned} R_{abcd} &= C_{abcd} \\ &\quad +\tfrac{1}{2}\left(g_{ac}R_{bd}+g_{bd}R_{ac}-g_{bc}R_{ad}-g_{ad}R_{bc}\right) \\ &\quad -\tfrac{1}{6}R\left(g_{ac}g_{bd}-g_{ad}g_{bc}\right)\,, \end{aligned} \tag{18}$$

where C_{abcd} is the conformal curvature (Weyl) tensor.

Through the EFE, the Ricci curvature determines the local gravitational field. The Weyl curvature is associated to the long range gravitational field, that is tidal forces and gravity waves.

The Weyl tensor splits as

$$\begin{aligned} C_{abcd} &= \left(g_{abqp}g_{cdsr}-\eta_{abqp}\eta_{cdsr}\right)u^q u^s E^{pr} \\ &\quad -\left(\eta_{abqp}g_{cdsr}+g_{abqp}\eta_{cdsr}\right)u^q u^s H^{pr}, \end{aligned} \tag{19}$$

where $g_{abcd}=g_{ac}g_{bd}-g_{ad}g_{bc}$ and

$$E_{ab}=C_{acbd}u^c u^d\,, \quad H_{ab}=\tfrac{1}{2}\epsilon_{acd}C^{cd}{}_{be}u^e\,, \tag{20}$$

are respectively the electric and magnetic components of C_{abcd} (with $E_{ab}u^b=0=H_{ab}u^b$).

Kinematical Evolution

The kinematical evolution derives from the Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d\,, \tag{21}$$

applied to the 4-velocity field u_a.

The trace, the symmetric trace-free and the antisymmetric parts of the above give

$$\begin{aligned} \dot{\Theta} + \tfrac{1}{3}\Theta^2 &= -\tfrac{1}{2}(\mu + 3p) - 2(\sigma^2 - \omega^2) \\ &\quad + \dot{u}_a\dot{u}^a + \mathrm{D}^a\dot{u}_a + \Lambda\,, \end{aligned} \tag{22}$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$, $\omega^2 = \omega_{ab}\omega^{ab}/2 = \omega_a\omega^a$, and

$$\begin{aligned} \dot{\sigma}_{\langle ab\rangle} &= -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} \\ &\quad + \mathrm{D}_{\langle a}\dot{u}_{b\rangle} + \tfrac{1}{2}\pi_{ab} - E_{ab}\,, \end{aligned} \tag{23}$$

$$\dot{\omega}_{\langle a\rangle} = -\tfrac{2}{3}\Theta\omega_a + \sigma_{ab}\omega^b - \mathrm{curl}\dot{u}_a\,, \tag{24}$$

which determine the evolution of the expansion scalar, the shear and the vorticity.

The first of the above, which is known as Raychaudhuri's formula, is the key equation of gravitational collapse.

The propagation equations are complemented by the constraints

$$\tfrac{2}{3}\mathrm{D}_a\Theta = \mathrm{D}^b\sigma_{ab} - \mathrm{curl}\omega_a - 2\epsilon_{abc}\dot{u}^b\omega^c + q_a\,, \quad (25)$$

$$\mathrm{D}^a\omega_a = \omega_a\dot{u}^a\,, \quad (26)$$

$$H_{ab} = \mathrm{curl}\sigma_{ab} + 2\dot{u}_{\langle a}\omega_{b\rangle} + \mathrm{D}_{\langle a}\omega_{b\rangle} \quad (27)$$

where by definition

$$\mathrm{curl}\sigma_{ab} = \epsilon_{cd\langle a}\mathrm{D}^c\sigma^d{}_{b\rangle}\,. \quad (28)$$

These constraints determine the relations between the covariant kinematical variables in the observer's instantaneous rest space.

Conservation Laws

From the Bianchi identities one arrives at the conservation law

$$\nabla^b T_{ab} = 0\,. \tag{29}$$

The timelike component of the above leads to the energy density conservation law

$$\dot{\mu} = -\Theta(\mu + p) - 2\dot{u}_a q^a - \mathrm{D}_a q^a - \sigma_{ab}\pi^{ab}\,. \tag{30}$$

The momentum density conservation comes from the spacelike part of (29)

$$\begin{aligned}(\mu + p)\dot{u}_a \;&=\; -\mathrm{D}_a p - \tfrac{4}{3}\Theta q_a - \dot{q}_{\langle a\rangle} - \sigma_{ab}q^b \\ &\quad -\mathrm{D}^b\pi_{ab} - \pi_{ab}\dot{u}^b + \epsilon_{abc}\omega^b q^c\,.\end{aligned} \tag{31}$$

For a perfect fluid the above reduce to

$$\begin{aligned}\dot{\mu} &= -\Theta(\mu + p) \\ \dot{u}_a &= -\frac{1}{\mu + p}\,\mathrm{D}_a p\end{aligned} \tag{32}$$

Gravitational waves

The long range gravitational field is monitored by means of the Bianchi identities

$$\nabla_{[a}R_{bc]de} = 0\,. \tag{33}$$

The above split into two propagation equations

$$\begin{aligned}\dot{E}_{ab} &= -\Theta E_{ab} + \mathrm{curl}H_{ab} - \tfrac{1}{2}(\mu+p)\sigma_{ab} - \tfrac{1}{6}\Theta\pi_{ab} \\ &\quad -\tfrac{1}{2}\dot{\pi}_{ab} - \tfrac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} - \dot{u}_{\langle a}q_{b\rangle} + 3\sigma_{c\langle a}E^{c}{}_{b\rangle} \\ &\quad -\tfrac{1}{2}\sigma_{c\langle a}\pi^{c}{}_{b\rangle} + 2\epsilon_{cd\langle a}\dot{u}^{c}H^{d}{}_{b\rangle} \\ &\quad -\epsilon_{cd\langle a}\omega^{c}E^{d}{}_{b\rangle} - \tfrac{1}{2}\epsilon_{cd\langle a}\omega^{c}\pi^{d}{}_{b\rangle}\,, \end{aligned} \tag{34}$$

$$\begin{aligned}\dot{H}_{ab} &= -\Theta H_{ab} - \mathrm{curl}E_{ab} + \tfrac{1}{2}\mathrm{curl}\pi_{ab} + 3\sigma_{c\langle a}H^{c}{}_{b\rangle} \\ &\quad +\tfrac{1}{2}\epsilon_{cd\langle a}\dot{u}^{c}E^{d}{}_{b\rangle} - \tfrac{1}{2}\epsilon_{cd\langle a}q^{c}\sigma^{d}_{b\rangle} - \tfrac{3}{2}\omega_{\langle a}q_{b\rangle} \\ &\quad -\epsilon_{cd\langle a}\omega^{c}H^{d}{}_{b\rangle}\,. \end{aligned} \tag{35}$$

Combining the above one obtains wave-like equations for E_{ab} and H_{ab}, which describe propagating gravitational radiation.

The Bianchi identities also provide the constraints

$$\mathrm{D}^b E_{ab} = \tfrac{1}{3}\mathrm{D}_a\mu - \tfrac{1}{3}\Theta q_a - \tfrac{1}{2}\mathrm{D}^b\pi_{ab} + \tfrac{1}{2}\sigma_{ab}q^b - 3H_{ab}\omega^b + \epsilon_{abc}\sigma^{bd}H^c{}_d + \tfrac{3}{2}\epsilon_{abc}\omega^b q^c\,, \qquad (36)$$

$$\mathrm{D}^b H_{ab} = -(\mu+p)\omega_a + 3E_{ab}\omega^b - \tfrac{1}{2}\pi_{ab}\omega^b - \tfrac{1}{2}\mathrm{curl}\,q_a - \epsilon_{abc}\sigma^{bd}E^c{}_d - \tfrac{1}{2}\epsilon_{abc}\sigma^{bd}\pi^c{}_d\,. \qquad (37)$$

The propagation and constraint equations of the Weyl field show a remarkable analogy to Maxwell's formulae, which explains the name of E_{ab} and H_{ab}.

Spatial Curvature

In the absence of rotation, the geometry of the observer's 3-D rest space is determined by the 3-Riemann tensor defined by

$$\begin{aligned}\mathcal{R}_{abcd} &= h_a{}^e h_b{}^f h_c{}^l h_d{}^m R_{eflm} \\ &\quad -v_{ac}v_{bd} + v_{ad}v_{bc}\,, \end{aligned} \tag{38}$$

where $v_{ab} = \mathrm{D}_b u_a$.

The contractions of the 3-Riemann tensor lead to $\mathcal{R}_{ab} = \mathcal{R}^c{}_{acb}$ and $\mathcal{R} = \mathcal{R}^a{}_a$. The former is determined by the Gauss-Codacci equation

$$\begin{aligned}\mathcal{R}_{ab} &= \tfrac{1}{3}\mathcal{R}h_{ab} - \tfrac{1}{3}\Theta\sigma_{ab} + \sigma_{c\langle a}\sigma^c{}_{b\rangle} + \tfrac{1}{2}\pi_{ab} \\ &\quad +E_{ab}\,, \end{aligned} \tag{39}$$

where

$$\mathcal{R} = 2\left(\mu - \tfrac{1}{3}\Theta^2 + \sigma^2 + \Lambda\right)\,. \tag{40}$$

The above is also the generalised Friedmann equation.

The Friedmann Universe

The isotropy and homogeneity of the Friedmann solution guarantees that the only nonzero covariant quantities are

$$\mu\,,\; p\,,\; \Theta \;\text{ and }\; \mathcal{R} = \frac{6k}{a^2}\,, \tag{41}$$

where $k = 0, \pm 1$ is the 3-curvature index.

The evolution of the model is determined by two propagation equations

$$\begin{aligned} \dot{\mu} &= -\Theta(\mu + p) \\ \dot{\Theta} + \tfrac{1}{3}\Theta^2 &= -\tfrac{1}{2}(\mu + 3p) + \Lambda\,, \end{aligned} \tag{42}$$

supplemented by the constraint

$$\frac{3k}{a^2} = \mu - \tfrac{1}{3}\Theta^2 + \Lambda \tag{43}$$

An alternative expression for the Raychaudhuri equation is

$$q = \tfrac{1}{2}(\mu + 3p) - \Lambda \tag{44}$$

where $q = -a\ddot{a}/\dot{a}^2$ is the deceleration parameter of the universe.

Friedmann's equation also reads

$$H^2 = \tfrac{1}{3}\mu - \frac{k}{a^2} + \tfrac{1}{3}\Lambda\,, \qquad (45)$$

with $H = \Theta/3$ representing the Hubble parameter.

Alternatively we may recast the above as

$$1 - \Omega = -\frac{k}{a^2 H^2} + \Omega_\Lambda\,, \qquad (46)$$

where $\Omega = \mu/3H^2$ is the density parameter and $\Omega_\Lambda = \Lambda/3H^2$.

The model is completely determined when the equation of state of the cosmic medium is given.

For conventional matter $p = 0$ corresponds to a non-relativistic (dust) component, when $p = \mu/3$ we have a radiative matter and for $p = \mu$ the so called "stiff" fluid.

On the other hand, de Sitter type inflation corresponds to $p = -\mu$.

Perturbed Friedmann Universes

The theory of linear cosmological perturbations is plagued by what is known as the "gauge problem".

This emerges from the fact that we actually deal with two spacetimes: the realistic perturbed universe and the fictitious background one. The latter is usually identified with the FRW model.

To proceed we need to specify the connection, the "gauge", between these two spacetimes.

The main problem is that by changing the gauge one can arbitrarily change the perturbed value of a physical quantity. For example, the density contrast given by

$$\delta = \frac{\mu - \bar{\mu}}{\bar{\mu}} \tag{47}$$

is set to zero by identifying the perturbed surfaces of constant density with the background surfaces of constant time.

The best way around the gauge problem is by using gauge-independent variables.

The Perturbation Variables

To linear order, scalars are gauge invariant when they are covariantly constant in the background. Tensors, on the other hand, are gauge-independent only if they have zero background values.

We describe density perturbations by means of the vector

$$X_a = \mathrm{D}_a\mu = h_a{}^b\nabla_b\mu\,, \tag{48}$$

which describes the density variation between two neighbouring fundamental observers.

The above leads to the dimensionless, comoving density gradient

$$\mathcal{D}_a = \frac{a}{\mu}X_a\,. \tag{49}$$

In addition we define

$$\mathcal{Z}_a = a\mathrm{D}_a\Theta \quad \text{and} \quad Y_a = \mathrm{D}_a p\,, \tag{50}$$

which describe perturbations in the expansion and the fluid pressure respectively.

All three variables vanish in a FRW background and are therefore gauge invariant.

Evolution of the Inhomogeneities

The propagation equations for $\mathcal{D}_a$ and $\mathcal{Z}_a$ are

$$\begin{aligned}\dot{\mathcal{D}}_{\langle a\rangle} &= w\Theta\mathcal{D}_a-(1+w)\mathcal{Z}_a \\ &\quad -(\sigma_{ab}-\omega_{ab})\mathcal{D}^b \qquad (51)\end{aligned}$$

and

$$\begin{aligned}\dot{\mathcal{Z}}_{\langle a\rangle} &= -\tfrac{2}{3}\Theta\mathcal{Z}_a-\tfrac{1}{2}\mu\mathsf{D}_a+a\Re\dot{u}_a+a\mathsf{D}_aA \\ &\quad -(\sigma_{ab}-\omega_{ab})\mathcal{Z}^b-2a\mathsf{D}_a(\sigma^2-\omega^2)\,, \qquad (52)\end{aligned}$$

where $w=p/\mu$, $A=\nabla_a\dot{u}^a$ and

$$\Re=\tfrac{1}{2}\mathcal{R}+A-3(\sigma^2-\omega^2)\,. \qquad (53)$$

The above system monitors the nonlinear evolution of density inhomogeneities in a general spacetime (containing a perfect fluid). Next we will linearise these equations about a FRW universe with flat spatial sections.

The Linear Regime

During linearisation, quantities that have nonzero background value are treated as zero order variables. Those that vanish in the background are first order and higher order terms are ignored. scheme

Under this scheme, the nonlinear equations given before linearise to

$$\dot{\mathcal{D}} = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a \tag{54}$$

and

$$\dot{\mathcal{Z}}_a = -\frac{2}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\mu\mathcal{D}_a + a\mathrm{D}_aA\,. \tag{55}$$

For **dust** ($p = 0$), $\Theta = 2/t$ and $\mu = 4/3t^2$. Then, the above system gives

$$\ddot{\mathcal{D}}_a + \frac{2}{3}\Theta\mathcal{D}_a - \frac{1}{2}\mu\mathcal{D}_a = 0\,, \tag{56}$$

with solution

$$\mathcal{D}_a = C_1t^{2/3} + C_2t^{-1} \tag{57}$$

where C_1 and C_2 are time independent quantities.

In the case of **radiation** ($p = \mu/3$), $\Theta = 3/2t$ and $\mu = 3/4t^2$. Therefore,

$$\begin{aligned} \ddot{\mathcal{D}}_a &= -(\tfrac{2}{3} - w)\Theta\dot{\mathcal{D}}_a + \tfrac{1}{2}(1-w)(1+3w)\mu\mathcal{D}_a \\ &\quad + c_{\mathrm{s}}^2 \mathrm{D}^2\mathcal{D}_a\,, \end{aligned} \tag{58}$$

with $\mathrm{D}^2 = \mathrm{D}_a\mathrm{D}^a$.

Then,on introducing the harmonic decomposition

$$\mathcal{D}_a = \sum_n \mathcal{D}_n Q_a^n\,, \tag{59}$$

with $\mathrm{D}_a\mathcal{D}_n = 0 = \dot{Q}_a^n$ and

$$\mathrm{D}^2 Q_a^n = -\frac{n^2}{a^2} Q_a^n \tag{60}$$

we have

$$\begin{aligned} \ddot{\mathcal{D}}_n &= -(\tfrac{2}{3} - w)\Theta\dot{\mathcal{D}}_n \\ &\quad + \left[\tfrac{1}{2}(1-w)(1+3w)\mu - c_{\mathrm{s}}^2\frac{n^2}{a^2}\right]\mathcal{D}_n \end{aligned} \tag{61}$$

The last term in the right-hand side of the above describes the counteracting effects of gravity and pressure. On large enough scales gravity dominates and the inhomogeneity grows. On small scales, however, pressure support forces the perturbation to oscillate like a sound wave.

The Jeans' Scale

The gravitational and the pressure effects balance each other at a critical wavelength, which is known as the "Jeans scale" and given by

$$\lambda_J = \frac{a}{n_J} = c_{\mathsf{s}}\sqrt{\frac{2}{(1-w)(1+3w)\mu}} \tag{62}$$

Wavelengths below the Jeans' scale are supported by pressure and oscillate like acoustic waves.

On scales much larger than λ_J, on the other hand, the pressure term in Eq. (61) is negligible and density inhomogeneities grow as

$$\mathcal{D}_n = C_1 t^{1/2} + C_2 t^{-1}\,. \tag{63}$$