A self-organized criticality model for the magnetic field in toroidal confinement devices

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Sani Beach, Chalkidiki, June 2009

Outline

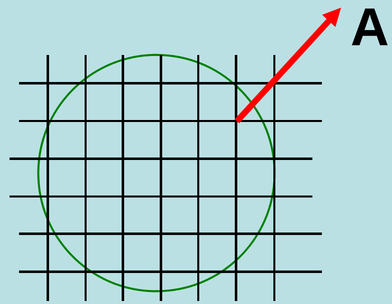
- INTRODUCTION
  - Cellular Automata (CA) and Self-Organized Criticality (SOC)
  - intended application: Reversed Field Pinch (RFP)

- The Extended CA model (X-CA) for the magnetic field in the RFP

- Results:
  - the SOC magnetic topology, flux surfaces,
  - current distribution
  - the safety factor profile,
  - magnetic energy dissipation

Cellular automata (CA)

- CA model complex systems, which consist in many sub-systems that interact with each other
- CA discrete systems, defined on a discrete grid (1D, 2D, 3D)
- Usually one basic grid variable, scalar or vector
- Time evolution in terms of (usually local) rules



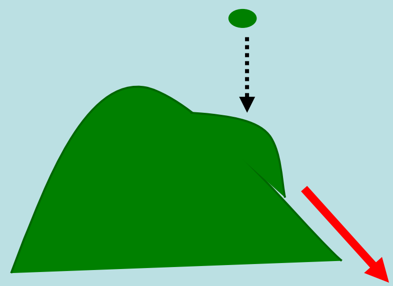
- application to systems where treatment with standard physical tools, such as DE, technically difficult due to complexity, e.g multi-scale physics
  (earth-quake modeling, traffic jam, solar flares, … confined plasma)
- Idea: try to reduce physics to (physically still meaningful) evolution rules
- In a plasma the grid-size can correspond to a typical smallest turbulent eddie size, or roughly equal correlation length
- Reminiscent of numerical solution of DE on a grid, but
  - grid-spacing not 'infinitesimally' small, just small
  - evolution according to rules, not solving DE

Self-Organized Criticality (SOC)

- SOC is a possible dynamical state for complex systems, typically modeled in the form of a CA
  with the following prerequisites:
  (i) the system is driven, stress (energy) is continuously added (in a systematic way)
  (ii) there is a local threshold dependent instability
  (iii) when the threshold is exceeded locally, a second, local process sets in that relaxes the instability in the local neighbourhood
- (iv) the relaxation process possibly is able to trigger instabilities in the neighbourhood of the primary, now relaxed instability
- Depending on the concrete rules, the system may reach the SOC state
  (a) chains of instabilities may occur that sweep through the entire system (avalanches)
  (b) statistical analysis of characteristics of the instabilities exhibit power-law distributions:
  size and duration of relaxation event, released energy etc,
  the system self-organizes, and
  the grid variable assume a characteristic global shape over the entire grid

SOC models for confined plasma

- Several SOC models for confined plasmas have been suggested, which with success model some aspects of plasma turbulence [e.g. 2,3,4,5]
- These models all use the sand-pile analogy, with variables such as the local height of the sand-pile, height differences
- Our aim is to construct SOC models for the usual physical variables (here the magnetic field), which is physically interpretable in a consistent way.
- Here, we will introduce a SOC model for the magnetic field in the form of a CA that is compatible with MHD
- Specific application is to the magnetic topology in the reversed field pinch
- Original SOC model by Bak, Tan & Wiesenfeld was a sand-pile model, i.e. the sand-pile as a paradigm for SOC (grains of sand are dropped on a sand-pile until it gets unstable and sand slides down)



- [1] Bak, P., Tang, C, Wiesenfeld, K., Phys. Rev. Lett. **59**, 381 (1987)
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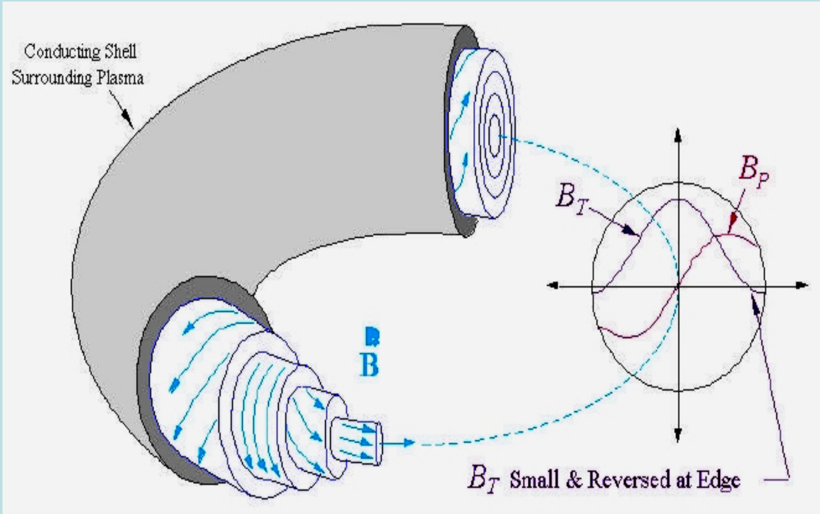
Reversed field pinch (RFP)



- The RFP is a torus-shaped confinement device, as the tokamak,
  (poloidal field coils and currents create toroidal field,and induced toroidal current creates poloidal field)
- differences to tokamak:
  - toroidal and poloidal field are of similar magnitude in the RFP
  , safety factor smaller than one
  - toroidal field changes direction (reverses , RFP) near the edge
- Characteristics:
  after an initial transient phase, plasma settles to a preferred state, where field profiles are independent of their time-history, the plasma actually self-organizes to a relaxed state
- Experiments:
  MST (USA), RFX (Italy), EXTRAP T2R (Sweden), TPE-RX (Japan)
- Theory of plasma relaxation originally developed by J.B. Taylor, Phys. Rev. Lett. **33**, 1139, (1974), Rev. Mod. Phys. **58**, 741 (1986)]

Plasma relaxation and self-organization: Theory

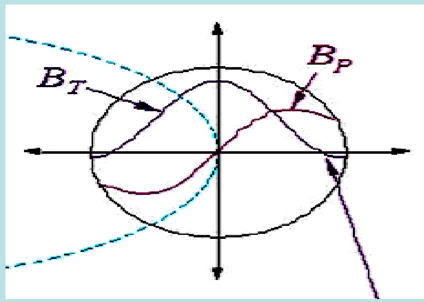
- Theory of plasma relaxation originally developed by J.B. Taylor, Phys. Rev. Lett. **33**, 1139, (1974), Rev. Mod. Phys. **58**, 741 (1986)]
- From Taylors' argumentation (based on induction equation, finite resistivity, and conservation of magnetic helicity)
  it follows that **B** must obey

  u § **B** = μ**B**

  i.e. **B** is force-free, with μ constant throughout the volume
- For circular cross-section torus of large aspect ratio, a solution in the cylindrical limit is

  $B_{tor} = B_0\, J_0(\mu\, r)$, $\qquad B_{pol} = B_0\, J_1(\mu\, r)$, $\qquad B_{rad} = 0$

  with J_0, J_1 Bessel functions of the 1st kind
  Bessel function model (BFM)
  (plasma neglected, refinements exist)



- In terms of the vector potential

$$\mathbf{A}^T(r,\theta) = (B_0/\mu)\, J_0(\mu r)\, \mathbf{e}_{\hat{\varphi}} + (B_0/\mu)\, J_1(\mu r)\, \mathbf{e}_{\hat{\theta}}$$

Plasma relaxation and self-organization: details

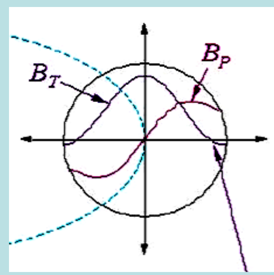
- Plasma in RFP settles to relaxed Taylor states
- In terms of the vector potential **A** (**B** = u §**A**), the magnetic field evolves acc. to MHD induction equation (here in Coulomb gauge, u ƒ**A** = 0) with **v** the velocity field, and η the resistivity

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} + \eta \frac{c^2}{4\pi} \nabla^2 \mathbf{A} + \nabla \chi$$

- If η = 0 (ideal MHD) then v$_{\text{fl}}$ **A** ƒ**B** dV over a flux-tube is an invariant (helicity)
- Taylors' argumentation:
  (0) η is not always and everywhere small, but it can be finite, even if small and in localized region
  (1) For small but finite η, v**A** ƒ**B** dV over the entire plasma is an invariant
  (2) The plasma settles to a minimum energy state, i.e. v(u §#**A**)^2 dV assumes a minimum
  (3) The constraint (1) and the condition (2) imply that **B** must obey u § **B** = μ**B**
  i.e. **B** is force-free, with μ constant throughout the volume
- For circular cross-section torus of large aspect ratio, a solution in the cylindrical limit is

$$B_{\text{tor}} = B_0\, J_0(\mu\, r), \qquad B_{\text{pol}} = B_0\, J_1(\mu\, r), \qquad B_{\text{rad}} = 0$$

with J_0, J_1 Bessel functions of the 1$^{\text{st}}$ kind
Bessel function model (BFM)
(plasma neglected, refinements exist)



A SOC model for the magnetic topology in the reversed field pinch

- To construct a SOC model in the form of a CA, we have to specify
  (1) the grid and the basic grid variable
  (2) the loading/driving process
  (3) the instability criterion
  (4) the relaxation process
- For a CA, the loading and relaxation process must be expressed in terms of (local) evolution rules
- And, since we use natural variable (**B**): MHD compatibility must be ensured
- Our particular aim is to motivate all the elements of the model physically and to make the evolution rules and the instability criterion based on and compatible with the underlying physics
- Since we present a SOC model for the evolution of the magnetic field,
  → we in particular demand compatibility with MHD
  e.g. the magnetic field should be divergence-free and always remain so

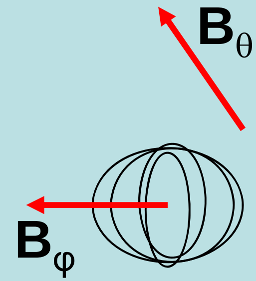
The model (1): MHD compatibility

- Our aim is to construct a SOC model for the magnetic field **B** in the form of a CA, in such a way as that it is also compatible with MHD:
  $\nabla \cdot \mathbf{B} = 0$ must hold, and the current must be determined as $\mathbf{J} = (c/4\pi)\, \nabla \times \mathbf{B}$
- Basic set-up in order to achieve MHD compatibility in the CA model:
  (1) The CA grid variable is the vector potential **A**
  (2) **A** evolves following CA rules for driving, instabilities, and relaxing (see below)
  (3) In order to calculate derivatives, **A** is interpolated, which allows to determine
  $$\mathbf{B} = \nabla \times \mathbf{A}$$
  $$\mathbf{J} = (c/4\pi)\, \nabla \times \mathbf{B}$$
  in the usual MHD way, so that e.g. $\nabla \cdot \mathbf{B} = 0$ is ensured (MHD compatibility)
- In this extended CA (X-CA), **B** is available every where in the simulation region, so that e.g. field lines can be drawn, or particles can be tracked
- In its simplest version, the MHD set-up does not interfere with the CA dynamics

Intended Application: RFP

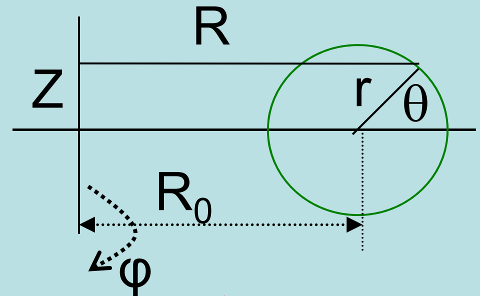
- As a guide for constructing the SOC model we need information on the topology as it is characteristic for the RFP (analytically or in numerical form, including data from experiments)
- Here, we use the Bessel function model of the relaxed Taylor state, for simplicity

$$\mathbf{B}^T(r,\theta) = B_0\, J_0(\mu r)\, \mathrm{e}_{\hat{\varphi}} + B_0\, J_1(\mu r)\, \mathrm{e}_{\hat{\theta}}$$



  which corresponds to simple circular flux surfaces
- Second, we need the topology to be expressed in terms of our grid-variable, the vector potential **A** (such that **B**=u § **A)**, which for the Bessel function model writes as

$$\mathbf{A}^T(r,\theta) = (B_0/\mu)\, J_0(\mu r)\, \mathrm{e}_{\hat{\varphi}} + (B_0/\mu)\, J_1(\mu r)\, \mathrm{e}_{\hat{\theta}}$$



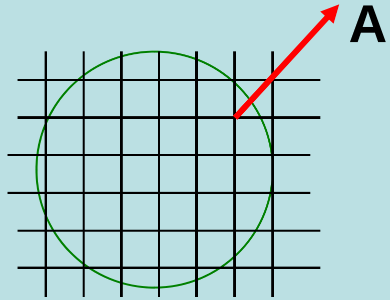
  and where A_θ yields the toroidal field B_φ,
  and A_φ yields the poloidal field B_θ
- Last, since physically the loading process is through driving the currents, we need the explicit connection of **A** to the current **I** ^ u § **B**

$$I_\varphi^T \propto -\frac{1}{r}\partial_r\left[r\partial_r A_\varphi^T\right], \quad I_\theta^T \propto -\partial_r\left[\frac{1}{r}\partial_r\left(rA_\theta^T\right)\right]$$

and where A_θ corresponds to the poloidal current I_θ
and A_φ corresponds to the toroidal current I_ϕ

$$\mathrm{I}^T(r,\theta) = B_0\mu\, J_0(\mu r)\, \mathrm{e}_{\hat{\varphi}} + B_0\mu\, J_1(\mu r)\, \mathrm{e}_{\hat{\theta}}$$

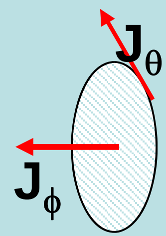
The model (1): initial set-up



- We concentrate on the 2-D poloidal plane and use a 2D grid for the CA
- The grid variable is the vector potential **A** (as a 3D vector)
- As initial condition $\mathbf{A}^{(0)}$, we can use any reasonable configuration, e.g. $\mathbf{A}^{(0)} = 0$, the state before the machine is switched on

The model (2): The driving (loading) mechanism (i)

- Basic idea: Physically, in the RFP
  - the poloidal field coils and plasma currents generate the toroidal field,
  - the induced (and self-generated) toroidal currents generate the poloidal field
- We implement this scenario and load the system through driving the toroidal and the poloidal current,
  - expressed though in rules and in terms of **A**
  - and following the structure of the Bessel function model BFM



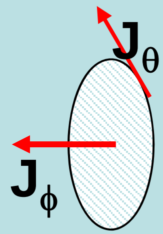
- In the BFM, two components of the current

$$I_\varphi^T \propto -\frac{1}{r}\partial_r\left[r\partial_r A_\varphi^T\right], \quad I_\theta^T \propto -\partial_r\left[\frac{1}{r}\partial_r\left(rA_\theta^T\right)\right]$$

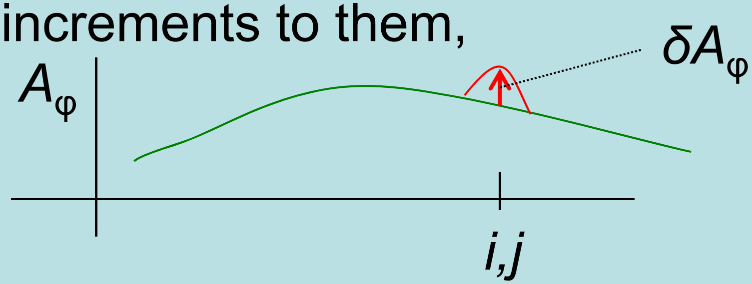
so that driving A_ϕ corresponds to driving I_ϕ
driving A_θ corresponds to driving I_θ

The model (3): the driving mechanism (ii)

- Physical driving scenario:
  I_θ, I_φ are driven , A_θ, A_φ evolves such that I_θ, I_φ increase



- In terms of CA rules
  we systematically increase A_φ and A_θ by adding increments to them,



$$
\begin{aligned}
A_\varphi(t+1,\vec{x}_{ij}) &= A_\varphi(t,\vec{x}_{ij}) + \delta A_\varphi(t,\vec{x}_{ij}),\\
A_\theta(t+1,\vec{x}_{ij}) &= A_\theta(t,\vec{x}_{ij}) + \delta A_\theta(t,\vec{x}_{ij}),
\end{aligned}
$$

  at one (usually random) grid site *ij*, at a time,
  (the toroidal/poloidal current increases because
  A_φ/A_θ increases or its local curvature increases)

- The driving process must be such that the system goes to a RFP topology,
  i.e. it must contain information about the RFP
  (else the system moves to an arbitrary and physically irrelevant state!)
  → we thus let the increments be in the direction of the Bessel function model for the Taylor state,
  with *s* a positive constant or random number

$$
\begin{aligned}
\delta A_\varphi &= s A_\varphi^{(T)}(r)\\
\delta A_\theta &= s A_\theta^{(T)}(r)
\end{aligned}
$$

The model (4): instabilities (i)

- Aim: implement a resistive, current driven instability
- The MHD induction equation in terms of the vector potential

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} + \eta \frac{c^2}{4\pi} \nabla^2 \mathbf{A} + \nabla \chi$$

with *η* the resistivity and *χ* an arbitrary function (Coulomb-gauge, u ƒ**A**=0)

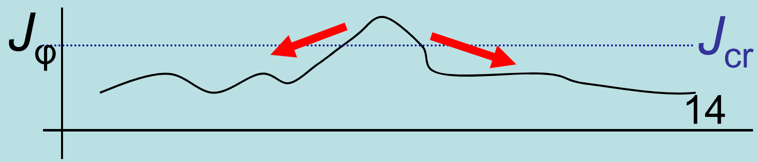
- Now **I** 2 -u 2**A**, and if we neglect the convective term and *χ*, **A** evolves as

$$\frac{\partial \mathbf{A}}{\partial t} = \eta \frac{c^2}{4\pi} \nabla^2 \mathbf{A} \propto -\eta \frac{c^2}{4\pi} \mathbf{I} \quad (1)$$

, the current causes a diffusive evolution of **A** if *η* is not zero

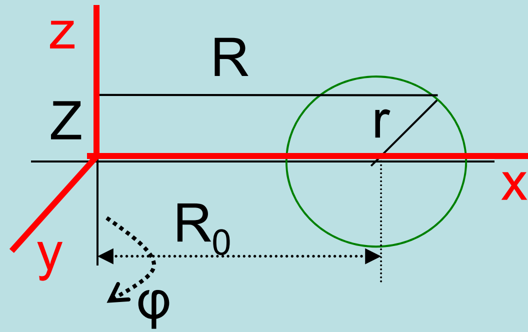
- Physical scenario: threshold dependent local diffusion
(0) define a threshold I_{cr}
(i) if the current is below the threshold, |**I**|<I_{cr}, nothing happens, the plasma is just further driven (*η* = 0 in Eq. (1))
(ii) if the current exceeds the threshold, |**I**|>I_{cr}, local diffusion sets in, according to Eq. (1), i.e. *η* has become finite (anomalous resistivity)

$ this has to be translated to CA rules …



The model (5): instabilities (ii)

- Simplification: we use a simple approximation d**A** to **I**:
  We turn to Cartesian coordinates *x,y,z,*
  and we use a difference scheme approximation for u ^{2}:

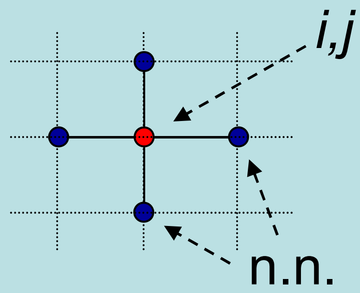


- From **I**^ -u ^{2}**A**, we have e.g. I_y ^ -u ^{2} A_y

$$\begin{aligned}\nabla^2 A_{y;i,j} &\approx (A_{y;i+1,j} - 2A_{y;i,j} + A_{y;i-1,j}) \\ &+ (A_{y;i,j+1} - 2A_{y;i,j} + A_{y;i,j-1})\end{aligned}$$

  we change the factors and sign, and we define $dA_{y,ij}$

$$dA_{y,ij} := A_{y,ij} - \frac{1}{4}\sum_{n.n.} A_{y,n.n.}$$



  The sum is over the four nearest neighbours (n.n.) in the 2D rectangular grid

  $dA_{y,ij}$ is the difference between the central value and the mean of its 4 neighbours.

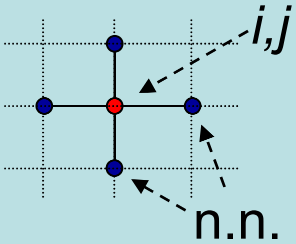
- After all, we use $d\mathbf{A}:=(dA_x,dA_y,dA_z)$ as approximation to the current **I**.
- We consider an instability to occur if
  (as a substitute for |**I**| > I_{cr})

$$\left|d\mathbf{A}_{ij}\right| > A_{cr},$$

- (In future versions, we will use directly **I** in the instability criterion, since it is consistently available)

The model (6): relaxation of the instability

- Physically, the local instability is relaxed since **A** locally diffuses according to the MHD induction equation, $$\frac{\partial \mathbf{A}}{\partial t} = -\eta \frac{c^2}{4\pi}(\nabla^2 \mathbf{A}) \propto \eta \frac{c^2}{4\pi}\mathbf{I}$$
  the diffusion process removes the cause of the diffusion, the current evolves from super-critical to sub-critical:
  $|d\mathbf{A}_{ij}| > A_{cr}$ $\$$ *relaxation (diffusion)* $\$$ $|d\mathbf{A}_{ij}| < A_{cr}$
- We apply the redistribution rules in the local neighbourhood



$$\mathbf{A}_{ij}(t+1) \longrightarrow \mathbf{A}_{ij}(t) - \frac{4}{5} d\mathbf{A}_{ij}$$ (central point)

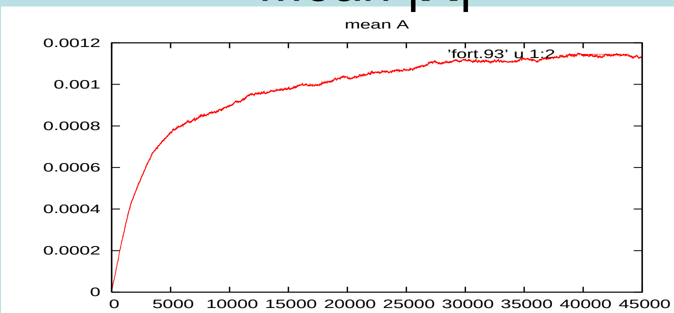
$$\mathbf{A}_{n.n.}(t+1) \longrightarrow \mathbf{A}_{n.n.}(t) + \frac{1}{5} d\mathbf{A}_{y,ij}$$ (4 nearest neighbours)

  which imply that $d\mathbf{A}_{ij}(t+1) = 0$, i.e. the instability is relaxed
- This relaxation actually corresponds to a local flattening of the **A** profile, as it is typical for diffusion (and whereby $\sum_{i,j+n.n.} A_{i,j+n.n.}$ is conserved)
- With the loading, instability criterion, and relaxation process, the CA/SOC model is completely specified
- Free parameter: threshold A_{cr}

Results: Reaching the SOC state

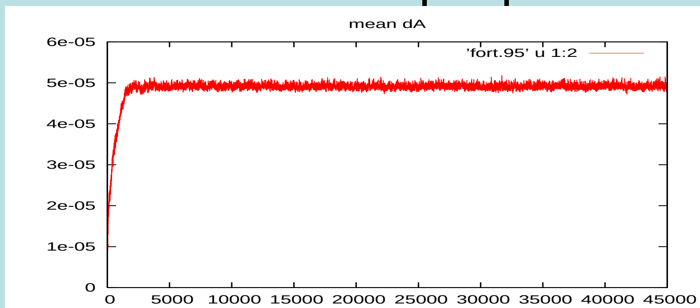
- System starts from initial condition **A**(t=0) = 0 and evolves in time: after a transient, build-up phase, the SOC state as a dynamical equilibrium state is reached
- E.g. number of instabilities (and relaxations) per grid-scan as a function of time, mean |**A**|, mean |d**A**|, total energy in the system ($\sum \mathbf{B}_{i,j}^2$)

Mean |**A**|



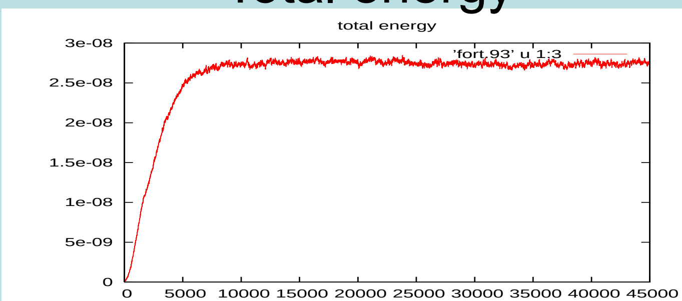
time

Mean |d**A**|



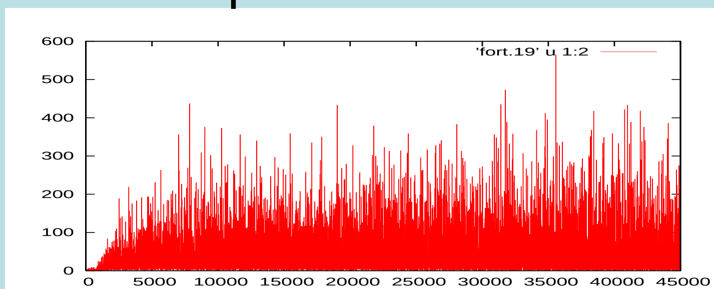
time

Total energy



time

Number of instabilities/relaxations per avalanche

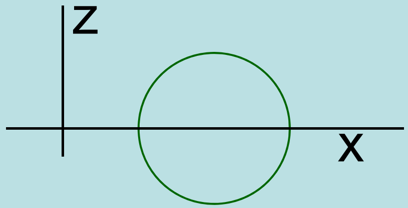


time

Also RFP is known to have transient phase before reaching a relaxed state

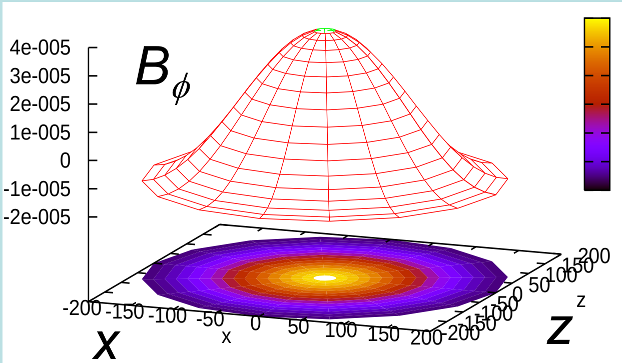
Structure of the magnetic field in SOC state

poloidal x-z-plane

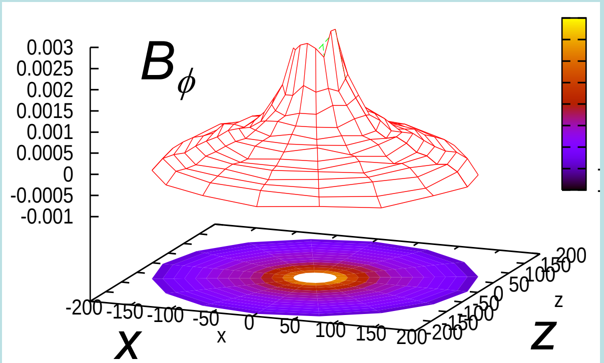


Toroidal field B_φ:

Bessel function model

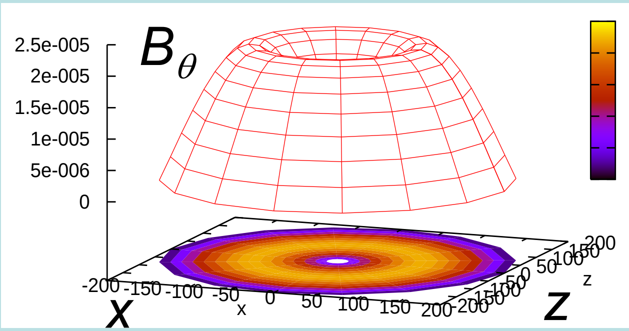


SOC model

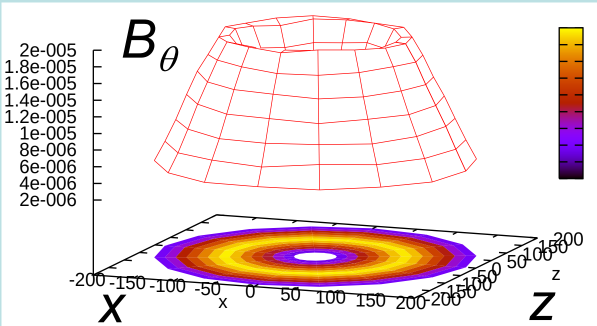


Poloidal field B_θ:

Bessel function model



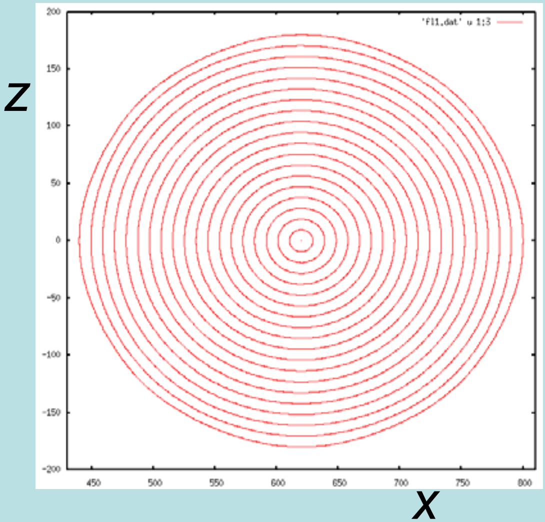
SOC model



- B_φ shows characteristic field reversal at the edges
- B_φ and B_θ are of similar magnitude
- B_ϕ and B_θ stay close to the characteristic shapes, they just slightly fluctuate about them ('noise')
- B_r fluctuating, not zero, but roughly 10 times smaller than B_ϕ

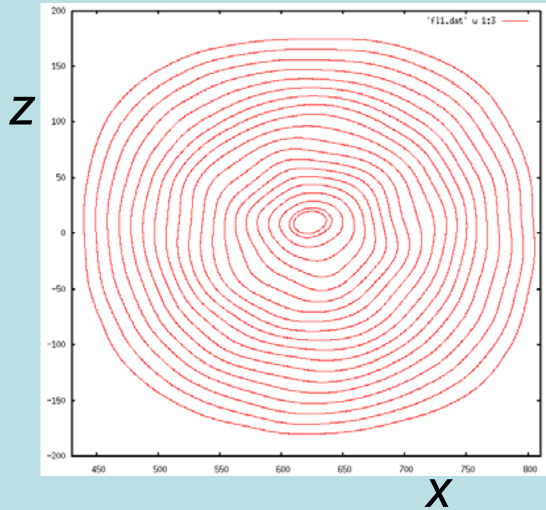
Flux surfaces

Bessel function model



Circular flux surfaces

during SOC (but stable)

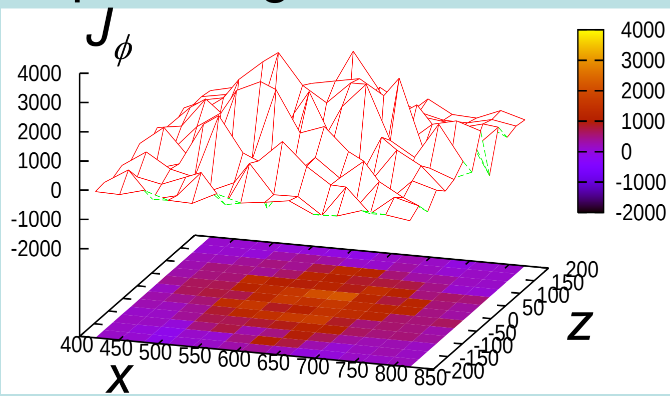


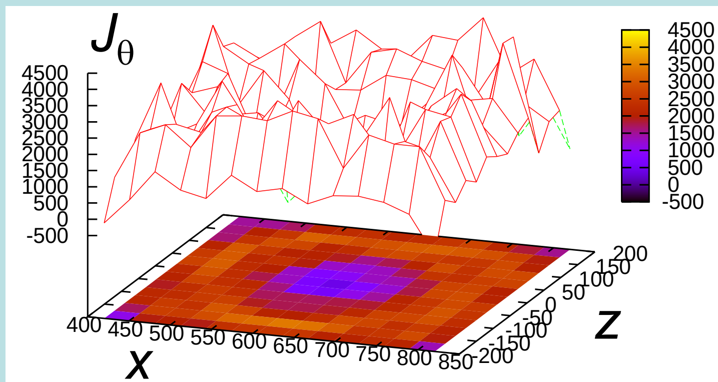
Perturbed, deformed ‘circles’

(rectangular grid !)

Current distributions:

spatial organization, as in the BFM, with strong fluctuations

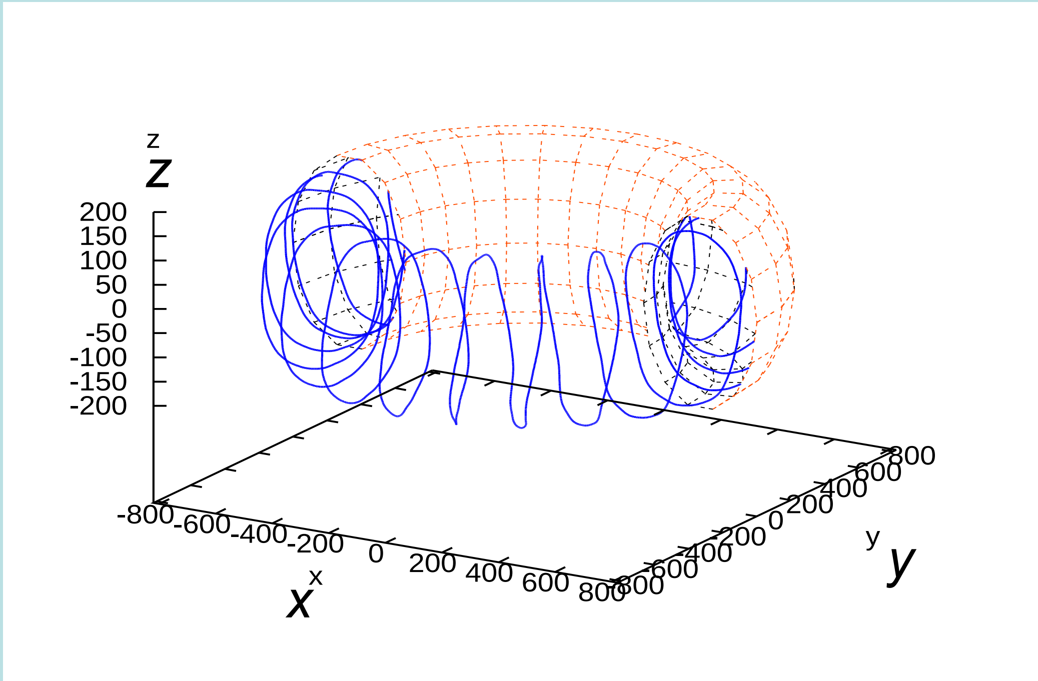




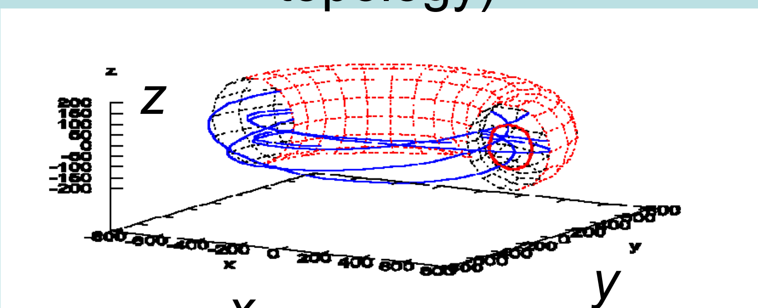
3D field line structures

assuming cylindrical symmetry, field lines can be followed in 3D space:
the field lines are well-behaved, they stay in the torus
(during SOC state, here in stable phase)

→ a 3D, cylindrically symmetric SOC model, where SOC is active in the perpendicular direction



(for comparison: tokamak type topology)



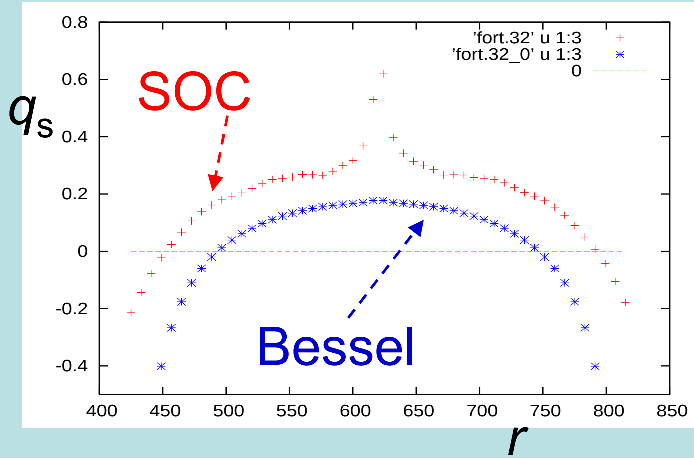
The safety factor profile

- From the evolution equation of the toroidal angle φ of a field line
  we can determine the safety factor q_s as

$$\frac{R\,d\varphi}{ds} = \frac{B_\varphi}{B_p},$$

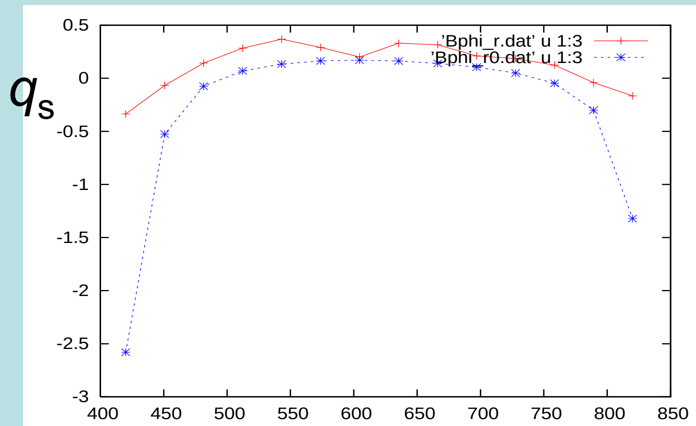
$$q_s = \frac{1}{2\pi}\int_C \frac{1}{R}\frac{B_\varphi}{B_p}\,ds,$$

- Safety factor smaller than 1,
- at the edge it is negative (field reversal)
- close to safety factor of Bessel function model



- Alternatively, assuming the flux-surfaces to be smooth:

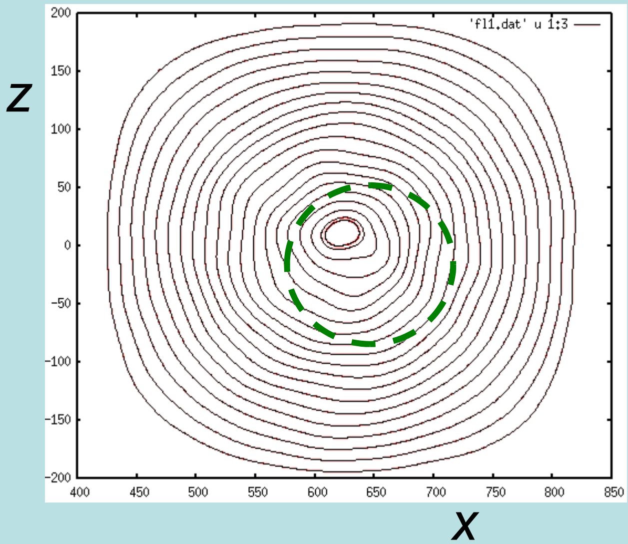
$$q_s = \frac{r}{R}\frac{B_\varphi}{B_p}$$

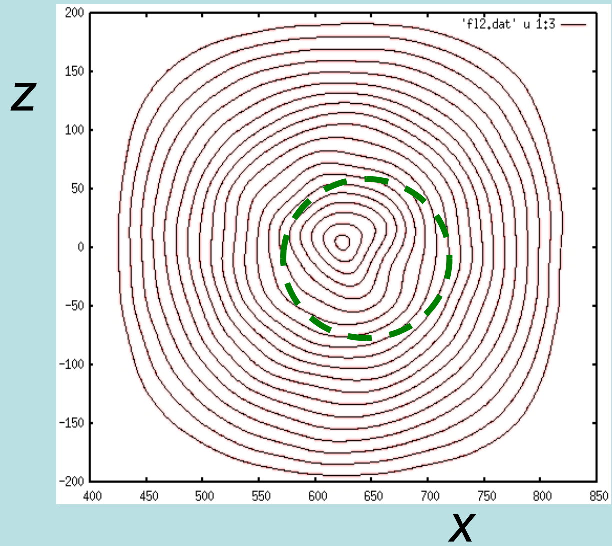


r

Flux surfaces during an avalanche

Instability, before relaxation … and after relaxation





Basically, deformations are smoothed, flattened out, which may lead to secondary deformations in the neighbourhood, i.e. new instabilities → chains of instabilities, as characteristic for SOC

Energy dissipation

- Magnetic energy defined as

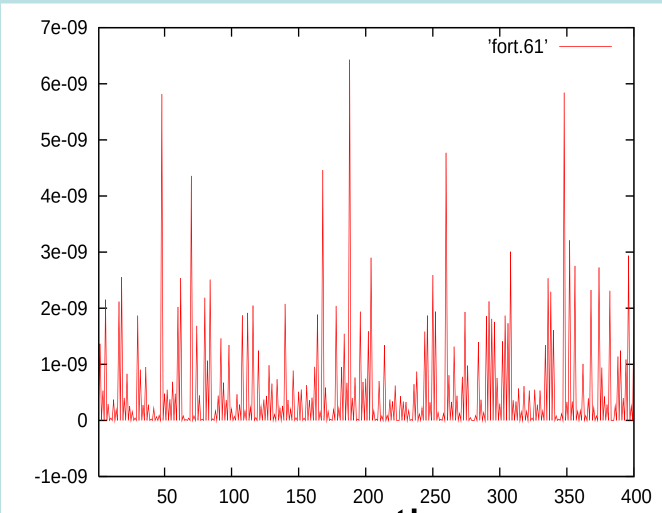
$$E_B = \frac{1}{8\pi}\int B^2\,dV$$

so that the energy dissipated in relaxation events can be calculated as

$$E_d = -\left(\frac{1}{8\pi}\int \mathbf{B(t+1,x)^2\,dV} - \frac{1}{8\pi}\int \mathbf{B(t,x)^2\,dV}\right)$$

(alternative forms in terms of current dissipation ηJ^2 are possible in the model)

Energy per avalanche E_A(t)

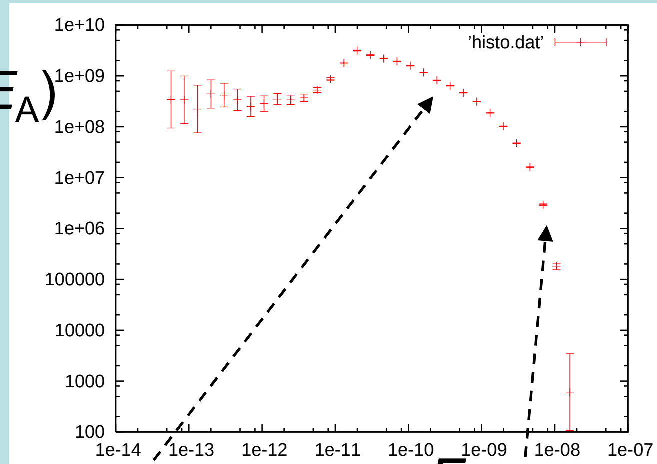


time

intermittent, with large spikes

Distribution of dissipated energy E_A

p(E_A)



E_A

slower than exponential fall-off

exponential decay

Conclusion

- We introduced a magnetic topology in the SOC state for the reversed field pinch (RFP), on the base of a CA, which is compatible with MHD
(in the tokamak the magnetic field is strongly controlled and much less self-organizing)
- the physics it implements is a magnetic field driven by toroidal and poloidal currents, with resistive instabilities that are relaxed in local diffusive events
- The SOC topology is qualitatively in agreement with topologies realized in the reversed field pinch

The model (5): instabilities (ii)

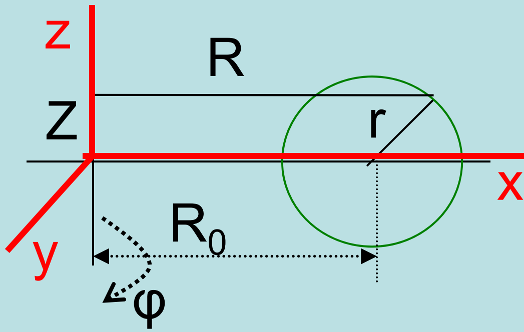
- Simplification: we use a simple approximation d**A** to **I**:
- We turn to Cartesian coordinates *x,y,z.*
  From **I**^ -u ^{2}**A**, we have e.g. I_y ^ -u ^{2} A_y
  and we use a difference scheme approximation for u ^{2} A_y,

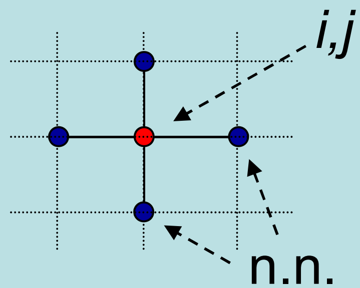
$$\begin{aligned}\nabla^2 A_{y;i,j} &\approx (A_{y;i+1,j} - 2A_{y;i,j} + A_{y;i-1,j}) \\ &+ (A_{y;i,j+1} - 2A_{y;i,j} + A_{y;i,j-1})\end{aligned}$$

  we change the factors and sign, and we define $dA_{y,ij}$

$$dA_{y,ij} := A_{y,ij} - \frac{1}{4}\sum_{n.n.} A_{y,n.n.}$$

  The sum is over the four nearest neighbours (n.n.) in the 2D rectangular grid:
  $dA_{y,ij}$ is the difference between the central value and the mean of its 4 neighbours.
  After all, we use $d\mathbf{A}:=(dA_x,dA_y,dA_z)$ as approximation to the current **I**.





- We consider an instability to occur if
  (as a substitute for |**I**| > I_{cr})

$$\left| d\mathbf{A}_{ij} \right| > A_{cr},$$

- (In future versions, we will use directly **I** in the instability criterion, since it is consistently available)

The model (6): relaxation of the instability

- Physically, the local instability is relaxed since **A** locally diffuses according to the MHD induction equation, $\frac{\partial \mathbf{A}}{\partial t} = -\eta\frac{c^2}{4\pi}(\nabla^2\mathbf{A}) \propto \eta\frac{c^2}{4\pi}\mathbf{I}$

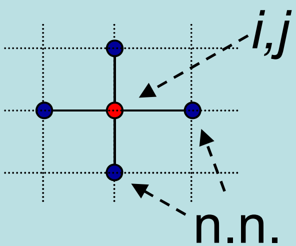
  the diffusion process removes the cause of the diffusion, the current evolves from super-critical to sub-critical:

  $$|\mathbf{I}| > J_{cr} \quad \$ \quad \textit{diffusion} \quad \$ \quad |\mathbf{I}| < J_{cr}$$

- The corresponding CA should mimic this in terms of $d\mathbf{A}_{ij}$, the current substitute in the CA,

  $$|d\mathbf{A}_{ij}| > A_{cr} \quad \$ \quad \textit{relaxation} \quad \$ \quad |d\mathbf{A}_{ij}| < A_{cr}$$

- We apply the redistribution rules in the local neighbourhood



$$\mathbf{A}_{ij}(t+1) \longrightarrow \mathbf{A}_{ij}(t) - \frac{4}{5}d\mathbf{A}_{ij} \qquad \text{(central point)}$$

$$\mathbf{A}_{n.n.}(t+1) \longrightarrow \mathbf{A}_{n.n.}(t) + \frac{1}{5}d\mathbf{A}_{y,ij} \qquad \text{(4 nearest neighbours)}$$

  which imply that $d\mathbf{A}_{ij}(t+1) = 0$, i.e. the instability is relaxed

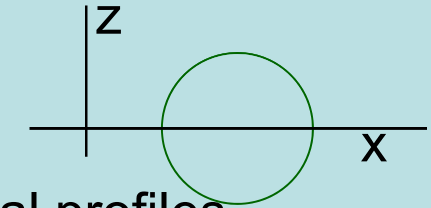
- This relaxation actually corresponds to a local flattening of the **A** profile, as it is typical for diffusion (and whereby $\sum_{i,j+n.n.} A_{i,j+n.n.}$ is conserved)
- With the loading, instability criterion, and relaxation process, the CA/SOC model is completely specified
- Free parameter: threshold A_{cr}

Stiffness of magnetic field profile

- Loading so-far was uniform, everywhere in the poloidal cross-section
- When driving the plasma only with an off-axis current, during the entire run (i.e. from the initial condition), again a SOC state is reached, with magnetic topology that basically is unrelated to the one in the RFP

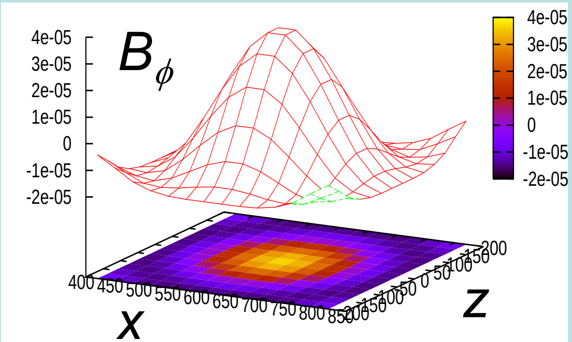
Structure of the magnetic field in SOC state

poloidal *x*-*z*-plane

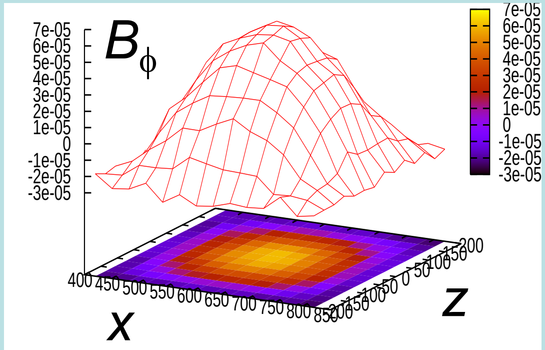


Toroidal field B_φ:

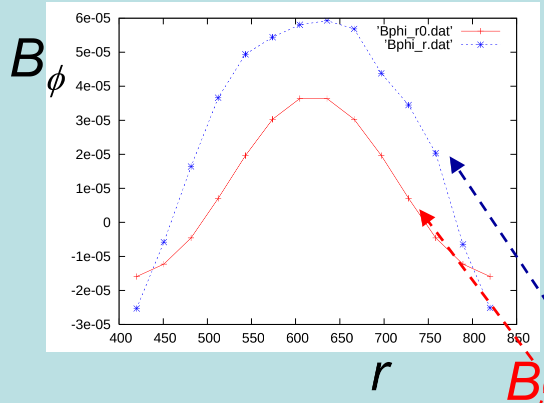
Bessel function model



SOC model



radial profiles

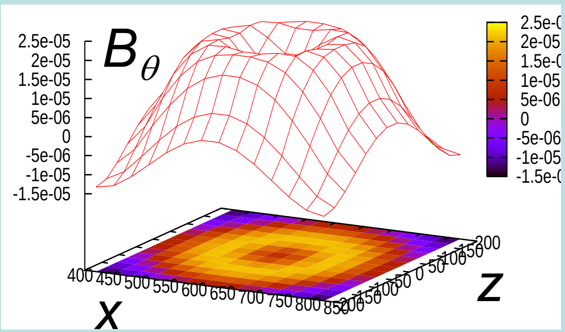


Bessel

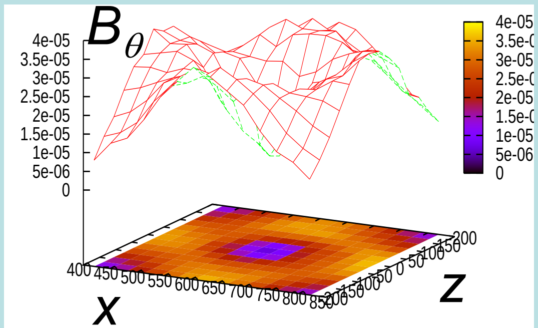
SOC

Poloidal field B_θ:

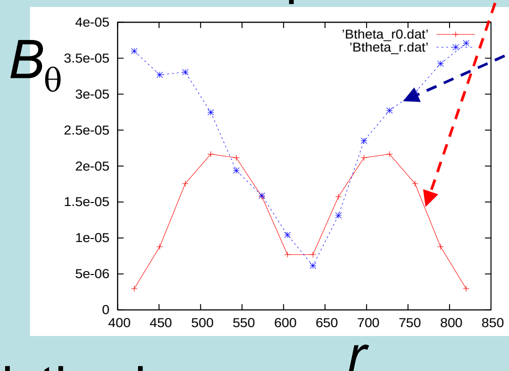
Bessel function model



SOC model



radial profiles



- In SOC state, B_ϕ and B_θ stay close to the characteristic shapes of the BFM, they just slightly fluctuate about them ('noise')
- B_ϕ shows characteristic field reversal at the edges
- B_ϕ and B_θ are of similar magnitude
- B_r fluctuating, not zero, but roughly 10 times smaller than B_ϕ

Flux surfaces during an avalanche