Pulsating relativistic stars

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1. Introduction

Pulsating stars have given important information in classical astronomy and are still an active research field, *e.g. asteroseismology* or *helioseismology*. By studying the rich oscillation stellar spectra, optical astronomy makes predictions for the stellar equations of state (EOS), the evolutionary age of the stars, and in the case of Cepheids one can make exact predictions for the distance of the object.

For relativistic astrophysics the pulsations of relativistic stars are of great importance since through them one can discuss stability properties of compact objects and, moreover, pulsating stars can be a promising source of gravitational waves. It is expected that during the first few seconds which follow a supernova explosion the newly formed neutron star will pulsate wildly and this pulsation will be mainly damped due to the emission of gravitational radiation which will carry away the characteristic signature of the collapsed object. The energy stored in the pulsation will be of the same order as the kinetic energy of the collapse and in this way a significant part of the original mass-energy of the newly formed neutron star will be radiated away as gravitational radiation. With the present sensitivity of the resonant detectors it is quite sure that the gravitational waves will be detected if a supernovae explosion takes place in our Galaxy. Depending on the details of the collapse (Bonazzola & Marck 1994), with the future generation of resonant detectors it is possible that events up to the distance of 1Mpc (*i.e.* our local group of galaxies) be detectable.

The study of stellar pulsations in General Relativity (GR) was initiated nearly thirty years ago by Thorne and his collaborators (Thorne & Campolattaro 1967; Thorne 1968; Price & Thorne 1969; Thorne 1969a, 1969b), and was concentrated mainly in the calculation of the f (fundamental) mode since this is the mode through which most of the gravitational radiation of the star is radiated away. All the other *fluid modes*, g (gravity), p (pressure), s (shear), t (toroidal), i (interface) modes, can be calculated with quite high accuracy with Newtonian dynamics since they don't emit significant amounts of gravitational radiation. This Newtonian like picture has changed significantly in the last ten years due to some new ideas by Kokkotas and Schutz (1986, 1992) and Chandrasekhar and Ferrari (1991a) and also due to subsequent work by Andersson and Kojima. The new approach has been based on the fact that the stellar pulsations in GR should not be treated in a semi-Newtonian manner because then some important features of the theory will be overlooked since they are related uniquely to the relativistic nature of the problem and do not exist in the Newtonian theory. These unique new features are:

- the existence of a new set of modes, the gravitational *w*ave modes (w-modes), which are purely spacetime modes and which cannot be seen in Newtonian theory (Kokkotas & Schutz 1986, 1992; Kojima 1988; Kojima *et al.* 1995; Andersson *et al.* 1995, 1996a, 1996b; Leins *et al.* 1993).
- that the stellar pulsations can be treated as a scattering problem (Chandrasekhar &

† Review talk for the proceedings of the Les Houches School on *Astrophysical Sources of Gravitational Waves* Eds J-A. Marck and J. P. Lasota, Cambridge University Press (1996)

Ferrari 1991a) *i.e.* in the perturbation equations only the variations of the gravitational field will enter while the fluid perturbations will be absent. This approach exists even in the limit of Newtonian stars (Ferrari & Germano 1994; Chandrasekhar & Ferrari 1995).

• that in the slow rotation limit the polar modes excite the axial ones due to the dragging of the inertial frames (Chandrasekhar & Ferrari 1991c; Kojima 1992, 1993a, 1993b)

• that the axial modes which were thought not to exist for non-rotating stars do exist (Chandrasekhar & Ferrari 1991b) and that they show a spectrum similar to the w-modes (Kokkotas 1994, 1995) and that, more or less, they can be considered on an equal footing as the polar modes (Kojima *et al.* 1995; Andersson *et al.* 1996a). Recently, it has been shown (Andersson & Kokkotas 1996) that these modes can be excited and emit as much energy as an oscillating black-hole.

These new developments in the last decade in the theory of relativistic stellar pulsations will be the subject of the following sections. We will not discuss stability since for non-rotating stars there is no significant progress in the last ten years and thus the reader can refer to a review article by Schutz (1987).

2. Fluid modes

The fluid modes are modes that exist in both Newtonian and GR treatment of stellar pulsation and their main characteristics do not depend on the specific gravitational theory. If we consider an unperturbed fluid it will be described in both theories by the mass and momentum conservation equations and an equation which describes the gravitational field (Poisson or Einstein equations). By assuming a small variation from sphericity we can describe all fluid and dynamical quantities in terms of spherical harmonics while a harmonic dependence on time can also be used. In Newtonian theory all the quantities which describe the variations of both the fluid and the potential are scalar quantities and their decomposition in spherical harmonics has only one parity. In GR, the variations of the spacetime metric are tensor quantities and thus their decomposition in spherical harmonics has two parities, *odd* and *even* ones. The *even* parity or *polar* perturbations are similar to the Newtonian ones and describe *spherical* deformations while the *odd* parity or *axial* ones (also called *toroidal*) are degenerate in Newtonian theory. They can set the star into a continuous non-varying differential rotation. Nevertheless, this degeneracy is erased in the presence of rotation, magnetic fields, or non-zero shear modulus. Thus, in principle, in the absence of rotation in both theories these modes are unimportant because they cannot describe time dependent motions and as a result there is no emission of gravitational radiation. We shall discuss these modes in the next section where we shall see that the Newtonian approach to the problem is not complete. The perturbation equations have the general form

$$\mathcal{D}_1 W + F\left(W', Z', W, Z; \ell, \omega\right) = 0 \tag{2.1}$$

$$\mathcal{D}_2 Z + G\left(W', Z', W, Z; \ell, \omega\right) = 0 \tag{2.2}$$

where $W(r)$ is a function of the radial, ξ_r, and tangential, ξ_θ, components of the fluid displacement vector. $Z(r)$ is a function of the components which describe the variation of the gravitational field †. ℓ is the harmonic index (m is not present because we have

† In the Newtonian theory they are the variation of the potential δU and its derivative, while in GR they are components of the perturbed part of the metric, *e.g.* for a certain choice of the gauge, h_{tt} and $h_{\theta\theta}$. In both cases the system of differential equations is of 4th order (Detweiler & Lindblom 1985), in the gauge used in (Chandrasekhar & Ferrari 1991a) the system

a $(2\ell+1)$-fold degeneracy due to the rotational symmetry of the equilibrium structure around any arbitrary axis) and ω is the pulsational frequency. $\mathcal{D}_1$ and $\mathcal{D}_2$ are both wave operators in GR with propagation velocities, the acoustic one for the first and the speed of light for the second equation (in Newtonian theory $\mathcal{D}_2$ is not a wave operator thus we have no emission of gravitational waves). In GR outside the star the equation (2.2) reduces to either Zerilli (polar) or Regge-Wheeler (axial) wave equations (Chandrasekhar 1983) depending on the parity. The equations (2.1) and (2.2) together with a set of appropriate boundary conditions formulate an eigenvalue problem with ω being the eigenvalue and $W(r)$ and $Z(r)$ the corresponding eigenfunctions. The boundary condition for the existence of quasi-normal modes (QNM) is that at infinity there is no incoming gravitation radiation. In this way a discrete spectrum of frequencies for a certain stellar model can be revealed.

The richness of the spectrum depends on the complexity of the stellar model and each new feature that we add to the stellar model results in a new family of modes †. The simplest possible model for a neutron star is a non-rotating fluid ball at zero temperature. If the fluid has constant density this model supports only one pulsation mode for each multipole ℓ. This mode, which in Newtonian theory has oscillation frequency

$$\omega(R^3/M)^{1/2} = \sqrt{2\ell(\ell-1)/(2\ell+1)} \approx 0.894 \quad \text{for } \ell = 2 \ , \tag{2.3}$$

where R and M are the radius and mass of the star in units where $c = G = 1$, was first studied by Kelvin (1863) and is usually referred to as the f-mode (fundamental mode). It is distinguishable by the fact that there are no nodes in the corresponding eigenfunctions inside the star. In a way, the f-mode is due to the interface between the star and its surroundings. The eigenfunctions of such modes would typically have maxima at the interface and fall off away from it. This is exactly the character of the f-mode: It reaches maximum amplitude at the surface of the star (see for example Figures 7–8 in (Kokkotas & Schutz 1992)). A comprehensive set of quadrupole f-mode calculations has been carried out for 13 equations of state by Lindblom and Detweiler (1983). The frequencies are in the range of 1.5-3.5 kHz and the damping times are of the order of 0.1-0.5 secs, *i.e.* the star undergoes a few hundred oscillations before it damps out.

A somewhat more realistic star consists of a perfect fluid. Then one must specify the equation of state, and most studies to date have been for simple polytropes. For this stellar model a second set of modes – the p-modes, the restoring force of which is pressure – exists. The oscillation frequencies of the p-modes depend directly on the travel time for an acoustic wave across the star. This would typically lead to higher oscillation frequencies starting from 5-6 kHz while their damping times are significantly longer than those of the f-modes and increase with the order of the mode.

When the temperature of the star is non-zero a further set of modes comes into play. These modes are mainly restored by gravity and are consequently referred to as g-modes. For a star in convection, *i.e.,* when the entropy is constant, the g-modes are all degenerate at zero frequency. In general, however, their oscillation frequencies depend directly on the central temperature of the star and are typically smaller than a few hundred Hz.

The three families of *fluid* modes discussed so far, the f-, p- and g-modes, all belong to the class of polar modes. For these models there are no *fluid* axial modes, since the stellar models discussed are all somewhat unrealistic. If even more realistic stellar models are

is of 5th order but it has been proven (Ipser & Price 1991) that by a gauge transformation it can always reduced to a 4th order one. A gauge invariant presentation by Moncrief (1974) has never been used up to now for normal mode calculations.

† The naming of the various families of fluid modes comes from a monumental work by Cowling in early '40s (Cowling 1941).

considered, for example it is expected that neutron stars will have a kilometer-thick crust which will crystallize, then, if the shear modulus in the crust is non-zero, axial modes do exist (Schumaker & Thorne 1983; Finn 1990). There will be also families of modes directly associated with the various interfaces inside the star which will not be discussed here and for a detailed discussion the reader should refer to an excellent presentation in McDermott *et al.* (1988).

The maximum pulsation energy is stored in the f-mode in which the fluid parameters undergo the largest changes, typical values of the nonspherical part of the internal kinetic energy of a pulsating neutron star are of the order of $\leq 10^{-3}\mathrm{M}_{\odot}$. If the gravitational radiation is the only damping mechanism of the pulsations (which is approximately true (McDermott *et al.* 1988)) then for the Newtonian ones the emission rate can be calculated by the quadrupole formula (Thorne 1969b; Osaki & Hansen 1973) and the damping time will be just $E/\dot{E}$ (for the analytic formula see (Balbinski & Schutz 1982)). In the GR approach the pulsation frequency is complex and the imaginary part corresponds to the damping of the pulsation. The calculations of the f-mode frequencies for the same mean density stars have shown that both Newtonian and GR predict more or less the same values. This picture is also true for all the other families of fluid modes and thus they can be studied in Newtonian theory (McDermott *et al.* 1988). The previous picture is partially true for the damping rate since when the star is fully relativistic (*e.g.* $M \approx 1.4M_{\odot}$ and $R \approx 10$Km) the Newtonian theory predicts 2-3 times faster damping of the oscillations (Balbinski *et al.* 1985). The post-Newtonian (pN) treatment of the problem reduces this gap significantly (Cutler 1991; Cutler & Lindblom 1992; Kokkotas & Schäfer 1996) and it is a promising approach especially for the study of rotating stars.

As it has been seen by equation (2.3) the f-mode frequency scales with the mean density of the star, while its damping time (due to radiation reaction) depends on the compactness of the star, *i.e.* the more compact the star the faster the oscillations damp out (for a discussion of the damping times including viscosity see Cutler *et al.* (1990)). These two properties of the f-mode frequency are useful for the estimation of the masses and radii of pulsating neutron stars.

We shall conclude this section by mentioning a very useful approximation for the study of the fluid modes, the so called *Cowling Approximation* in which the perturbations of the potential (or the metric) are set to zero and thus the whole problem is described with just one wave equation for the fluid. In this approximation both frequencies and damping times can be revealed with an error usually less than 10% (Robe 1968; McDermott *et al.* 1983; Finn 1988). An additional advantage is that the simple wave equation can be studied even analytically in the WKB approximation and the properties of the various families of modes described above can be visualized in an elegant way (Unno *et al.* 1989; Gautschy & Saio 1995).

3. Spacetime or w-modes

As it has been shown in the previous section the treatments of stellar pulsations in Newtonian theory and in GR do not differ significantly. Actually, when the pN correction was considered even the gap in the estimation of the damping times was considerably reduced. Thus the natural question is what is actually the rôle of equation (2.2) being a wave one in GR? Is it just for the direct calculation of the damping times directly from the solution of the eigenvalue problem and not afterwards from the ratio $E/\dot{E}$? The answer is clearly no. Spacetime has it own dynamics and does not play a passive rôle as just the medium in which gravitational waves propagate; moreover, it has its own unique spectrum.

Let us start by assuming a simplified version of the problem. We shall consider the inverse of the Cowling Approximation (ICA) (Andersson *et al.* 1996b) which has been discussed earlier, but now instead of *freezing* the spacetime perturbations we shall *freeze* the fluid ones. Then the whole problem is described by just one second order wave equation both inside and outside the star (as for the axial perturbations), which together with the appropriate boundary conditions (regularity of the perturbed quantities in the center and only outgoing waves at infinity) forms a well posed eigenvalue problem. The spectrum should have, in principle, a lot of similarities with that of QNMs of black holes (Andersson 1992), but since the boundary conditions at the center of the star are different than those at the horizon for a black-hole, the two spectra should not be identical. Nevertheless, the characteristic times of the perturbation described by the wave equation in the ICA will be proportional to the wave speed which in this case is the speed of gravitational waves and thus the perturbations should be of higher frequency and they should be damped out very fast (we remind you that for the fluid pulsation the characteristic time was related to the speed of the acoustic waves). In this way we reveal a new family of modes which is related directly to the spacetime and which can never be seen through the Newtonian approach to the problem.

It took us nearly 20 years (from the original work by Thorne) to recognize this natural GR extension of the stellar pulsation spectrum. This approach was initiated ten years ago when Kokkotas and Schutz (1986) studied the QNMs of a simple "toy" model consisting of two strings, one finite and one semi-infinite coupled together with a spring; this system mimics very well the star-spacetime system. They found that there exist two families of modes, one with slow damping (like the fluid modes) which were mainly modes of the finite string (star) and another family with very fast damping which was mainly modes of the semi-infinite string (spacetime). The damping time of the QNMs of this second family becomes shorter as the coupling (the spacetime curvature) between the two strings becomes looser. Other such instructive "toy" models can now easily be constructed and all of them suggest the existence of this second family of modes (Kokkotas 1985; Baumgarte & Schmidt 1993; Andersson 1996).

A few years later this new family of spacetime modes was found by the same authors (Kokkotas & Schutz 1992) for stars. They named these new modes "gravitational *w*ave" modes or w-modes †. The characteristic properties of these modes, as was expected by the previous analysis and "toy" model, were: (a) *high frequencies* (8-12 kHz), (b) *fast damping times* (0.02-0.1 msecs) which decrease as the order of the mode and/or with the compactness of the star (in accordance with the "toy" model (Kokkotas & Schutz 1986)), and (c) practically *no significant fluid motions* inside the star.

The w-modes exist for both polar and axial perturbations since they do not depend on the perturbations of the fluid, actually in the black-hole perturbations the spectra in the two cases coincide. Nevertheless, for stars the boundary conditions near the center are different than those for black-holes (as well as the form of the potential inside the star) and naturally we should not expect coincidence, but instead a similarity of the axial and polar w-mode spectra as it has been shown in (Kojima *et al.* 1995; Andersson *et al.* 1996a, 1996b), see also Figure 1. So in what follows we shall not discriminate between the axial and polar perturbations.

A way to understand the nature of these modes is by the "trapping" of gravitational

† It should be mentioned that Kojima (1988) had earlier found a few of the w-modes but he had not revealed the whole spectrum and its properties.

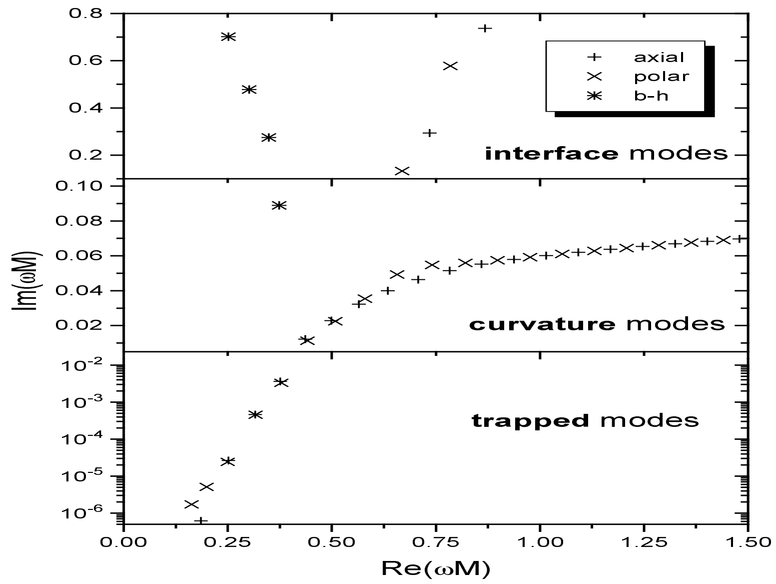

FIGURE 1. A graph which shows all the w-modes: *curvature*, *trapped* and *interface* both for axial and polar perturbations for a very compact uniform density star with $M/R = 0.44$. The black-hole spectrum is also drawn for comparison. As the star becomes less compact the number of *trapped* modes decreases and for a typical neutron star ($M/R = 0.2$) they disappear. The $Im(\omega) = 1/damping$ of the *curvature* modes increases with decreasing compactness and for a typical neutron star the first curvature mode nearly coincides with the fundamental black-hole mode. The behaviour of the *interface* modes slightly changes with varying compactness (cf. Figure 3). The similarity of the axial and polar spectra is apparent.

waves by the spacetime curvature inside the star. It is easy to see how this may happen if one plots the gravitational-wave speed as measured by an observer at infinity: g_{tt} as a function of r. This has a minimum at the center of the star, and the interior w-modes would be trapped in this "bowl" of curvature. Moreover, such modes would naturally be concentrated at the center of the star, which agrees well with the eigenfunctions constructed in Leins *et al.* (1993) (see also an extensive discussion in Andersson *et al.* (1996a)). Hence, it makes sense to refer to these modes as "*curvature modes*".

A naive but useful argument leads to standing wave solutions inside the star, essentially of the form $\sin(\omega r)$. If this were the true form of the solutions, and the modes only leak out slowly through the surface, we would get

$$\omega_n R = n\pi \quad , \quad n = 1, 2, ... \tag{3.4}$$

where R is the stellar radius. This argument is far too simplistic, but it is rewarding to find that two of its predictions agree well with the results for the curvature modes: There would exist an infinite sequence of such modes, and their pulsation frequencies $\mathrm{Re}(\omega_n R)$ would be separated by π (cf. Figure 5 in Andersson *et al.* (1996a)). This dependence on the size of the star was also predicted by the simple "toy" model in Kokkotas & Schutz (1986).

When the star is made increasingly compact, a few of the *curvature* modes clearly change character and should be considered as trapped in the potential well that arises inside the black-hole potential barrier ($R < 3M$). In this regime even the fluid f-mode should also be considered as a trapped mode and share the properties of the w-modes (cf. Figure 1 in Andersson *et al.* (1996a)). It will be instructive to name all the modes which are trapped because of the potential well as *trapped* spacetime modes. Their number is finite and depends on the depth of the potential well. The existence of trapped modes was first discussed by Detweiler (1975) but interest in them has been revived with the work of Chandrasekhar and Ferrari (1991b) for the axial stellar perturbations. The existence of *trapped* modes for both polar and axial perturbations has been only recently shown (Kojima *et al.* 1995; Andersson *et al.* 1996a, 1996b) but they are unlikely to be of any great astrophysical relevance, since such compact stars will probably never form.

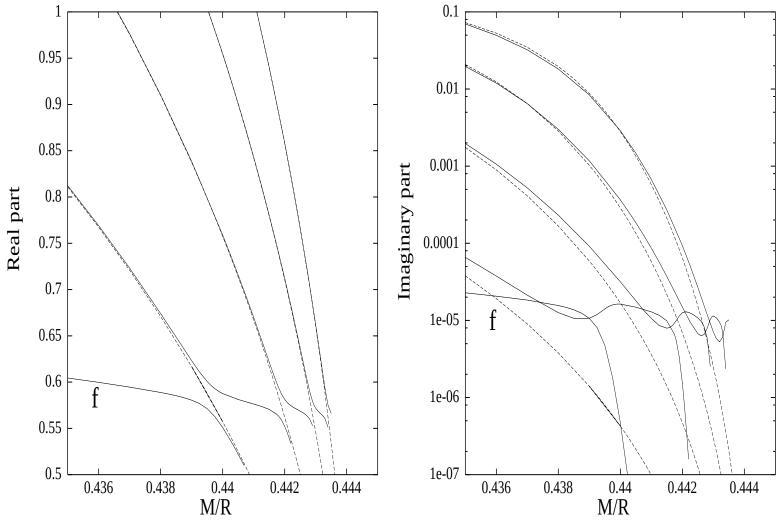

FIGURE 2. (left) Re $\omega(R^3/M)^{1/2}$ and (right) Im $\omega(R^3/M)^{1/2}$ as functions of the compactness of the star for the f-mode (solid and denoted by f), and the first few w-modes for polar (solid) and axial (dashed) perturbations. Notice the beautiful example of avoided crossings in the pulsation frequencies.

The other family of w-modes discovered by Leins *et al.* (1993) was not predicted before by any "toy" model, and it is characterized by extremely fast damping and quite low frequencies; they named them w_{II}-modes. The existence of this family of w-modes may be more directly due to the discontinuity at the surface of the star. Then the mode-eigenfunctions need not be localized in the star. Rather, these modes would be similar to modes for acoustic waves scattered of a hard sphere. One would typically expect such modes to be short-lived compared to modes trapped inside the star. Only a finite number of modes exist for each multipole ℓ, and there may be purely imaginary ones. The latter feature suggests that one should perhaps not be surprised to find stellar modes "emerge" from points on the imaginary-frequency axis as in Figure 3 here. This evidence seems to be compelling and makes the association of the second family of w-modes and the interface at the stellar surface likely. Hence, it will be instructive to call these modes *interface modes*. In principle, the conjecture that only a finite number of these modes exist should be testable. But at present numerical difficulties restrict calculations to $\mathrm{Im}(\omega M) < 1.25$ or so. Much better numerical schemes, or other formulations of the problem, are required to test this prediction.

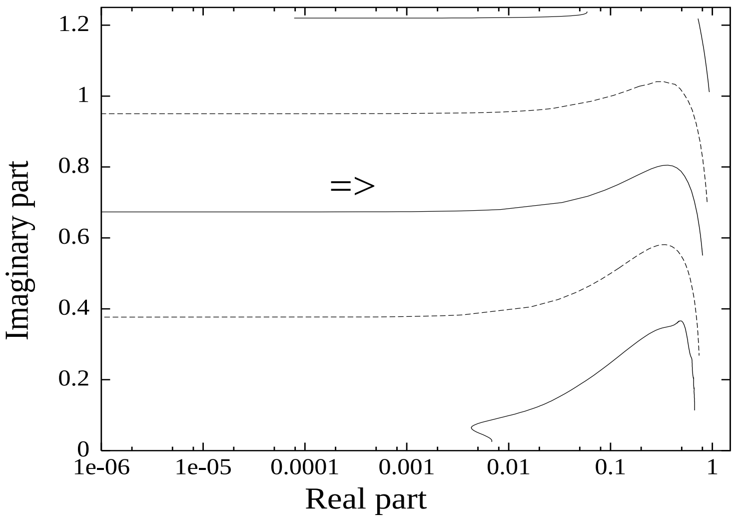

FIGURE 3. Im(ωM) vs Re(ωM) for the polar (solid) and axial (dashed) interface modes. Only the first of these modes exist for all values of stellar compactness. Further modes arise for sufficiently compact stars. The numerical calculation becomes difficult for large Im(ωM) and consequently we have only partial data for the third of the polar modes modes. The arrow indicate the direction of increasing compactness.

Finally, we shall discuss some of the new results induced by the slow rotation of neutron stars. As we have mentioned earlier the eigenfrequencies are degenerate with respect to m at fixed ℓ. In the presence of rotation, however, the degeneracy with respect to m is removed and different modes are mixed with each other. The frequency and damping times of the co-rotating f-mode increase with the stellar angular velocity while those of the counter-rotating mode decrease (Kojima 1993b). Moreover, the axial mode with ℓ,m is coupled with the polar mode with $\ell \pm 1$, m and vice versa (Chandrasekhar & Ferrari 1991c; Kojima 1992, 1993a). This results from the dragging of the inertial frames (Lense-Thirring effect) by the rotation of the star.

4. Excitation and parameter estimation

In the previous sections we have seen all the new developments in our understanding of the role of the spacetime for pulsating stars. Nevertheless, many important questions remain, and the most important of all of these is whether the w-modes can contribute to observable gravitational waves, and thus play a role in astrophysics. To test this, Andersson and Kokkotas (1996) have studied scattering of axial gravitational wave-packets by a compact star. The problem is similar to the black hole one that was studied by Vishveshwara in the early '70s (Vishveshwara 1970). The axial problem corresponds to a single wave equation with an effective potential. It is remarkable that what must be considered a basic exercise in numerical analysis can provide us with new and important information.

The result of a typical simulation is shown in Figure 4. The exponential ring-down at late times (from $t \approx 100M$) corresponds to the first axial w-mode (Figure 4B-D). A theoretical waveform based on the slowest damped axial w-mode for this star fits the numerical one perfectly. After the QNM ring-down (at very late times) the waveform is dominated by a power-law tail. This is exactly what one would expect (Ching *et al.* 1995).

It is important to be mentioned that the last part of the signal would be the same if the star was replaced by a black hole of the same mass.

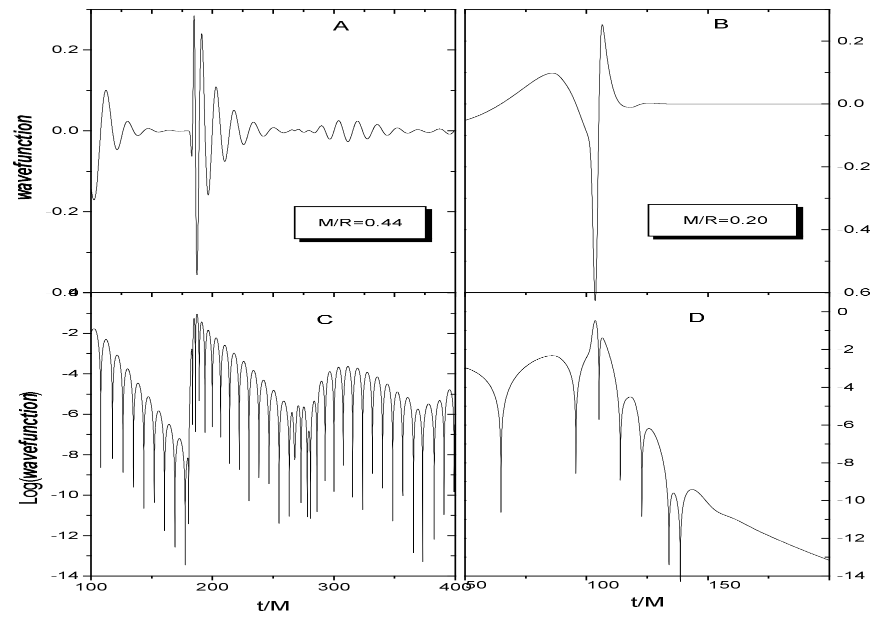

FIGURE 4. The response of two uniform density stars ($M/R = 0.44$ left and $M/R = 0.20$ right) to a Gaussian pulse of gravitational waves. The top shows the actual form of the axial perturbation as seen by a distant observer, while the lower panel shows the same function on a logarithmic scale. The QNM ringing that is apparent after $t \approx 100M$ (right) corresponds to the first axial *curvature* w-mode, while there are no *trapped* w-modes. The power-law tail dominates after $t \approx 150M$. In the left picture we can see a series of QNMs which correspond both to *curvature* and *trapped* w-modes.

Similar simulations for ultracompact stars ($R < 3M$) have been also performed (Figure 4A-C), for these there will be a few *trapped* modes. In this case we generally find that the first mode "above the peak" [with $(\omega M)^2 > V_{\max}$, this is not a *trapped* mode] dominates the radiation initially. Most of the energy goes into this mode, but since it is more rapidly damped than the trapped ones, the latter dominate the radiation at later times. But as we mentioned earlier the ultracompact stars are interesting only from a theoretical point of view: we can learn little astrophysics from them †. An extension of the above work would be to study the excitation of both fluid and spacetime modes (Kind *et al.* 1993), *i.e.* polar perturbations, there one can exactly determine the amount of energy radiated due to fluid (f, p-modes) and spacetime modes (Allen *et al.* 1996).

† Recently, Borelli and Ferrari (1995) came to similar results about the excitation of the *trapped* axial modes by a mass falling onto a compact star. Their method is similar to the one employed by Kojima (1987) for the study of stellar resonances due to orbiting particles.

Simulations suggest that the w-modes will be relevant in many dynamical processes involving neutron stars. The modes ought to be excited when a stellar core collapses to form a neutron star. Much of the initial deformation of spacetime could then be radiated away in terms of w-modes. The modes should also be excited when two neutron stars collide or at the final stages of binary neutron star coalescence.

It seems clear that the w-modes can play a rôle in many astrophysical scenarios, but will we be able to observe them with future gravitational-wave detectors? This is a key question, the answer of which demands much more detailed calculations than the ones presented here and needs 3D fully GR codes for gravitational collapse or fully GR evolution codes for the binary neutron star coalescence. But let us nevertheless suggest a handwaving argument: a spectral analysis (see Figure 5) shows that the w-modes will be excited to roughly the same level as the modes of a black hole would be in a similar situation. The black-hole modes are expected to be detectable (Finn 1992), so the situation looks promising also in the case of stars.

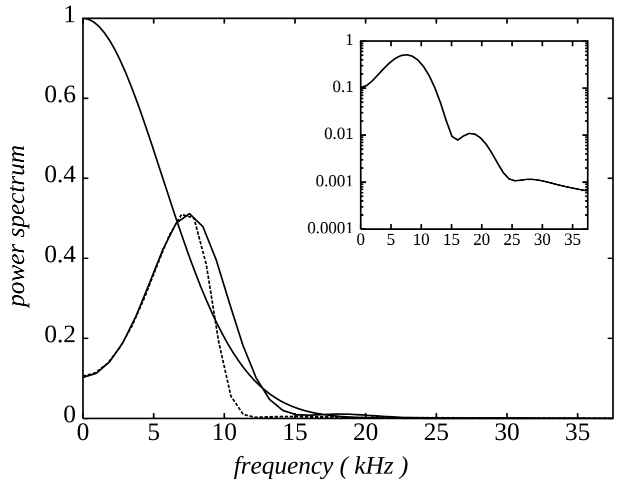

FIGURE 5. The power spectrum for the simulation in Figure 4B-D. The axial power spectrum (solid line) is compared to the corresponding one for a black hole (dashed line). We also show the power in the initial Gaussian pulse (another solid line) that has been used to normalize the other spectra. Note that the star mode is excited to roughly the same level as the black-hole one. A significant part of the initial energy clearly goes into quasinormal-mode ringing. The insertion shows the axial power spectrum in logarithmic scale. From this it is clear that the first three w-modes are excited (the corresponding frequencies are $\omega M \approx 0.35$, 0.82 and 1.25 which corresponds to $\omega \approx 8.36$, 19.6 and 29.8 kHz)

In fact, the frequencies of the f-modes (around 1-2 kHz) make them well suited for detection by the operating resonant bar detectors and the recently proposed spherical solid-mass detectors, *e.g.* TIGA (Johnson & Merkowitz 1993; Merkowitz & Johnson 1995). The detection of the w-modes (and even of p-modes) which have frequencies of the order of 8-12 kHz will be possible by the proposed arrays of small resonant detectors (Frasca 1995).

Suppose that we detect a gravitational-wave signal from a compact star, what can we hope to learn from it? In what follows we shall suggest a possible scheme that looks very promising. The idea is the following: Assume that we detect a signal and manage to extract both the fundamental polar w-mode and the fluid f-mode from it. The discussion in the previous two sections suggests that:

• the oscillation frequency of the f-mode scales with the mean density of the star $\sqrt{M/R^3}$ (cf. Figure 6-I),

• the damping rate of the w-mode scales with the compactness ratio of the star M/R (Kokkotas & Schutz 1992) (cf. Figure 6-IV).

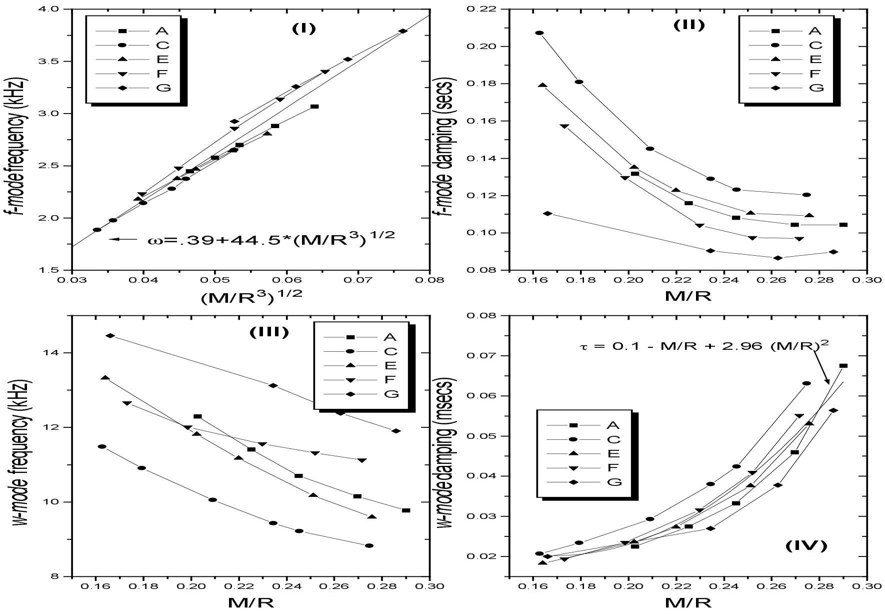

FIGURE 6. (I) The pulsation frequency of the f-mode as a function of the mean density of the star, (II) the damping of the same f-mode as function of the stellar compactness, (III) the frequency of the slowest damped w-mode as function of the stellar compactness and (IV) the damping rate of the same w-mode as a function of the stellar compactness. The symbols stand for the corresponding EOS *i.e.* A for Pandharipandi (1971), C for Bethe and Johnson (1974), E for Moszkowski (1974), F for Arponen (1972) and G for Canuto and Chitre (1974).

In principle, one should therefore be able to infer both the mass and the radius of the star from the observed data. Additionally, the information from the damping time of the f-mode (Figure 6-II) and the frequency of a w-mode (Figure 6-III) will pose strict constraints on the possible neutron star EOS. We are not aware of any other scheme that would enable us to extract such precise information, and put as strong constraints on the nuclear equation of state. Moreover, if the star is rotating the spectrum changes and the degree of deviation from the non-rotational case could help us to estimate the rotation rate of the star just as for black-holes (Finn 1992).

The idea seems simple enough, but will it be useful in practice? We shall test it for a number of different neutron star models with realistic EOS from those listed in Lindblom

TABLE 1. Test of the estimation of parameters hypothesis on polytropic models with N=0.8, 1 and 1.2, the values in parentheses are the estimated values from the relations (4.5) and (4.6).

N	R (Km)	$M/M_\odot$	M/R	ω_f (kHz)	τ_w (msecs)
0.8	10.21 (10.54)	0.92 (0.97)	0.133 (0.135)	1.933	0.0152
0.8	9.49 (9.61)	1.36 (1.35)	0.211 (0.207)	2.532	0.0238
1	8.86 (8.35)	1.27 (1.16)	0.206 (0.211)	2.870	0.0236
1	7.42 (7.28)	1.35 (1.31)	0.266 (0.269)	3.668	0.0493
1.2	10.48 (9.89)	1.44 (1.48)	0.203 (0.221)	2.540	0.0276
1.2	8.97 (9.45)	1.46 (1.45)	0.240 (0.254)	3.128	0.0422

& Detweiler (1983) for which we have determined the f-mode and the slowest damped polar w-modes. The relevant data are graphed in Figure 6. The theoretical spectra are well fitted by the following two relations: The oscillation frequency varies with the average density of the star as

$$\omega(kHz) = 0.39 + 44.45 \left(\frac{M}{R^3}\right)^{1/2} \tag{4.5}$$

while the damping rate of the w-mode behaves as

$$\tau(msec) = 0.1 - \frac{M}{R} + 2.96 \left(\frac{M}{R}\right)^2 \tag{4.6}$$

(M, R in kms). If the two relations (4.5) and (4.6) provide reasonably accurate estimates for both M and R for all stars in our dataset, then this idea passes a first test. The procedure passes this simple test with flying colors: the mass and the radius of each star can be determined to within 5% by inverting (4.5) and (4.6). The results of such a test are shown in Table 1, where we have tested whether the above relations can estimate the masses and radii of various polytropes.

The evidence provided here is, of course, only an indication that this idea can be useful in a real detection situation. To investigate this in more detail one must incorporate the estimated effect of statistical and measurement errors. It will, for example, be much more difficult to infer the w-mode damping rate from a data set than to find the f-mode pulsation frequency (the observability of a periodic signal buried in noise scales roughly as the square-root of the number of observed cycles), and also to obtain fits similar to (4.5) and (4.6) for all known realistic EOS. If such relations prove as robust as the ones we have obtained here then the suggested scheme looks truly promising. In principle it will be possible even to identify exotic types of stars like boson stars which also show a w-mode spectrum (Yoshida *et al.* 1994).

5. Conclusions

In this review we have shown that the study of stellar pulsations as sources of gravitational waves should be performed in GR since in Newtonian gravity apart from deviations in the strong field regime, a lot of useful physical information is overlooked.

The existence of new families of modes (not seen in Newtonian theory), which are not just another branch of the possible mathematical solutions of the eigenvalue problem but instead are modes which can be excited and detected, enrich the information that we can get from the signals of pulsating stars.

The assertions presented here must be tested by more detailed, fully general relativistic simulations. This is, in fact, an interesting challenge for numerical relativity. A relativistic description of gravitational collapse to form a neutron star, or the merger of two stars, should tell us whether the w-modes are of observational relevance or not. Given the present importance for TIGA, fully relativistic simulations are urgently required.

Moreover, there is still a lot of work to be done on the understanding of stellar pulsations. For example, we have not discussed the rotating case, which in GR is tractable only in the limit of slow rotation. Thus a natural question is: do the w-modes induce new instabilities or enhance the classical gravitational radiation instability for fast rotating stars? The answer to this question is of great importance for the gravitational wave astronomy and, due to the lack of full GR rotating star solutions, one should try to attack the problem via the pN approach (Schäfer 1990).

I acknowledge the helpful suggestions and stimulating discussions with Gerhard Schäfer which have improved the form of this article. The collaboration with Bernard Schutz, Nils Andersson, Yasafumi Kojima and Gabrielle Allen has greatly enhanced my understanding of this field. This work was supported by an exchange programme from the British Council and the Greek GSRT and also by a visiting fellowship from the German DAAD. A visit at the Max-Planck Research-Unit for Gravitational Theory at the University of Jena during which this work was completed is gratefully acknowledged.

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