Gravitational waves and pulsating stars: What can we learn from future observations?

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We present new results for pulsating stars in general relativity. First we show that the so-called gravitational-wave modes of a neutron star can be excited when a gravitational wave impinges on the star. Numerical simulations suggest that the modes may be astrophysically relevant, and we discuss whether they will be observable with future gravitational-wave detectors. We also discuss how such observations could lead to estimates of both the radius and the mass of a neutron star, and thus put constraints on the nuclear equation of state.

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Pulsating stars have been studied within Einstein's general theory of relativity for thirty years. Hence it is somewhat surprising that recent results have altered our understanding of these problems considerably. We now know that there are oscillation modes that are directly associated with the curvature of spacetime. But it is not yet clear whether these modes are of any relevance to astrophysics, or if they are simply a curiosity of the mathematical theory. In order to demonstrate their relevance one must show that the modes are excited in a realistic astrophysical situation, such as gravitational collapse to form a neutron star. Unfortunately, we do not yet have the ability to perform such calculations in a relativistic framework. While waiting for full 3D numerical relativity to provide a definitive statement regarding the relevance of the gravitational-wave modes of a compact star, we must settle for suggestive calculations. The purpose of this letter is to describe one such calculation: We consider the scattering of gravitational-wave packets off a uniform density star. In this case we find that the gravitational-wave modes are indeed excited.

Recent work on the pulsation properties of compact stars has shown that general relativity plays an important role: A relativistic star exhibits two distinct sets of pulsation modes. One is slowly damped and corresponds to the well-known fluid modes [1–3], while the second is rapidly damped and has no analogue in Newtonian theory. The new modes have been termed w-modes because they are closely associated with gravitational waves [4–7]. We have recently managed to put the w-modes in context and provide a better understanding of the physics involved [8–11].

The equations describing perturbed nonrotating stars in general relativity split into two classes [2]. Polar perturbations correspond to zonal compressions of the star, whereas axial ones induce differential rotation in the fluid. In general, the polar problem corresponds to two coupled wave equations: one that represents the fluid motion and one for the gravitational waves [12,13]. The fluid pulsation modes (eg. the f-mode) belong to the polar class of perturbations. Meanwhile, axial perturbations are described by a single, homogeneous wave equation [2]. The axial problem is thus considerably simpler. Nevertheless, it attained very little attention until recently. This is mainly because an axial perturbation can only induce pulsations in the stellar fluid through a non-zero shear modulus [15], or through coupling to the polar perturbations if the star is rotating [14]. However, the axial problem is interesting also for the simplest stellar models: If the star can be made ultracompact ($R < 3M$ in geometrical units $c = G = 1$) the peak of the exterior curvature potential barrier (that is familiar from black-hole perturbation theory) will be unveiled. Then gravitational waves that impinge on the star can be trapped. That such trapped modes exist has been demonstrated [16,17], but they are unlikely to be of any great astrophysical relevance: A sufficiently compact star will probably never form.

More importantly, one can argue that the axial problem is relevant also for less compact stars (the canonical values $R = 10$ km and $M = 1.4M_\odot$ for neutron stars lead to $M/R \approx 0.2$), especially as far as gravitational waves are concerned. For black holes, when the gravitational waves are the only active agent, the axial problem leads to a pulsation spectrum that is identical to the polar one [18]. Knowing this, the results of a detailed mode-survey for uniform density stars do not come as a complete surprise: The axial and the polar w-mode spectra are remarkably similar [9]. This is an important result because of its implications for physical interpretations of the w-modes. Since the axial modes do not couple to the stellar fluid one cannot invoke the fluid in an explanation of them. The w-modes must arise because gravitational waves can be temporarily trapped in the "bowl of spacetime curvature" provided by the mass of the star [9].

Are the w-modes excited?. — Our understanding of the role of a dynamic spacetime for pulsating stars has been improved, but the most important question remains. As yet it has not been established that the w-modes can contribute to observable gravitational waves, and thus play a role in astrophysics. The modes should be excited when a neutron star is formed through gravitational collapse, or when two neutron stars merge at the final stage of a binary systems evolution, but how much energy goes into the pulsation? At the present time we cannot answer such questions: To perform the required calculations in a relativistic framework is simply beyond our means.

As a first step towards understanding the issue we have studied scattering of axial gravitational wave-packets by a compact star. Since axial perturbations are governed by a single wave equation with an effective potential [2,16], this problem is in many respects identical to that for black holes [19].

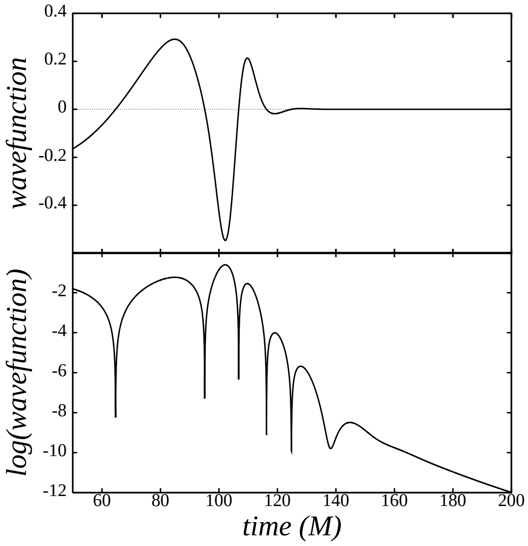

FIG. 1. The response of a uniform density star ($M/R = 0.2$) to a Gaussian pulse of axial gravitational waves. The top panel shows the axial perturbation as seen by a distant observer, while the lower panel shows the same function on a logarithmic scale.

The result of a typical simulation is shown in Figure 1. The exponential ringdown at late times (from $t \approx 100M$) corresponds to the first axial w-mode. A theoretical waveform based on the slowest damped axial mode for this star fits the numerical result perfectly. Moreover, the power spectrum for the data in Figure 1 shows that the first three axial modes are excited. A significant part of the initial energy clearly goes into quasinormal-mode ringing. After the normal-mode ringdown (at very late times) the waveform is dominated by a power-law tail [21]. This last part of the signal would be identical if the star was replaced by a black hole of the same mass. To study the relation between the fluid modes and the w-modes we must turn to the polar problem. Preliminary studies [22] indicate that i) a polar perturbation typically excites both the fluid modes and the w-modes and ii) a considerable amount of energy is released through the w-modes.

Our simulations suggest that the w-modes may be relevant in many dynamical processes involving neutron stars. We believe that the modes ought to be excited when a stellar core collapses to form a neutron star. Most of the initial deformation of spacetime could then be radiated away in terms of w-modes. For the mode-excitation to be considerable the collapse must, of course, be asymmetric. To estimate the asymmetries in a realistic supernova is difficult, but the evidence is that the average velocity of radio pulsars is large. Asymmetries in the core collapse would explain such high velocities [23]. Moreover, since it is the asymmetries of the spacetime that are relevant for the excitation of the w-modes the situation for these modes is not too different from that for black holes. It is well-known that both axial and polar quasinormal modes are generic features of collapses that form a black hole [24], and that the radiation following the collision of two black holes is dominated by quasinormal-mode ringing [25]. Hence, it does not seem unreasonable to assume that the w-modes will be excited when a neutron star is created or when two such stars collide.

One important conclusion that follows from our simple simulations is that *studies of dynamical processes involving neutron stars must be performed using general relativity.* This may seem obvious, but at the present time a considerable effort is invested in calculations using Newtonian gravity. The gravitational radiation is typically extracted by means of the quadrupole formula. The w-modes will never appear in such calculations and thus this feature of the full problem will be unaccounted for.

It is clear that the stellar pulsation modes can play a role in many astrophysical scenarios, but will we be able to observe them with future gravitational-wave detectors? In the high-frequency regime where both the fluid f-mode and the w-modes reside (1-2 kHz and 8-12 kHz, respectively) one would not expect too much from the new generation of laser-interferometric gravitational-wave detectors. These will probably not be sensitive enough at such high frequencies (although their performance in this regime may be much improved through dual recycling). But neutron stars pulsating in the f-mode are ideal sources for the resonant bar detectors that are currently operating. The frequency of the f-mode also makes it well suited for detection by the recently proposed spherical solid-mass detectors, *e.g.*, TIGA [26]. Meanwhile, the w-modes provide interesting sources for the proposed array of smaller bar detectors, which should be sensitive in the few kHz regime [27].

To obtain rough estimates for the typical gravitational-wave amplitudes from a pulsating star we use the standard relation for the gravitational-wave flux [28]

$$F = \frac{c^3}{16\pi G}|\dot{h}| = \frac{1}{4\pi r^2}\frac{dE}{dt}, \quad (1)$$

which is valid far away from the star. Combining this with i) $dE/dt = E/2\tau$ where τ is the e-folding time of the pulsation and E is the available energy ii) the assumption that the signal is monochromatic (with frequency f) and iii) the knowledge that the effective amplitude achievable after matched filtering scales as the square-root of the number of observed cycles: $h_{\rm eff} = h\sqrt{n} = h\sqrt{f\tau}$, we get the estimates

$$h_{\rm eff} \sim 3\times 10^{-21}\left(\frac{E}{10^{-6}M_\odot c^2}\right)^{1/2}\left(\frac{2\text{kHz}}{f}\right)^{1/2}\left(\frac{50\text{kpc}}{r}\right), \quad (2)$$

for the f-mode, and

$$h_{\rm eff} \sim 1\times 10^{-21}\left(\frac{E}{10^{-6}M_\odot c^2}\right)^{1/2}\left(\frac{10\text{kHz}}{f}\right)^{1/2}\left(\frac{50\text{kpc}}{r}\right), \quad (3)$$

for the fundamental w-mode. Here we have used typical parameters for the pulsation modes, and the distance scale used is that to SN1987A. In this volume of space one would not expect to see more than one event per ten years or so. However, the assumption that the energy release in gravitational waves in a supernova is of the order of $10^{-6}M_\odot c^2$ is very conservative [28]. If a substantial fraction of the binding energy of a neutron star were released through the pulsation modes we could hope to see such events out to the Virgo cluster (and perhaps as many as a few per year).

What can we learn from observations?. — If they were detected, the stellar pulsation modes carry key information about the nature of neutron stars. Suppose that we detect a gravitational-wave signal from a compact star, what can we hope to learn from it? Again, a detailed answer requires much further study. But it is useful to speculate on the possibilities.

Assume that we detect a signal and manage to extract both the fluid f-mode and the slowest damped polar (or axial) w-mode from it. Then spectral studies suggest that: i) the oscillation frequency of the f-mode scales with the average density of the star through $\sqrt{M/R^3}$, and ii) the damping rate of the w-mode depends linearly on the compactness ratio M/R of the star. These properties are illustrated in Figures 2a,b. In principle, one should be able to infer the average density of the star from an observation of the f-mode. Similarly, an observation of a w-mode leads to an estimate of the stars compactness. Although important, the information available in either case would not provide detailed information about the stellar parameters. But the situation is markedly different if both modes are observed. Then both the mass and the radius of the star can be extracted from the observed data. This information puts strong constraints on the nuclear equation of state [29].

The idea seems simple enough, but will it be useful in practice? Let us give an example for stars with polytropic equations of state: We have constructed a set of independent polytropic stellar models ($p = K\rho^{1+1/n}$) with varying polytropic index ($n = 0.5; 0.8; 1; 1.2; 1.5$). We have determined the f-mode and the slowest damped polar w-mode for each of these models (the results would be similar if we use the fundamental axial w-mode). The relevant data is graphed in Figures 2a,b. The theoretical spectra are well fitted by the following two relations: The oscillation frequency of the f-mode varies with the average density of the star as

$$\omega_f(\text{kHz}) \approx 0.17 + 2.30\sqrt{\left(\frac{10\text{km}}{R}\right)^3\left(\frac{M}{1.4M_\odot}\right)}, \quad (4)$$

while the damping rate of the w-mode behaves as (τ_w is the e-folding time of the pulsation)

$$\frac{1}{\tau_w}(\text{MHz}) \approx 0.107 - 0.069\left(\frac{10\text{km}}{R}\right)\left(\frac{M}{1.4M_\odot}\right). \quad (5)$$

These approximations are shown in Figures 2a,b. The crucial question is: How robust are these fits for manifestly independent stellar models? The two relations (4) and (5) must provide reasonably accurate estimates for both M and R for all stars in our dataset in order for the idea to be useful. The scheme passes this first simple test with flying colours: By inverting (4) and (5), the mass and the radius of each star can be determined at least as accurately as can the average density and the stellar compactness. The error in M and R is, in fact, typically smaller than 10% .

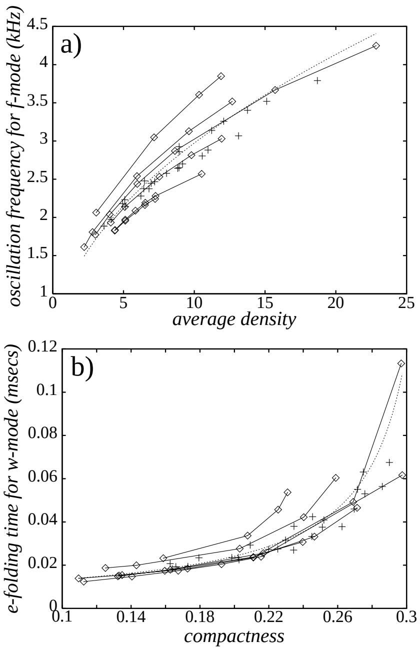

FIG. 2. (a) The pulsation frequency of the f-mode as a function of the average density (in units 10^{14}g/cm^3) of the star , and (b) the e-folding time for the slowest damped polar w-mode as a function of the stellar compactness (dimensionless). The behaviour is not strongly dependent on the polytropic model used, and therefore ideal for parameter extraction. The polytrope data that is used in the discussion in the text is shown as diamonds connected with solid lines. The corresponding least-square fits are shown as dashed curves. Also indicated (by crosses) are preliminary results for a few realistic equations of state.

This is clear evidence that our idea can be useful in a real detection situation, and that it deserves to be investigated in more detail. One must: i) incorporate the estimated effects of statistical errors and measurement ones. It will, for example, be much more difficult to infer the w-mode damping rate from a data set than to find the f-mode pulsation frequency (remember that the observability of a periodic signal buried in noise scales roughly as the square-root of the number of observed cycles). ii) obtain fits similar to (4) and (5) for more realistic equations of state. If such relations prove as robust as the ones we have obtained for polytropes then the suggested scheme looks truly promising.

We are presently investigating these questions and will return to them in the future. As yet, we have done preliminary calculations for seven of the equations of state that were used by Lindblom and Detweiler [3] (their models A, B, C, E, F, G and I). These results, which have been included in Figures 2a,b, are in impressive agreement with the polytrope data used in our example. As a further demonstration of the robustness of the suggested scheme we have considered the f-mode data that Cutler, Lindblom and Splinter obtained for several realistic equations of state [30]. One can use the stellar parameters M and R from Table 2 in [30] together with our relation (4). This leads to approximate f-mode frequencies that should be compared to the data in Table 3 of [30]. Despite being derived for polytropes, our equation (4) approximates almost all the $\ell = 2$ frequencies in Table 2 of [30] to well within 10 %.

Concluding remarks. — In this letter we have presented suggestive results for problems that can have direct impact on the future detection of gravitational waves from neutron stars.

First we have shown that the gravitational-wave modes of a neutron star will be excited in a dynamical situation. The modes should be excited when a neutron star is formed in a gravitational collapse, a process when a considerable amount of energy is released. But at present it is difficult to estimate the amount of energy that goes into the pulsation modes. What is absolutely clear (and this is a very important conclusion) is that we are asking questions that can never be answered within Newtonian gravity. *The assertions presented here must be tested by more detailed, fully general relativistic, simulations.* This is a challenge for numerical relativity. A relativistic description of gravitational collapse to form a neutron star, or the merger of two stars, should tell us whether the w-modes are of observational relevance or not. Furthermore, the pulsation properties of a slightly perturbed star can be used as a powerful test of the reliability of a numerical code.

The second part of our work shows that the stellar pulsation modes can, if observed, provide us with accurate information about the stellar parameters. We have illustrated how a simple scheme can lead to accurate estimates of both the mass and the radius of the star. This is a very useful suggestion, since it can put strong constraints on the nuclear equation of state.

Given the obvious importance for high-frequency gravitational-wave detectors, further studies of the issues raised here are urgently required.

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