Class. Quantum Grav. **17** (2000) 2215–2241. Printed in the UK PII: S0264-9381(00)50001-X

Cosmological perturbations on a magnetized Bianchi I background

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Received 13 December 1999

Abstract. Motivated by the isotropy of the CMB spectrum, all existing studies of magnetized cosmological perturbations employ FRW backgrounds. However, it is important to know the limits of this approximation and the effects one loses by neglecting the anisotropy of the background magnetic field. We develop a new treatment, which fully incorporates the anisotropic magnetic effects by allowing for a Bianchi I background universe. The anisotropy of the unperturbed model facilitates the closer study of the coupling between magnetism and geometry. The latter leads to a curvature stress, which accelerates positively curved perturbed regions and balances the effect of magnetic pressure gradients on matter condensations. We argue that the tension carried along the magnetic force lines is the reason behind these magneto-curvature effects. For a relatively weak field, we also compare with the results of the almost-FRW approach. We find that some of the effects identified by the FRW treatment are in fact direction-dependent, where the key direction is that of the background magnetic field vector. Nevertheless, the FRW-based approach to magnetized cosmological perturbations remains an accurate approximation, particularly on large scales, when one looks at the lowest-order magnetic impact on gravitational collapse. On small scales, however, the accuracy of the perturbed Friedmann framework may be compromised by extra shear effects.

PACS numbers: 9880H, 0440N, 4775, 9530Q, 9862E, 0420

1. Introduction

Over the years, the implications of primordial magnetic fields for the formation and the evolution of the observed structure have been the subject of continuous theoretical investigation. The majority of these studies are Newtonian, or semi-relativistic, and all of them are confined within perturbed Friedmann-Robertson-Walker (FRW) models. Mathematically speaking, however, the spatial isotropy of the Friedmann universe is not compatible with the presence of large-scale magnetic fields. Nevertheless, one may still preserve the FRW symmetries by postulating a random background field (e.g. [1,2]), or by allowing for a uniform field that is too weak to destroy the Robertson–Walker isotropy (e.g. [3–5]). In either approach the anisotropy induced by the magnetic field to the zero-order stress tensor of the universe is treated as a perturbation. Both treatments, apart from their different initial set-up, proceed essentially in the same way and they should reach the same conclusions.

A strong motivation for adopting an FRW background comes from observation. Current measurements of the Cosmic Microwave Background (CMB) anisotropy suggest an energy density for the primeval field much smaller than that of matter at recombination [6]. Similar limits, though less stringent, are placed at nucleosynthesis by Helium-4 measurements [7].

One may also use dissipative arguments to limit the strength of primordial magnetic fields on small scales [8]. Intuitively, the impact of such a weak field on the isotropy of the smooth FRW spacetime should be negligible. Nevertheless, pending more detailed treatments, one might still ask:

- What is the exact nature of the effects induced by the anisotropic magnetic stresses?
- How important are the deviations from the results based on perturbed FRW models?

We address these questions by employing a Bianchi I background, the simplest anisotropic cosmology that can naturally accommodate large-scale magnetic fields. We also use the covariant Lagrangian perturbation formalism [9], which has proved powerful in dealing with issues of similar complexity in the past. This article extends the work of [5], where the exact equations of a general magnetized cosmology were derived and subsequently linearized about an FRW spacetime. Here, we linearize the same equations about a Bianchi I universe supplied with a large-scale magnetic field and a perfectly conducting ideal fluid. Our main aim is to identify and analyse the kinematic and dynamic effects of the anisotropic magnetic stresses. In the process we will also test the accuracy of the FRW-based treatments of magnetized cosmological perturbations.

We provide the full set of linear equations and then isolate the effects of the background magnetic stresses, by treating the field as an eigenvector of the shear tensor to zero order. In this way we can identify the deviations from the almost-FRW results that are caused purely by the background magnetic stresses. Note that background anisotropies that are not related to the magnetic presence are not relevant to our study and therefore not included. At the linear level, however, the field is not assumed to be a shear eigenvector. Thus, in the perturbed universe the shear is no longer the pure result of the magnetic stresses. We consider a universe filled with a single perfectly conducting barotropic fluid and compare to the predictions of the FRW-based treatments. Generally speaking, the impact of a cosmological magnetic field has an isotropic as well as an anisotropic component. The former results from the magnetic contributions to the total energy density and isotropic pressure, whereas the latter is due to the magnetic stresses. Our analysis shows that although the Friedmannian limit identifies the isotropic component exactly, it can only address the 'average' anisotropic effects of the field. Note that the 'average' is taken over the three linearly independent spatial directions rather than over the whole space. Thus, as one should expect, the isotropy of the background spacetime sets certain limits to the accuracy of the standard treatment. In this report we show that some of the magnetic effects identified by the FRW approximation are in fact direction-dependent, with the key direction determined by the zero-order field vector. Even when the field is relatively weak, certain quantities, such as the curvature, continue to contribute differently parallel to the magnetic force lines than orthogonal to them. Clearly, in the linear regime, such directional dependence results from the background anisotropy and it is not accessible through an FRW background.

Of particular interest are the magnetohydrodymanical effects on the average volume expansion of the universe. We find that density perturbations co-linear to the background field contribute more efficiently to the deceleration rate. Also, only spatial curvature perturbations parallel to the zero-order magnetic vector affect the volume expansion. The latter effect is of geometrical origin, resulting from the vectorial nature of the magnetic field. It was first identified in [10], although there the isotropy of the FRW background meant that the effect was independent of direction. Here, as in [10], the magneto-curvature term in the Raychaudhuri equation decelerates negatively curved regions, but accelerates those with positive local spatial curvature. The crucial difference made by the background anisotropy, is that it reveals the prominent role played by the field lines themselves. We can now attribute the unconventional magneto-geometrical effects to the negative pressure, the tension, experienced along the magnetic lines of force. This property of the field, together with the coupling between magnetism and geometry, gives rise to a *curvature stress*. The latter reacts to the effect of local magnetic pressure gradients by modifying the expansion rate of the region accordingly. This relativistic effect is analogous to the classical curvature stress exerted by field lines with a non-zero local radius of curvature.

Direction-dependent contributions are also identified in the evolution equations of the density gradients. Curvature effects, for example, are again confined in the direction of the background field. At the weak field limit and on super-horizon scales, this change in the curvature contribution is what separates the Bianchi I perturbation equations from their FRW counterparts. Nevertheless, such directional effects can only alter the lowest-order magnetic impact during the radiation era. Even then, the predictions of the FRW approximation still hold, at least in certain critical cases. Our results verify the point made in [11], that magneto-curvature effects tend to counterbalance the 'pure' magnetic impact on the evolution of density gradients. In particular, for zero curvature input the field was found to suppress slightly the growth of the density contrast [10], in agreement with the Newtonian treatment of [3]. However, when curvature contributions were included in the equations, this inhibiting effect vanished [11]. Here, we verify these statements and argue that the magneto-geometric contributions can also reverse the pure magnetic impact on gravitational collapse. When the curvature perturbation along the field lines is maximum, we find that the density contrast grows faster than in standard non-magnetized cosmological models. Thus, the combined magneto-curvature action opposes the effect of the pure field on local matter condensations. Again, the explanation comes from the tension experienced along force lines of the field. When coupled to geometry, this magnetic property leads to a curvature stress, which always tries to balance the effect of the magnetic pressure gradients.

On sub-horizon scales, we find that curvature complexities are accompanied by similar direction-dependent shear effects, even when the field is weak. These shear contributions result from the kinematical implications of the background magnetic stresses, and do not appear in an almost-FRW treatment. One might be able to invoke fluid 'backreaction' arguments and ignore, to first order, the aforementioned small-scale shear effects. In this case, one recovers the standard FRW linear equations, provided that the curvature perturbation is averaged. Nevertheless, one should be rather careful when discussing the small-scale magnetic effects within perturbed FRW cosmologies. In contrast, super-horizon weakly magnetized density perturbations proceed basically as predicted by the Friedmann approximation.

This paper is organized as follows: section 2 provides a brief review of the covariant and gauge-invariant approach to cosmology with references for further reading. In section 3 we linearize the exact equations about a magnetized Bianchi I background. Particular properties of the unperturbed model are discussed in section 4. In section 5 we apply the linear equations to the case of a barotropic medium, and examine the field impact on the model's kinematic and dynamic evolution. Section 6 addresses the weak field limit and section 7 discusses solutions of the perturbation equations. Finally, we summarize our conclusions in section 8.

2. The covariant formalism

In the covariant Lagrangian approach to cosmology (see [12] for an up-to-date review) one makes a physical choice for the 4-velocity u_a of the matter. Then, one uses $h_{ab} \equiv g_{ab} - u_a u_b$, where g_{ab} is the spacetime metric, to project into the instantaneous rest spaces of the co-moving observers†. Note that these rest spaces form an integrable three-dimensional hypersurface only

† We employ geometrized units with $c = 1 = 8\pi G$ and our metric signature is $(- + + +)$.

for zero vorticity. Nevertheless, we will keep referring to them as 'spatial' or '3-spaces' for simplicity. Thus, the fluid 4-velocity introduces a unique 1+3 splitting of the spacetime into time and space. We can now decompose tensor fields and tensor equations into their timelike and spacelike parts. Without going into much detail, we provide some key decompositions essential to our analysis.

2.1. The basic equations

The kinematical properties of the motion arise from splitting the covariant derivative of u_a into its irreducible parts

$$\nabla_b u_a = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \varepsilon_{abc}\omega^c - \dot{u}_a u_b\,, \tag{1}$$

where ε_{abc} is the rest-space volume element, $\Theta \equiv \tilde{\nabla}^a u_a$ is the rate of the volume expansion, $\sigma_{ab} \equiv \tilde{\nabla}_{\langle a} u_{b\rangle}$ is the shear, $\omega_a \equiv -\frac{1}{2}\mathrm{curl}\, u_a$ is the vorticity and $\dot{u}_a \equiv u^b\tilde{\nabla}_b u_a$ is the acceleration†. The volume expansion defines an average length scale S via $\dot{S}/S = \frac{1}{3}\Theta$‡.

The medium is characterized by its energy–momentum tensor. Relative to a co-moving observer, the stress tensor of an imperfect fluid decomposes as

$$T^*_{ab} = \mu^* u_a u_b + p^* h_{ab} + 2q^*_{(a}u_{b)} + \pi^*_{ab}\,, \tag{2}$$

where μ^* and p^* are respectively the relativistic energy density and the isotropic pressure, q^*_a is the energy flux vector and π^*_{ab} is the PSTF tensor representing anisotropic pressures. The detailed physics is encoded in the equations of state. For a perfect fluid, with zero flux and anisotropic stresses, equation (2) reduces to

$$T_{ab} = \mu u_a u_b + p h_{ab}\,. \tag{3}$$

Similarly, in the absence of electric fields, the stress tensor of a pure magnetic field H_a splits as

$$T_{ab} = \tfrac{1}{2}H^2 u_a u_b + \tfrac{1}{6}H^2 h_{ab} + \Pi_{ab}\,, \tag{4}$$

where $H^2 \equiv H_a H^a$. The latter provides a measure of the magnetic energy density and isotropic pressure, while

$$\Pi_{ab} \equiv \tfrac{1}{3}H^2 h_{ab} - H_a H_b \equiv -H_{\langle a}H_{b\rangle} \tag{5}$$

is the PSTF tensor that represents the anisotropic magnetic stresses.

The Weyl tensor C_{abcd} represents the long-range gravitational field, namely tidal forces and gravity waves. Together with the Ricci tensor R_{ab}, which is decided locally by matter, it determines the Riemann curvature tensor R_{abcd} completely. For a co-moving observer we have [14]

$$C_{ab}{}^{cd} = 4\left(u_{[a}u^{[c} + h_{[a}{}^{[c}\right)E_{b]}{}^{d]} + 2\varepsilon_{abe}u^{[c}H^{d]e} + 2u_{[a}H_{b]e}\varepsilon^{cde}\,, \tag{6}$$

where $E_{ab} = C_{acbd}u^c u^d$ and $H_{ab} = \frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e$ are respectively the 'electric' and 'magnetic' parts of C_{abcd}.

† Following [13], we use angled brackets for the projected, symmetric, trace-free (PSTF) part of tensors and for the orthogonal projections of vectors. The same notation also represents orthogonally projected covariant time derivatives.
‡ For comparison reasons, the current article adopts the notation of [5] with a number of changes motivated by [12] and [13]. In particular, totally projected covariant derivatives are now represented by $\tilde{\nabla}$ instead of ${}^{(3)}\nabla$, Π_{ab} has replaced M_{ab} as the anisotropic magnetic pressure tensor, the 3-Ricci tensor and the 3-Ricci scalar have changed from ${}^{(3)}R_{ab}$ and K to $\mathcal{R}_{ab}$ and $\mathcal{R}$ respectively, while the scalar $\mathcal{B}$ now represents the dimensionless ratio $S^2\tilde{\nabla}^2H^2/H^2$ rather than the Laplacian $\tilde{\nabla}^2H^2$. Other changes include a difference in the definition of the Alfvén speed (see section 4.2.2) and the use of the vorticity vector ω_a instead of the antisymmetric tensor ω_{ab}. Finally, we employ the 'streamline' definitions for the covariant curls of spacelike vectors and tensors (see [13]).

2.2. The basic gauge-invariant variables

The covariant Lagrangian formalism is the foundation for a covariant and gauge-invariant perturbation theory [9], which provides an alternative to the metric-based gauge-invariant formalism [15]. In the covariant approach, one deals with the inhomogeneous and anisotropic spacetime directly, instead of perturbing away from the smooth background universe. Thus, one obtains the exact (fully non-linear) equations first, before linearizing them about a chosen background. This is clearly a major advantage, because it allows one to address cosmological models more general than the perturbed FRW universes. Indeed, the covariant approach has already been employed to study perturbed Bianchi I cosmologies filled with a non-magnetized perfect fluid [16]. Here, we use the same formalism to analyse, for the first time, almost Bianchi I spacetimes permeated by a large-scale magnetic field.

The covariant approach utilises the Stewart–Walker lemma [17], to define a set of gauge-independent tensors. These describe inhomogeneities in the key quantities and have a transparent geometrical and physical interpretation. For our purposes, the basic gauge-invariant variables are [5]

$$\mathcal{D}_a \equiv \frac{1}{\mu} S\tilde{\nabla}_a \mu\,, \qquad \mathcal{Z}_a \equiv S\tilde{\nabla}_a \Theta \qquad \text{and} \qquad \mathcal{M}_{ab} \equiv S\tilde{\nabla}_b H_a\,. \tag{7}$$

The above respectively represent spacelike variations in the energy density of the fluid, the volume expansion and the magnetic field vector, as these are seen by a pair of neighbouring fundamental observers†. They vanish in an exact Bianchi I spacetime, thus satisfying the Stewart–Walker criterion for gauge invariance. Note that gradients in the fluid pressure are represented by the gauge-independent variable $Y_a \equiv \tilde{\nabla}_a p$. For a barotropic fluid, however, Y_a is always given in terms of $\mathcal{D}_a$.

3. Perturbed magnetized Bianchi I universes

The non-linear equations that describe density perturbations of a perfectly conducting medium in the presence of a large-scale magnetic field have been derived in [5]. Here, we linearize these equations about a Bianchi I spacetime. Covariantly, the dynamics of the model is characterized by the energy density (μ) and the pressure (p) of the matter. The magnetic presence also brings into play the field's energy density and isotropic pressure (both expressed via H^2), as well as anisotropic stresses represented by Π_{ab}. Kinematically, the Bianchi I cosmology is determined by the average expansion scalar (Θ) and by the shear tensor (σ_{ab}), which describes departures from the isotropic expansion. Anisotropies in the spacetime curvature propagate via the electric Weyl tensor (E_{ab}), while the spatial sections of the Bianchi I spacetime are flat. During linearization all these variables are treated as zero-order. First-order quantities are those that vanish in the background and therefore satisfy the requirement for gauge invariance [17]. These are the acceleration ($\dot{u}_a$), the vorticity (ω_a), all variables representing spatial curvature, the magnetic Weyl tensor (H_{ab}) and the inhomogeneity variables $\mathcal{D}_a$, Y_a, $\mathcal{Z}_a$ and $\mathcal{M}_{ab}$. Terms of higher perturbative order will be disregarded.

3.1. The medium

According to equations (3) and (4), the energy-momentum tensor for a magnetized spacetime filled with a single perfectly conducting ideal fluid is

$$T_{ab} = \left(\mu + \tfrac{1}{2}H^2\right) u_a u_b + \left(p + \tfrac{1}{6}H^2\right) h_{ab} + \Pi_{ab}\,, \tag{8}$$

† In a perturbed Bianchi I spacetime the aforementioned geometrical interpretation of the key inhomogeneity variables holds only for weak shear (see the appendix).

where the assumption of infinite conductivity ensures the absence of electric fields [5]. Comparing equations (8) and (2) we see that our magnetized medium corresponds to an imperfect fluid with $\mu^* = \mu + \frac{1}{2}H^2$, $p^* = p + \frac{1}{6}H^2$, $q^*_a = 0$ and $\pi^*_{ab} = \Pi_{ab} \equiv -H_{\langle a}H_{b\rangle}$. Thus, magnetism contributes to the total energy density and isotropic pressure, while it is solely responsible for the anisotropic stresses. Note that zero electric field implies a vanishing Poynting vector, which in turn explains the absence of an energy flux vector in equation (8).

3.2. The linear equations

Next, we provide the key relations that describe the linear evolution of our magnetized Bianchi I cosmology. Unless stated otherwise, all equations derive from the non-linear formulae given in [5]†. In particular, we employ a combination of propagation and constraint equations. These are:

(i) The fluid conservation laws, represented by the continuity equation

$$\dot{\mu} = -\mu(1+w)\Theta \tag{9}$$

and by Euler's formula

$$\mu\left(1+w+\tfrac{2}{3}h\right)\dot{u}_a = -Y_a - \varepsilon_{abc}H^b\text{curl}\,H^c - \dot{u}^b\Pi_{ba}\,, \tag{10}$$

where $w \equiv p/\mu$, $h \equiv H^2/\mu$ and $\text{curl}\,H_a \equiv \varepsilon_{abc}\tilde{\nabla}^bH^c$. Generally, w is allowed to vary with

$$\dot{w} = -(1+w)\left(c_{\rm s}^2 - w\right)\Theta\,, \tag{11}$$

where $c_{\rm s}^2 \equiv \dot{p}/\dot{\mu}$. Therefore, when $\dot{w} = 0$, then $c_{\rm s}^2 = w = constant$. Note the total absence of magnetic terms in equation (9), since the field's energy density is conserved separately as a consequence of Maxwell's equations.

(ii) The kinematic propagation equations, comprising Raychaudhuri's formula

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\mu\,(1+3w+h) - 2\sigma^2 + A + \Lambda\,, \tag{12}$$

where $A = \tilde{\nabla}^a\dot{u}_a$ (to first order) and Λ is the cosmological constant, the vorticity propagation equation

$$\dot{\omega}_a = -\tfrac{2}{3}\Theta\omega_a + \sigma_{ab}\omega^b - \tfrac{1}{2}\text{curl}\,\dot{u}_a \tag{13}$$

and the shear propagation equation

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} + \tilde{\nabla}_{\langle a}\dot{u}_{b\rangle} + \tfrac{1}{2}\Pi_{ab} - E_{ab}\,. \tag{14}$$

The above are supplemented by an equal number of constraints. In the nomenclature of [12], these are the (0α)-equation

$$\tfrac{2}{3}\tilde{\nabla}_a\Theta = \tilde{\nabla}^b\sigma_{ab} + \text{curl}\,\omega_a\,, \tag{15}$$

the vorticity divergence constraint

$$\tilde{\nabla}^a\omega_a = 0 \tag{16}$$

and the H_{ab}-equation

$$H_{ab} = (\text{curl}\,\sigma)_{ab} + \tilde{\nabla}_{\langle a}\omega_{b\rangle}\,, \tag{17}$$

where $(\text{curl}\,S)_{ab} \equiv \varepsilon_{cd\langle a}\tilde{\nabla}^cS^d{}_{b\rangle}$ for every PSTF tensor S_{ab} [13]. Note that constraint (16) ensures that the ω_a is a solenoidal vector to first order.

† For a quick cross-check, the reader may compare our equations to those of section 2 in [12]. The latter must be adapted to the particular imperfect fluid defined by equation (8).

(iii) The magnetic equations, obtained from the covariant Maxwell equations [19]. For a perfectly conducting medium, the latter decompose into one propagation equation

$$\dot{H}_{\langle a\rangle} = -\tfrac{2}{3}\Theta H_a + \sigma_{ab}H^b + \varepsilon_{abc}H^b\omega^c\,, \tag{18}$$

namely the magnetic induction equation, and three constraints

$$\varepsilon_{abc}\dot{u}^b H^c + \mathrm{curl}\, H_a = J_{\langle a\rangle}\,, \tag{19}$$

$$2\omega_a H^a = \epsilon\,, \tag{20}$$

$$\tilde{\nabla}^a H_a = 0\,, \tag{21}$$

where $J_{\langle a\rangle}$ is the projected current density and $\epsilon = -J_a u^a$ is the charge density. According to equation (20), an overall electrical neutrality (i.e. $\epsilon = 0$) ensures that the vorticity vector and the magnetic field are orthogonal. Also, equation (21) guarantees that H_a is a solenoidal. Contracted with H_a, the induction equation provides the conservation law of the magnetic energy density

$$\left(H^2\right)^{\cdot} = -\tfrac{4}{3}\Theta H^2 - 2\sigma^{ab}\Pi_{ab}\,, \tag{22}$$

where the background anisotropy has added the last term of the right-hand side. Thus, unlike perturbed FRW models, the magnetic energy density no longer drops as S^{-4}. Also, from equation (27) below we obtain the expression

$$\Pi_{ab} = \tfrac{2}{3}\Theta\sigma_{ab} - 2\sigma_{c\langle a}\sigma^c{}_{b\rangle} - 2E_{ab} + \mathcal{R}_{\langle ab\rangle}\,, \tag{23}$$

that connects Π_{ab} to the rest of the linear anisotropic sources.

Note that, so far, all equations have been derived from the fully non-linear relations given in [5]. Next we provide a new pair of linear equations that govern the behaviour of Π_{ab}. In particular, the time derivative of (5) and equations (18) and (22) lead to the propagation equation of the anisotropic magnetic stresses

$$\dot{\Pi}_{\langle ab\rangle} = -\tfrac{4}{3}\Theta\Pi_{ab} + 2\Pi_{c\langle a}\left(\sigma^c{}_{b\rangle} + \varepsilon^{cd}{}_{b\rangle}\omega_d\right) - \tfrac{2}{3}\mu h\sigma_{ab}\,. \tag{24}$$

Similarly, by projecting the gradient of (5) orthogonal to the fluid flow we arrive at the constraint

$$\tilde{\nabla}^b\Pi_{ab} = \varepsilon_{abc}H^b\mathrm{curl}\, H^c - \tfrac{1}{6}\tilde{\nabla}_a H^2\,. \tag{25}$$

(iv) The equations that determine the geometry of the observer's rest space. The latter is characterized by the 3-Riemann tensor $\mathcal{R}_{abcd}$, which when contracted provides the spatial Ricci tensor $\mathcal{R}_{ab} = \mathcal{R}^c{}_{acb}$ and Ricci scalar $\mathcal{R} = \mathcal{R}^a{}_a$†. For our purposes, the key expression is the Gauss–Godazzi equation

$$\mathcal{R}_{ab} = \tfrac{1}{3}\mathcal{R}h_{ab} + \tfrac{1}{2}\Pi_{ab} - \tfrac{1}{3}\Theta\left(\sigma_{ab} + \omega_{ab}\right) + \sigma_{c\langle a}\sigma^c{}_{b\rangle} + 2\sigma_{c[a}\omega^c{}_{b]} + E_{ab}\,, \tag{26}$$

implying that the 3-Ricci tensor has also an antisymmetric part. In fact, $\mathcal{R}_{ab}$ decomposes into

$$\mathcal{R}_{(ab)} = \tfrac{1}{3}\mathcal{R}h_{ab} + \tfrac{1}{2}\Pi_{ab} - \tfrac{1}{3}\Theta\sigma_{ab} + \sigma_{c\langle a}\sigma^c{}_{b\rangle} + E_{ab} \tag{27}$$

and

$$\mathcal{R}_a = -\tfrac{1}{3}\Theta\omega_a - \sigma_{ab}\omega^b\,, \tag{28}$$

† We remind the reader that in rotating spacetimes there are no proper spacelike hypersurfaces orthogonal to the fluid flow. Here, we use the term 'spatial', when referring to the instantaneous rest space of a fundamental observer, for convenience only. Also, when $\omega_a \neq 0$, $\mathcal{R}_{abcd}$ does not share the full symmetries of the spacetime Riemann tensor [18].

where $\mathcal{R}_a \equiv \frac{1}{2}\varepsilon_{abc}\mathcal{R}^{bc}$ represents the skew part of $\mathcal{R}_{ab}$. The average spatial curvature is determined by the sign of the 3-Ricci scalar. The latter follows from the generalized Friedmann equation

$$\mathcal{R} = 2\mu\left(1 + \tfrac{1}{2}h\right) - \tfrac{2}{3}\Theta^2 + 2\sigma^2 + 2\Lambda\,, \tag{29}$$

and evolves according to

$$\dot{\mathcal{R}} = -\tfrac{2}{3}\Theta\mathcal{R} + 2\left[\left(\sigma^2\right)^{\cdot} + 2\Theta\sigma^2\right] - 2\sigma^{ab}\Pi_{ab} - \tfrac{4}{3}\Theta A\,. \tag{30}$$

For the Weyl tensor components, we use equations (24) and (25) to obtain a set of two propagation formulae, namely the $\dot{E}$-equation

$$\dot{E}_{\langle ab\rangle} = -\Theta\left(E_{ab} - \tfrac{1}{2}\Pi_{ab}\right) + 3\left(\sigma_{c\langle a} + \tfrac{1}{3}\varepsilon_{cd\langle a}\omega^d\right)\left(E^c{}_{b\rangle} - \tfrac{1}{2}\Pi^c{}_{b\rangle}\right) - \tfrac{1}{2}\mu(1+w)\sigma_{ab} + (\mathrm{curl}\, H)_{ab} \tag{31}$$

and the $\dot{H}$-equation

$$\dot{H}_{ab} = -\Theta H_{ab} + 3\sigma_{c\langle a}H^c{}_{b\rangle} - (\mathrm{curl}\, E)_{ab} + \tfrac{1}{2}(\mathrm{curl}\,\Pi)_{ab} + 2\varepsilon_{cd\langle a}E^c{}_{b\rangle}\dot{u}^d\,, \tag{32}$$

which are accompanied by an equal number of constraints, the $(\mathrm{div}E)$-equation

$$\tilde{\nabla}^b E_{ab} = \frac{\mu}{3S}\left(\mathcal{D}_a + \tfrac{3}{4}h\mathcal{B}_a\right) - \tfrac{1}{2}\varepsilon_{abc}H^b\mathrm{curl}\, H^c + \varepsilon_{abc}\sigma^b{}_d H^{cd}\,, \tag{33}$$

and the $(\mathrm{div}H)$-equation

$$\tilde{\nabla}^b H_{ab} = \mu\left(1 + w + \tfrac{2}{3}h\right)\omega_a - \varepsilon_{abc}\sigma^b{}_d\left(E^{cd} + \tfrac{1}{2}\Pi^{cd}\right) + 3\left(E_{ab} - \tfrac{1}{6}\Pi_{ab}\right)\omega^b\,, \tag{34}$$

where $\mathcal{B}_a \equiv S\tilde{\nabla}_a H^2/H^2$. The latter is an additional gauge-invariant variable representing inhomogeneities in the magnetic energy density.

(v) The linear propagation equations of the key inhomogeneity variables $\mathcal{D}_a$, $\mathcal{Z}_a$ and $\mathcal{M}_{ab}$. The former describes gradients in the fluid energy density and evolves according to

$$\dot{\mathcal{D}}_a = w\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a + \frac{\Theta S}{\mu}\varepsilon_{abc}H^b\mathrm{curl}\, H^c + \tfrac{2}{3}h\Theta S\dot{u}_a - \sigma_{ab}\mathcal{D}^b + \frac{\Theta S}{\mu}\Pi_{ab}\dot{u}^b\,, \tag{35}$$

obtained from equation (71) in [5]. Note the last two terms in the right-hand side of the above, which reflect the background anisotropy. Following equation (72) in [5], the expansion gradients propagate as

$$\dot{\mathcal{Z}}_a = -\tfrac{2}{3}\Theta\mathcal{Z}_a - \tfrac{1}{2}\mu\mathcal{D}_a + SA_a - 3S\sigma^2\dot{u}_a + \tfrac{3}{2}S\varepsilon_{abc}H^b\mathrm{curl}\, H^c - \tfrac{1}{2}\mu h\mathcal{B}_a - \sigma_{ab}\mathcal{Z}^b + \tfrac{3}{2}S\Pi_{ab}\dot{u}^b - 2S\tilde{\nabla}_a\sigma^2\,, \tag{36}$$

where $A_a \equiv \tilde{\nabla}_a A$ and the last three terms are due to the anisotropic background. Finally, inhomogeneities in the magnetic field vector are governed by

$$\dot{\mathcal{M}}_{ab} = -\tfrac{2}{3}\Theta\mathcal{M}_{ab} - \tfrac{2}{3}H_a\mathcal{Z}_b - \tfrac{1}{3}\Theta S\left(H_{[a}\dot{u}_{b]} + 3H_{\langle a}\dot{u}_{b\rangle}\right) + SH^c\tilde{\nabla}_b(\sigma_{ac} + \varepsilon_{acd}\omega^d) - S\varepsilon_{acd}H^cH^d{}_b + \sigma_{ac}\mathcal{M}^c{}_b - \sigma_{bc}\mathcal{M}_a{}^c + SH^c\dot{u}_c\sigma_{ab} + 2SH^c\sigma_{c[a}\dot{u}_{b]}\,, \tag{37}$$

which derives from equation (75) in [5]. Here, the background anisotropy is represented by the last four terms of the right-hand side.

4. The Bianchi I background

It has long been known that anisotropic exact Bianchi cosmologies of type I can accommodate spatially homogeneous ordered magnetic fields. We refer the reader to [20–23] for a sample of such studies during the 1960s and the early 1970s. All these authors assume, either directly or indirectly, that the magnetic field is aligned along the shear eigenvector, and the same is also true for our analysis. Note, however, that the field equations do not necessarily impose any such condition upon the magnetic vector in Bianchi I models [24]. In fact, a magnetized Bianchi I universe, where the field is not a shear eigenvector, has been studied by means of dynamical system techniques [25].

Here, we focus upon the anisotropic magnetic stresses and their impact on cosmological perturbations. For our purposes, the whole of the background anisotropy must result from the presence of the field. This, in turn, will isolate the effects of the magnetic stresses and facilitate their analysis.

4.1. The anisotropy

The key effect of any large-scale magnetic field is the introduction of a preferred direction, which is clearly manifested by the eigenvalues of the Π_{ab} tensor. According to equation (5), these eigenvalues are $-\frac{2}{3}H^2$ along the magnetic direction and $\frac{1}{3}H^2$ in the orthogonal plane. In other words there is a negative pressure, a *tension*, parallel to the field lines. Thus, we see the prominent role played by the magnetic lines of force. Each small flux tube is like a rubber band under tension, and infinitely elastic [26]. We will return to this property in sections 5 and 7, when discussing the field effects on the kinematics of the universe and on the growth of density perturbations.

Given that $(\Pi_{ab}\Pi^{ab})^{1/2} \sim H^2$, the dimensionless energy density ratio $h = H^2/\mu$ offers a particularly useful measure of the magnetically induced anisotropy, provided that matter is distributed isotropically. Clearly, the degree of distortion inflicted on the background isotropy by the magnetic presence depends on the field's strength relative to the fluid, that is on the value of h with respect to unity. When the field is relatively weak, that is for $h \ll 1$, one might treat the background universe as isotropic to leading order. Strictly speaking, however, only anisotropic spacetimes can naturally accommodate large-scale cosmological magnetic fields.

Let us assume that the entire background anisotropy is due to the magnetic presence. This scheme may not represent the most general case, but it helps to isolate the effects of the magnetic stresses. Therefore, to zero order, both shear and Weyl anisotropies result from the presence of the field. Mathematically, we achieve this by demanding that the background magnetic field is aligned in the direction of a shear eigenvector.

$$\sigma_{ab}H^b = \tfrac{2}{3}\lambda H_a\,, \tag{38}$$

where $\tilde{\nabla}_a\lambda = 0$ to zero order. Now, we introduce a shear eigenframe with the background field vector along its x^1-axis. In this frame, hereafter referred to as the magnetic frame, both σ_{ab} and Π_{ab} are simultaneously diagonalizable (recall that H_a is the natural eigenvector of Π_{ab}). Axial symmetry then implies

$$\sigma_{ab} = -\frac{\lambda}{H^2}\Pi_{ab}\,, \tag{39}$$

thus guaranteeing that shear anisotropies result directly from the magnetic stresses. The above resembles the Navier–Stokes equation, so that $H^2/2\lambda$ may be thought of as the magneto-shear viscocity coefficient†. Following equation (38), the shear eigenvalues are $\frac{2}{3}\lambda$, parallel to the field vector, and $-\frac{1}{3}\lambda$ orthogonal to it. Also, relations (38) and (39) imply that

$$\sigma^2 \equiv \tfrac{1}{2}\sigma_{ab}\sigma^{ab} = \tfrac{1}{3}\lambda^2\,, \tag{40}$$

to zero order.

We monitor the impact of the shear, relative to the average volume expansion, via the dimensionless shear-impact parameter $\zeta \equiv \lambda/\Theta$. The latter measures the degree of the kinematic anisotropy. As an example, consider the expansion rate along the field lines. Then, on using (38), we find

$$\Theta_{ab}n^a n^b = \tfrac{1}{3}(1+2\zeta)\Theta\,, \tag{41}$$

where $\Theta_{ab} \equiv \sigma_{ab} + \frac{1}{3}\Theta h_{ab}$ is the zero-order expansion tensor and $n_a \equiv H_a/\sqrt{H^2}$ is the unit vector in the direction of the field. Although the shear effect depends on the specific form of ζ, it becomes insignificant when $\zeta \ll 1$. The same condition guarantees that the shear impact is also weak normally to the field lines. In fact, result (40) ensures that the shear effect, relative to the volume expansion, is always negligible provided $\zeta \ll 1$‡.

4.2. The evolution

In this section we discuss key features of the background Bianchi I spacetime, assuming that the magnetic field is the sole source of its anisotropy. Unless stated otherwise, the equations presented here are obtained from the linearized formulae of section 3 by keeping their zero-order terms only.

4.2.1. Energy conservation. The fluid and the magnetic energy densities obey the familiar conservation laws. These are respectively represented by the equations of continuity

$$\dot{\mu} = -(1+w)\mu\Theta \tag{42}$$

for the fluid, and

$$\left(H^2\right)^{\cdot} = -\tfrac{4}{3}(1-\zeta)\Theta H^2 \tag{43}$$

for the field. The latter derives from equation (22) on using condition (38).

4.2.2. Volume expansion. To zero-order, result (40) simplifies Raychaudhuri's equation (12) to

$$\dot{\Theta} = -\tfrac{1}{3}\left(1+2\zeta^2\right)\Theta^2 - \tfrac{1}{2}\mu(1+3w+h) + \Lambda\,, \tag{44}$$

where the magnetic and the shear effects, respectively represented by h and ζ, assist the matter in focusing the fluid flow-lines. Similarly to the FRW models, we may introduce an average scale factor (S) via $\dot{S}/S = \frac{1}{3}\Theta$, which also defines the mean Hubble rate.

† The direct dependence of the shear, and subsequently of the electric Weyl tensor, on the magnetic stresses holds in the background only. No relation analogous to (38) or (39), between any two anisotropic sources, is ever imposed in the perturbed spacetime.

‡ The overall weakness of the shear is also necessary, if the co-moving gradients that describe the inhomogeneities are to maintain the desirable geometrical and physical interpretation they have within an almost-FRW universe (see the appendix).

4.2.3. Shear anisotropies. The spatial flatness of the Bianchi I spacetime and the Gauss–Godazzi equation (see equation (83) in [5]) provide an extra evolution equation for the background shear

$$\dot{\sigma}_{ab} = -\Theta\sigma_{ab} + \Pi_{ab}\,, \tag{45}$$

thus supplementing the standard zero-order propagation formula for the shear. The latter is obtained from equation (14) by means of condition (39) and has the form

$$\dot{\sigma}_{ab} = -\tfrac{2}{3}\left(1 + \tfrac{1}{2}\zeta\right)\Theta\sigma_{ab} + \tfrac{1}{2}\Pi_{ab} - E_{ab}\,, \tag{46}$$

which guarantees that the zero-order shear evolution is shaped by magnetic anisotropies and by long-range gravitational forces. The flatness of the unperturbed spatial sections and equation (29) also supply a zero-order expression for the magnitude of the shear tensor

$$\sigma^2 = -\mu\left(1 + \tfrac{1}{2}h\right) + \tfrac{1}{3}\Theta^2 - \Lambda\,, \tag{47}$$

which is simply the background Friedmann equation. Note that on using equation (40) the above can be written as

$$\mu\left(1 + \tfrac{1}{2}h\right) - \tfrac{1}{3}\Theta^2\left(1 - \zeta^2\right) + \Lambda = 0\,, \tag{48}$$

an expression that will prove useful later. Finally, employing (39) and (40), the zero-order component of equation (30) gives

$$\dot{\sigma} = -\Theta\sigma - \tfrac{1}{\sqrt{3}}\,\mu h\,, \tag{49}$$

which, on using (40), is recast as

$$\dot{\lambda} = -\Theta\lambda - \mu h\,, \tag{50}$$

that monitors the evolution of the λ-parameter.

4.2.4. Magnetic stresses. Equations (45) and (46) combine to provide the zero-order expression

$$\Pi_{ab} = \tfrac{2}{3}(1 - \zeta)\Theta\sigma_{ab} - 2E_{ab}\,, \tag{51}$$

which is also obtained by substituting (39) into the background component of equation (23). The above provides a unique relation between Π_{ab}, σ_{ab} and E_{ab}. This fine-tuning of the unperturbed anisotropic sources results from the spatial flatness of the Bianchi I model and ensures that equations (45), (46) are simultaneously satisfied. Using condition (39), equations (24) gives

$$\dot{\Pi}_{ab} = -\tfrac{4}{3}(1 - \zeta)\Theta\Pi_{ab} \tag{52}$$

to zero order. The latter is also obtained from equation (39) by means of (39), (43) and (49).

4.2.5. Weyl anisotropies. In the unperturbed universe, equation (51) guarantees the direct dependence of the Weyl anisotropies on the magnetic stresses. In fact, equations (38) and (51) confirm that H_a is also an eigenvector of E_{ab}, while equations (39) and (51) ensure that

$$E_{ab} = -\frac{\eta}{H^2}\Pi_{ab}\,, \tag{53}$$

where $\eta \equiv \tfrac{1}{2}\mu h + \tfrac{1}{3}(1 - \zeta)\lambda\Theta$ has the dimensions of E_{ab}. The background evolution of the electric Weyl tensor is given by the zero-order part of equation (31)

$$\dot{E}_{\langle ab\rangle} = -\Theta\left(E_{ab} - \tfrac{1}{2}\Pi_{ab}\right) + 3\sigma_{c\langle a}\left(E^c{}_{b\rangle} - \tfrac{1}{2}\Pi^c{}_{b\rangle}\right) - \tfrac{1}{2}\mu(1 + w)\sigma_{ab}\,. \tag{54}$$

Hence, condition (38) and axial symmetry guarantee the direct dependence of the whole background anisotropy on the magnetic presence. Once the parameter ζ has been established, the zero-order evolution of both σ_{ab} and E_{ab} is determined by equation (52).

4.3. The Alfvén speed

Disturbances in a magnetized medium propagate via the Alfvén speed. For an arbitrarily strong magnetic field we define

$$c_{\rm a}^2 \equiv \frac{h}{1+w+h}\,, \tag{55}$$

so that $c_{\rm a}^2 < 1$ always. This coincides with the definitions in [5] and [10] for a weak field in a dust-dominated universe (i.e. $h \ll 1$ and $w \equiv 0$), but deviates when the fluid has non-vanishing pressure. The reader must take this difference into account when comparing our pre-recombination equations to those in [5] or [10].

On using equations (42) and (43), we obtain the evolution law for the ratio between the magnetic and the fluid energy densities

$$\dot{h} = -\tfrac{1}{3}(1-3w-4\zeta)\Theta h\,. \tag{56}$$

Note that when $w \neq 1/3$ and the field is weak (i.e. $\zeta \ll 1$), the last term in the right-hand side of (56) is negligible. Hence, in the dust era, h evolves exactly as in weakly magnetized FRW cosmologies. In the radiation era, however, $w = 1/3$ and therefore $\dot{h} = 4\lambda h/3$. In this case the behaviour of h depends on the specific form of λ, that is on the relation between σ_{ab} and Π_{ab}. This complication, which was first pointed out in [27], corresponds to the 'critical case' discussed in [28, 29].

The Alfvén speed also shows the same subtle behaviour during the radiation epoch. Indeed, combining definition (55) with equations (11) and (56) we obtain

$$\left(c_{\rm a}^2\right)^{\cdot} = -\tfrac{1}{3}\left(1-3c_{\rm s}^2-4\zeta\right)\left(1-c_{\rm a}^2\right)\Theta c_{\rm a}^2\,. \tag{57}$$

Clearly, when $c_{\rm s}^2 = 1/3$, the evolution of the Alfvén speed couples to the shear impact parameter, even for $c_{\rm a}^2 \ll 1$.

5. The case of a barotropic fluid

The set of equations (35)–(37), which holds for a general perfect fluid with pressure, does not contain an evolution formula for the pressure gradients. The reason being that the propagation of Y_a is determined directly form (35), once the material content of the universe has been specified. Here we consider a universe filled with a single barotropic perfect fluid. This has the simple equation of state $p = p(\mu)$ which implies that

$$SY_a = \mu c_{\rm s}^2 \mathcal{D}_a\,. \tag{58}$$

5.1. The scalar variables

The gauge-invariant variables defined next, allow us to distinguish between inhomogeneities of different types. To begin with, we decompose the co-moving spatial gradient of $\mathcal{D}_a$ as

$$S\tilde{\nabla}_b\mathcal{D}_a \equiv \Delta_{ab} = \tfrac{1}{3}\Delta h_{ab} + \varepsilon_{abc}W^c + \Sigma_{ab}\,, \tag{59}$$

where $\Delta \equiv \Delta^a{}_a$, $W_a \equiv \frac{1}{2}\varepsilon_{abc}\Delta^{bc}$ and $\Sigma_{ab} \equiv \Delta_{\langle ab\rangle}$. The scalar Δ describes the average gravitational clumping of the fluid, the vector W_a represents rotational instabilities and Σ_{ab} describes volume-true infinitesimal distortions in clustering [18, 30]. We also apply similar decompositions to $\mathcal{Z}_{ab} \equiv S\tilde{\nabla}_b\mathcal{Z}_a$, for the expansion inhomogeneities, and to $\mathcal{B}_{ab} \equiv S\tilde{\nabla}_b\mathcal{B}_a$, which is associated with gradients in the magnetic energy density.

Given that density perturbations (i.e. scalar matter aggregations) are the focus of this analysis the following scalars are crucial

$$\Delta \equiv \Delta^a{}_a = \frac{S^2}{\mu}\tilde{\nabla}^2\mu\,, \tag{60a}$$

$$\mathcal{Z} \equiv \mathcal{Z}^a{}_a = S^2\tilde{\nabla}^2\Theta\,, \tag{60b}$$

and

$$\mathcal{B} \equiv \mathcal{B}^a{}_a = \frac{S^2}{H^2}\tilde{\nabla}^2 H^2\,, \tag{60c}$$

where the equalities hold in the linear regime only. The above variables, which are all gauge-invariant, are supplemented by the first-order scalar

$$\mathcal{K} \equiv S^2\mathcal{R}\,, \tag{61}$$

representing spatial curvature perturbations. Note that only Δ, $\mathcal{B}$ and $\mathcal{K}$ are dimensionless, while $\mathcal{Z}$ has inverse time dimensions.

5.2. The kinematic evolution

5.2.1. Acceleration. Employing the barotropic relation (58), the momentum density conservation law (see equation (10)) gives

$$\dot{u}_a = -\frac{c_{\rm s}^2}{(1+w)S}\left(1-\tfrac{2}{3}c_{\rm a}^2\right)\mathcal{D}_a - \frac{c_{\rm a}^2}{\mu h}\varepsilon_{abc}H^b{\rm curl}\,H^c + \frac{c_{\rm s}^2c_{\rm a}^2}{\mu h(1+w)S}\Pi_{ab}\mathcal{D}^b\,, \tag{62}$$

with the background anisotropy reflected in the last term on the right. Now, expression (5) recasts equation (62) into

$$\dot{u}_a = -\frac{c_{\rm s}^2}{(1+w)S}\left(1-c_{\rm a}^2\right)\mathcal{D}_a - \frac{c_{\rm a}^2}{\mu h}\varepsilon_{abc}H^b{\rm curl}\,H^c - \frac{c_{\rm s}^2c_{\rm a}^2}{(1+w)S}n_an^b\mathcal{D}_b\,, \tag{63}$$

where the last term acts only along the direction of the field and the second last is transverse. Hence, the acceleration components are

$$\dot{u}_a = \begin{cases} -\frac{c_{\rm s}^2}{(1+w)S}\mathcal{D}_a & \text{parallel to } H_a\,, \\ -\frac{c_{\rm s}^2}{(1+w)S}\left(1-c_{\rm a}^2\right)\mathcal{D}_a - \frac{c_{\rm a}^2}{\mu h}\varepsilon_{abc}H^b{\rm curl}\,H^c & \text{orthogonal to } H_a\,, \end{cases} \tag{64}$$

confirming that the field does not affect the fluid motion along its lines of force. Note that when the field is weak (i.e. for $c_{\rm a}^2 \ll 1$), the acceleration components reduce to their FRW counterparts (compare to equation (31) in [10]).

Equation (62) verifies that, in the absence of fluid pressure (i.e. when $c_{\rm s}^2 \equiv 0$), $\dot{u}_a$ depends only on magnetic gradients and is always normal to the field vector. Clearly, this is not the case when $p \neq 0$. For non-vanishing pressure, one might also say that $\dot{H}_a$ is no longer normal to the fluid flow. We verify these statements by simply contracting (62) with H_a. We find that

$$H^a\dot{u}_a = \dot{H}^au_a = -\frac{c_{\rm s}^2}{(1+w)S}H^a\mathcal{D}_a\,, \tag{65}$$

where generally $H^a\mathcal{D}_a \neq 0$. The above condition, which is independent of the field's strength, is identical to that obtained by the almost-FRW treatment [10].

According to the kinematic equations (12)–(14), the acceleration gradients affect all aspects of the expansion. They also have an impact on the spatial curvature (see equation

(30)). Consequently, the following linear decomposition is particularly useful

$$\tilde{\nabla}_b\dot{u}_a = -\frac{c_{\rm s}^2}{(1+w)S^2}\left(1-\tfrac{2}{3}c_{\rm a}^2\right)\Delta_{ab} - \frac{c_{\rm a}^2}{2S^2}\mathcal{B}_{ab} + \frac{c_{\rm a}^2}{\mu h S}H^c\tilde{\nabla}_c\mathcal{M}_{ab}$$
$$-\tfrac{4}{9}(1-\zeta)c_{\rm a}^2\Theta\varepsilon_{abc}\omega^c + \tfrac{1}{3}c_{\rm a}^2\mathcal{R}_{ab} + \frac{c_{\rm s}^2c_{\rm a}^2}{\mu h(1+w)S^2}\Pi_a{}^c\Delta_{cb}$$
$$-\frac{4c_{\rm a}^2\Theta}{3\mu h}(1-\zeta)\Pi_a{}^c\varepsilon_{bcd}\omega^d - \frac{c_{\rm a}^2}{\mu h}\Pi^{cd}\mathcal{R}_{acbd}\,, \qquad (66)$$

where the last three terms result from the anisotropic background. Note the magneto-curvature contributions in the right-hand side of the above. Their presence manifests the vectorial nature of the magnetic field and underlines the coupling between magnetism and geometry. Using equation (5), the two magneto-curvature terms reduce to one given by $c_{\rm a}^2H^cH^d\mathcal{R}_{acbd}/\mu h$. The latter represents a force transverse to the background field lines, as the symmetries of $\mathcal{R}_{abcd}$ confirm. This term, which appears inevitably when the 3-gradients of the field vector commute (see equation (B9) in [16]), can trigger a range of rather unexpected effects.

Next, the trace, the skew part and the symmetric trace-free component of equation (66) will be used to analyse the magnetohydrodynamical effects upon the kinematics, the dynamics and the geometry of our cosmological model.

5.2.2. Deceleration parameter. On using expression (5), the linearized trace of (66) gives

$$A = -\frac{c_{\rm s}^2}{(1+w)S^2}\left[\Delta_{ab}n^an^b + \left(1-c_{\rm a}^2\right)\Delta_{ab}v^{ab}\right] - \frac{c_{\rm a}^2}{2S^2}\mathcal{B} + c_{\rm a}^2\mathcal{R}_{ab}n^an^b\,, \qquad (67)$$

where $A = \tilde{\nabla}^a\dot{u}_a$ to first order (see equation (12)) and $v_{ab} \equiv h_{ab} - n_an_b$ projects into the 2-plane orthogonal to H_a. The scalars $\Delta_{ab}n^an^b$ and $\mathcal{R}_{ab}n^an^b$ respectively represent density and 3-curvature perturbations parallel to the background field lines. On the other hand, $\Delta_{ab}v^{ab}$ describes matter aggregations orthogonal to H_a, so that $\Delta_{ab}n^an^b + \Delta_{ab}v^{ab} = \Delta$. Note that $S_{ab}n^an^b + S_{ab}v^{ab} = S^a{}_a$ for any spacelike tensor S_{ab}, while $S_{ab}n^an^b = S_{11}$ and $S_{ab}v^{ab} = S_{22} + S_{33}$ in the magnetic frame.

Substituting the above result into equation (12) we obtain the Raychaudhuri equation for a magnetized almost Bianchi I universe filled with a perfectly conducting ideal fluid. The resulting formula is then transformed into the following alternative expression

$$\tfrac{1}{3}\Theta^2{\rm q} = \tfrac{1}{2}\mu(1+3w+h) + \frac{c_{\rm s}^2}{(1+w)S^2}\left[\Delta_{ab}n^an^b + \left(1-c_{\rm a}^2\right)\Delta_{ab}v^{ab}\right] + \frac{c_{\rm a}^2}{2S^2}\mathcal{B}$$
$$- c_{\rm a}^2\mathcal{R}_{ab}n^an^b + 2\sigma^2 - \Lambda\,, \qquad (68)$$

where ${\rm q} \equiv -\ddot{S}S/\dot{S}^2$ is the dimensionless deceleration parameter. Clearly, the sign of the right-hand side determines the state of the expansion, with positive terms slowing it down and negative accelerating it. As expected, the field slows the universe down by adding to the total energy density, and the shear has a similar decelerating impact. Comparing to equation (36) in [10], we see that the effect from local increases in the magnetic energy density (i.e. for $\mathcal{B} > 0$) remains unchanged. However, the anisotropic background has left its signature through a number of direction-dependent effects. In particular, for relatively strong magnetic fields (i.e. when $c_{\rm a}^2 \sim 1$), matter aggregations contribute differently along the field direction than perpendicularly to H_a. In fact, as the magnetic field gets stronger (i.e. as $c_{\rm a}^2 \to 1$), the decelerating effect from $\Delta_{ab}v^{ab}$ tends to zero.

According to equation (68), only curvature perturbations parallel to the magnetic force lines have an influence on the expansion rate. This effect was originally identified in [10], although there, the isotropy of the background FRW model meant that it was independent of direction. Here, as in [10], negative curvature perturbations add to the total deceleration, whereas a positive $\mathcal{R}_{ab}n^an^b$ tends to accelerate the universe. This sounds odd at first, given that positive curvature is traditionally associated with gravitational collapse. However, the tension carried along the direction of the field, means that small magnetic flux tubes behave like elastic rubber bands [26]. When coupled to geometry, this property of the field gives rise to a curvature stress that opposes the action of the magnetic energy density gradients. Hence, in equation (68), *the magneto-geometrical term reacts to local changes in the spatial curvature by modifying the expansion rate of the perturbed region accordingly*. Intuitively, the coupling between magnetism and geometry has injected the elastic properties of the field into space itself. Note that the effect of the relativistic magneto-curvature term $c_{\mathrm{a}}^2\mathcal{R}_{ab}n^an^b$ in equation (68), bares a striking resemblance to the classical (non-relativistic) curvature stress exerted by a distorted field. Following [26], the tension from field lines with a local radius of curvature R, exerts a transverse force $\sim H^2/R$ per unit volume. Here, the distortion in the field pattern is triggered by perturbations in the spatial geometry itself.

Comparing to equation (36) in [10], we note that the magneto-curvature effect identified in equation (68) reduces to that predicted by the FRW-based analysis when $\mathcal{R}_{ab}n^an^b = \frac{1}{3}\mathcal{R}$. Thus, for a weak field, the two equations become indistinguishable when the curvature perturbation along H_a takes the (directional) average value. Generally, however, $\mathcal{R}_{ab}n^an^b$ has a rather complicated evolution that also depends on magnetic, shear and Weyl contributions. In fact, contracting equation (26) with the field vector we find

$$\mathcal{R}_{ab}n^an^b = \tfrac{1}{3}\mathcal{R} - \tfrac{1}{3}\mu h - \tfrac{2}{3}\sigma^2 - \tfrac{1}{3}(1-2\zeta)\Theta\sigma_{ab}n^an^b + E_{ab}n^an^b\,, \tag{69}$$

where $\sigma_{ab}n^an^b$ and $E_{ab}n^an^b$ are respectively the components of the shear and the electric Weyl tensors parallel to the filed lines.

The anisotropy of the Bianchi I background has introduced directional effects to the expansion dynamics, along and normally to the background magnetic field lines. As we shall show next, similar direction-dependent complications affect almost every aspect of the evolution. None of these effects could have been identified by an FRW-based treatment, where the isotropy of the unperturbed spacetime allowed only the average of these effects to be measured.

5.2.3. Vorticity. Substituting the background relation (39) to the vorticity propagation equation (13) we have

$$\dot{\omega}_a = -\tfrac{2}{3}\left(1+\tfrac{1}{2}\zeta\right)\Theta\omega_a - \tfrac{1}{2}\mathrm{curl}\,\dot{u}_a\,, \tag{70}$$

where ζ always represents the magnetically induced kinematic anisotropies. Using equations (5), (28), (38), (39), (43) and (59), the commutator (B8) from [16], the symmetries of $\mathcal{R}_{abcd}$ (see equation (81) in [18]) and the linear relation $W_a = -(1+w)\Theta S^2\omega_a$, we obtain an expression for the skew part of equation (66). Then, given that $\mathrm{curl}\,\dot{u}_a = \varepsilon_{abc}\tilde{\nabla}^b\dot{u}^c$, the vorticity propagation equation is recast as

$$\begin{aligned}\dot{\omega}_a &+ \tfrac{2}{3}\left[\left(1+\tfrac{1}{2}\zeta\right) - \tfrac{3}{2}\left(1-\tfrac{1}{2}c_{\mathrm{a}}^2\right)c_{\mathrm{s}}^2\right]\Theta\omega_a \\ &= -\frac{c_{\mathrm{a}}^2}{2\mu h}H^b\tilde{\nabla}_b\mathrm{curl}\,H_a + \frac{c_{\mathrm{s}}^2c_{\mathrm{a}}^2}{2\mu h(1+w)S^2}\varepsilon_{abc}\Pi^{bd}\Sigma^c{}_d\,,\end{aligned} \tag{71}$$

where $\Sigma_{ab} = (S^2/\mu)\tilde{\nabla}_{\langle a}\tilde{\nabla}_{b\rangle}\mu$ to first order. The above holds for an electrically neutral medium with $H^a\omega_a = 0$. As predicted by the almost-FRW treatment (see equation (37) in [10]), magnetism becomes a source of vorticity when $\mathrm{curl}\,H_a$ varies in the direction of the field vector. Note that before recombination distortions in the density distribution can also generate vorticity. According to equation (71), a relatively strong magnetic field with ζ, $c_{\rm a}^2 \sim 1$ can modify the vortex dilution caused by the volume expansion.

5.2.4. Shear. The symmetric trace-free part of (66) transforms equation (14) into

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - \frac{c_{\rm s}^2}{(1+w)S^2}\left(1-\tfrac{2}{3}c_{\rm a}^2\right)\Sigma_{ab} - \frac{c_{\rm a}^2}{2S^2}\mathcal{B}_{\langle ab\rangle} + \tfrac{1}{3}c_{\rm a}^2\mathcal{R}_{\langle ab\rangle} + \tfrac{1}{2}\Pi_{ab} + \frac{c_{\rm a}^2}{\mu h S}H^c\tilde{\nabla}_c\mathcal{M}_{\langle ab\rangle} - E_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} + \frac{c_{\rm s}^2c_{\rm a}^2}{\mu h(1+w)S^2}\Pi_{c\langle a}\Delta^c{}_{b\rangle} - \frac{c_{\rm a}^2}{\mu h}\Pi_{cd}\mathcal{R}_{\langle a}{}^c{}_{b\rangle}{}^d - \frac{4c_{\rm a}^2\Theta}{3\mu h}(1-\zeta)\Pi^c{}_{\langle a}\varepsilon_{b\rangle cd}\omega^d\,. \tag{72}$$

Comparing with equation (38) in [10], we identify the effects of the background anisotropy in the last four terms. The extra magnetic input to the Σ_{ab}-term, which is negligible for relatively weak fields, results from the magnetic contribution to the total energy density of the universe. Any other changes are due to differences in the Alfvén speed definitions.

An additional useful relation is the propagation equation of the shear magnitude. We obtain it by contracting equation (83) in [5] with σ_{ab}. We then employ results (66) and (67), the symmetries of $\mathcal{R}_{abcd}$ and the constraint $\mathcal{M}^a{}_a = 0$ to find

$$\left(\sigma^2\right)^{\cdot} = -2\Theta\sigma^2 - \mu h\sigma_{ab}n^an^b - \tfrac{1}{3}\zeta\Theta\left[2\left(1+\tfrac{1}{2}c_{\rm a}^2\right)\mathcal{R}_{ab}n^an^b - \mathcal{R}_{ab}v^{ab}\right] - \frac{\zeta c_{\rm s}^2\Theta}{3(1+w)S^2}\left[2\Delta_{ab}n^an^b - \left(1-c_{\rm a}^2\right)\Delta_{ab}v^{ab}\right] - \frac{\zeta c_{\rm a}^2\Theta}{6S^2}\left(2\mathcal{B}_{ab}n^an^b - \mathcal{B}_{ab}v^{ab}\right)\,, \tag{73}$$

where $\sigma_{ab}n^an^b$ satisfies equation (69). Also, $\mathcal{B}_{ab}n^an^b$ and $\mathcal{B}_{ab}v^{ab}$ respectively represent magnetic energy density gradients parallel and orthogonal to the background field lines. Similarly, $\mathcal{R}_{ab}v^{ab}$ is the 3-curvature perturbation in the plane normal to H_a. Note that the anisotropic effects, manifested by the direction-dependent terms of the right-hand side, remain even when the field is relatively weak. The above result will prove useful in section 5, when studying spatial curvature perturbations.

5.3. The dynamic evolution

5.3.1. Perturbations in the matter density. In the linear regime, magnetized density gradients propagate according to equation (35). For a barotropic perfectly conducting medium, the latter becomes

$$\dot{\mathcal{D}}_a = \left(w - c_{\rm s}^2c_{\rm a}^2 + \tfrac{1}{3}\zeta\right)\Theta\mathcal{D}_a - (1+w)\mathcal{Z}_a + \frac{\Theta S}{\mu}\left(1-c_{\rm a}^2\right)\varepsilon_{abc}H^b\mathrm{curl}\,H^c - \left(c_{\rm s}^2c_{\rm a}^2 - \zeta\right)\Theta n_an^b\mathcal{D}^b\,, \tag{74}$$

on using (5), the zero-order relation (39), the barotropic expression (62) and result (65). We can now see that the field affects density perturbations along as well as normally to its own force lines. Parallel to H_a, the density gradients are influenced by the magnetically induced kinematical anisotropies, with their presence manifested by the shear impact parameter ζ.

The evolution of linear matter aggregations is determined by the 3-divergence of equation (74). Using the commutation law between time derivatives and 3-gradients (see formula (B18) in [16]), we obtain

$$\dot{\Delta} = \left(w - \tfrac{4}{3}\zeta\right)\Theta\Delta_{ab}n^an^b + \left(w - c_{\rm s}^2c_{\rm a}^2 + \tfrac{2}{3}\zeta\right)\Theta\Delta_{ab}v^{ab} - (1+w)\mathcal{Z} + \tfrac{1}{2}(1+w)c_{\rm a}^2\Theta\mathcal{B} - (1+w)c_{\rm a}^2\Theta\mathcal{K}_{ab}n^an^b\,. \tag{75}$$

Note that different quantities contribute differently parallel than perpendicular to the field lines. Also, the magneto-shear effects, which are imprinted in the impact parameter ζ, vanish when $\Delta_{ab}n^an^b$ takes the directional average value $\frac{1}{3}\Delta$.

5.3.2. Perturbations in the expansion. Starting from equation (36) we employ expression (5), the zero-order relations (39), (38) and (47), together with the linear equations (62), (65) and (67) to arrive at the propagation formula of the expansion gradients in a barotropic fluid environment

$$\begin{aligned}\dot{\mathcal{Z}}_a = &-\tfrac{2}{3}\left(1-\tfrac{1}{2}\zeta\right)\Theta\mathcal{Z}_a - \tfrac{1}{2}\mu\left[1+4c_{\rm s}^2c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]\mathcal{D}_a\\ &-\tfrac{1}{2}\mu h\mathcal{B}_a + \tfrac{3}{2}\left[1-\tfrac{4}{3}c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]S\varepsilon_{abc}H^b\text{curl}\,H^c\\ &-\frac{c_{\rm s}^2}{(1+w)S}\tilde{\nabla}_a\left[\Delta_{ab}n^an^b+\left(1-c_{\rm a}^2\right)\Delta_{ab}v^{ab}\right]-\frac{c_{\rm a}^2}{2S}\tilde{\nabla}_a\mathcal{B}\\ &-\zeta\Theta n_an^b\mathcal{Z}_b+\frac{3\mu hc_{\rm s}^2}{2(1+w)}\left[1-\tfrac{4}{3}c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]n_an^b\mathcal{D}_b\\ &+c_{\rm a}^2S\tilde{\nabla}_a\left(\mathcal{R}_{ab}n^an^b\right)-2\zeta\Theta S\tilde{\nabla}_a\left(\sigma_{ab}n^an^b\right)\,,\end{aligned}\tag{76}$$

where the dimensionless parameter $r \equiv \frac{1}{h}\left[1-\frac{1}{\mu}\left(\frac{1}{3}\Theta^2-\Lambda\right)\right]$ will prove useful later.

Taking the co-moving spatial divergence of equation (76), using the commutator between time derivatives and spatial gradients of first order spacelike vectors (see equation (B18) in [16]), together with expression (5) and the background relation (39) we find that, to first order,

$$\begin{aligned}\dot{\mathcal{Z}} = &-\tfrac{2}{3}\Theta\left[(1+2\zeta)\mathcal{Z}_{ab}n^an^b+(1-\zeta)\mathcal{Z}_{ab}v^{ab}\right]\\ &-\tfrac{1}{2}\mu\left\{\left[1+\frac{c_{\rm s}^2h}{1+w}(1+6r)\right]\Delta_{ab}n^an^b+\left[1+4c_{\rm s}^2c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]\Delta_{ab}v^{ab}\right\}\\ &+\tfrac{1}{4}\mu h\left[1-4c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]\mathcal{B}-\tfrac{3}{2}\mu h\left[1-\tfrac{4}{3}c_{\rm a}^2\left(1+\tfrac{3}{2}r\right)\right]\mathcal{K}_{ab}n^an^b\\ &-\frac{c_{\rm s}^2}{1+w}\tilde{\nabla}^2\left[\Delta_{ab}n^an^b+\left(1-c_{\rm a}^2\right)\Delta_{ab}v^{ab}\right]-\tfrac{1}{2}c_{\rm a}^2\tilde{\nabla}^2\mathcal{B}\\ &+c_{\rm a}^2\tilde{\nabla}^2\left(\mathcal{K}_{ab}n^an^b\right)-2\zeta\Theta S^2\tilde{\nabla}^2\left(\sigma_{ab}n^an^b\right)\,,\end{aligned}\tag{77}$$

where $\mathcal{Z}_{ab}n^an^b$ and $\mathcal{Z}_{ab}v^{ab}$ are respectively the expansion gradients parallel and orthogonal to the zero-order magnetic vector. Note that most of the anisotropic effects in (76), (77) die away at the weak magnetic field limit. Those that remain are the directional dependence of the curvature contribution, together with the curvature and shear Laplacians.

5.3.3. Perturbations in the magnetic energy density. For the linear evolution of the magnetic energy density gradients we start from the definition of $\mathcal{B}$. Taking the time derivative of equation (60*c*), we employ expression (5), the linear conservation law (22), the background relations (39) (guaranteeing that $\sigma_{ab}n^an^b = \frac{2}{3}\lambda$ to zero-order) and (43), the linear result (67) and the propagation equation (75) to find

$$\begin{aligned}\dot{\mathcal{B}} = &\frac{4}{3(1+w)}\dot{\Delta}-\frac{4\Theta}{3(1+w)}\Big\{\left[w-c_{\rm s}^2(1-\zeta)-\tfrac{4}{3}\zeta\right]\Delta_{ab}n^an^b\\ &+\left[w-c_{\rm s}^2\left(1-\zeta\left(1-c_{\rm a}^2\right)\right)+\tfrac{2}{3}\zeta\right]\Delta_{ab}v^{ab}\Big\}\\ &-\tfrac{4}{3}\Theta\left[\zeta\left(1+\tfrac{1}{2}c_{\rm a}^2\right)\mathcal{B}_{ab}n^an^b-\tfrac{1}{2}\zeta\left(1-c_{\rm a}^2\right)\mathcal{B}_{ab}v^{ab}\right]\\ &+\tfrac{4}{3}c_{\rm a}^2\zeta\Theta\mathcal{K}_{ab}n^an^b+2S^2\tilde{\nabla}^2\left(\sigma_{ab}n^an^b\right)\,,\end{aligned}\tag{78}$$

where the shear Laplacian is the only effect of the anisotropic background that remains at the weak magnetic field limit. In deriving equation (78) we have employed the linear commutation law

$$\left(\tilde{\nabla}^2 f\right)^{\cdot} = \tilde{\nabla}^2 \dot{f} - \tfrac{2}{3}\Theta\tilde{\nabla}^2 f - 2\sigma^{ab}\tilde{\nabla}_a\tilde{\nabla}_b f + \dot{f}A\,, \tag{79}$$

where $\tilde{\nabla}_a f$ vanishes in the background. The above follows from linearizing the 3-divergence of (B18) in [16].

5.3.4. Perturbations in the spatial curvature. On using the barotropic expressions (67) and (73), equation (30) is recast as

$$\begin{aligned}\dot{\mathcal{R}} = -\tfrac{2}{3}\Theta\left\{\left[1 + 2c_{\rm a}^2\left(1 + \tfrac{1}{2}\zeta\right) + 2\zeta\right]\mathcal{R}_{ab}n^a n^b + (1-\zeta)\mathcal{R}_{ab}v^{ab}\right\} \\ + \frac{4c_{\rm s}^2\Theta}{3(1+w)S^2}\left[(1-\zeta)\Delta_{ab}n^a n^b + \left(1 - c_{\rm a}^2\right)\left(1 + \tfrac{1}{2}\zeta\right)\Delta_{ab}v^{ab}\right] \\ + \frac{2c_{\rm a}^2\Theta}{3S^2}\left[(1-\zeta)\mathcal{B}_{ab}n^a n^b + \left(1 + \tfrac{1}{2}\zeta\right)\mathcal{B}_{ab}v^{ab}\right]\,,\end{aligned} \tag{80}$$

which reduces to equation (40) in [10] at the weak field limit. For a relatively strong field with $c_{\rm a}^2, \zeta \sim 1$, the kinematic anisotropies (represented by ζ) complicate the overall magnetic impact on curvature. One faces a highly anisotropic situation, where disturbances parallel to the background field increase the average curvature, while perturbations orthogonal to H_a have the opposite effect and vice versa. If we ignore all ζ-contributions in equation (80), the magneto-curvature effect on $\mathcal{R}$ is imprinted in the direction-dependent term $2c_{\rm a}^2\mathcal{R}_{ab}n^a n^b$. The latter boosts the smoothing effect of the expansion on 3-curvature perturbations in agreement with [10]. In fact, the two effects coincide when $\mathcal{R}_{ab}n^a n^b = \frac{1}{3}\mathcal{R}$.

It should be emphasized that all our equations refer to an arbitrarily strong magnetic field. So far, the only restriction is that the background field is an eigenvector of the shear tensor, which translates into a simple algebraic dependence of the zero-order shear and electric Weyl tensors on Π_{ij} (see equations (39) and (53)). Next, we will apply these formulae to the case of a relatively weak background magnetic field.

6. The weak magnetic field limit

The observed anisotropy of the CMB imposes strict limits on the strength of any potential primordial magnetic field at the time of decoupling. Also, Helium-4 measurements restrict the magnitude of such field at the time of nucleosynthesis. All point towards a relatively weak magnetic field, with current estimates arguing for a field strength of $\sim 10^{-8}G$ at present. Here, on the basis of these observations, we assume a background magnetic field weak compared to the dominant isotropically distributed matter component (i.e. $h, c_{\rm a}^2 \simeq h/(1+w) \ll 1$). Given that magnetism is the sole source of the background anisotropy, it is reasonable to assume that the shear impact is also weak to zero order (i.e. $\zeta \ll 1$). At this limit, the unperturbed and linear equations simplify considerably.

6.1. Background evolution

Given that the source of the anisotropy is weak, one expects that certain key figures of our Bianchi-I background will approach their Friedmannian counterparts. Indeed, when $\zeta \ll 1$ the magnetic energy density (see equation (43)) evolves unaffected by the background anisotropy

$$\left(H^2\right)^{\cdot} = -\tfrac{4}{3}\Theta H^2\,. \tag{81}$$

The same is also true for the fluid energy density, given the absence of magnetic terms in equation (42)†.

The weakness of the field also means that the average volume expansion of our Bianchi I background approaches that of the Friedmann model. When $h,\ \zeta \ll 1$, the zero-order Raychaudhuri equation (44) gives

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}\mu(1+3w) + \Lambda\,, \tag{83}$$

as in isotropic FRW cosmologies. The generalized Friedmann equation (48) also acquires an FRW profile

$$\mu - \tfrac{1}{3}\Theta^2 + \Lambda = 0\,, \tag{84}$$

given the spatial flatness of the Bianchi I spacetime. This guarantees that the dimensionless parameter r in equations (76), (77) vanishes at the weak field limit.

It should be emphasized that these results do not, in any case, suggest that the unperturbed Bianchi I model has now been reduced to a simple FRW cosmology. Our background universe is still anisotropic, supplied with magnetic stresses, shear and an electric Weyl tensor. However, the weakness of the magnetic field means that, to leading order, the anisotropy no longer affects certain key evolutionary aspects, such as the rate of the average volume expansion.

6.2. Linear evolution

6.2.1. Kinematics. For a weak background magnetic field, Raychaudhuri's equation, written in terms of the deceleration parameter, becomes

$$\tfrac{1}{3}\Theta^2\mathrm{q} = \tfrac{1}{2}\mu(1+3w) + 2\sigma^2 + \frac{c_{\mathrm{s}}^2}{(1+w)S^2}\Delta + \frac{c_{\mathrm{a}}^2}{2S^2}\mathcal{B} - c_{\mathrm{a}}^2\mathcal{R}_{ab}n^an^b - \Lambda\,. \tag{85}$$

Relative to equation (68), all the directional dependences in the contributions from matter aggregations have been smoothed out. On the other hand, comparing to equation (36) in [10], we notice the extra shear term, which adds to the total deceleration. The anisotropic background has also refined the curvature contribution, which is now confined along the field lines only. In agreement with the FRW treatment, the magneto-geometrical term accelerates models with positive local curvature, but slows down negatively curved regions. The effect observed here reduces exactly to that identified by the FRW analysis when $\mathcal{R}_{ab}n^an^b = \frac{1}{3}\mathcal{R}$. Generally, however,

$$\mathcal{R}_{ab}n^an^b = \tfrac{1}{3}\mathcal{R} - \tfrac{1}{3}\mu h - \tfrac{2}{3}\sigma^2 - \tfrac{1}{3}\Theta\sigma_{ab}n^an^b + E_{ab}n^an^b\,. \tag{86}$$

Note that the impact of the σ^2-term in equation (85) becomes negligible provided that the weak field restriction is imposed upon the actual universe, as well as the background.

Given the weakness of the background magnetic field, the vorticity propagation equation reduces to

$$\dot{\omega}_a + \tfrac{2}{3}\left(1 - \tfrac{3}{2}c_s^2\right)\Theta\omega_a = -\frac{1}{2\mu(1+w)}H^b\tilde{\nabla}_b\mathrm{curl}\,H_a + \frac{c_{\mathrm{s}}^2c_{\mathrm{a}}^2}{2\mu h(1+w)S^2}\varepsilon_{abc}\Pi^{bd}\Sigma^c{}_d\,. \tag{87}$$

† For radiation, $w = \frac{1}{3}$ and equation (56) implies that $\dot{h} = \frac{4}{3}\Theta\zeta h$. Also, the shear impact parameter evolves as $\dot{\zeta} = -\frac{1}{3}\Theta(\zeta + h)$, given that $\zeta \equiv \lambda/\Theta$ and using equations (50), (83), (84) and $\Lambda = 0$. Thus, when there is no magnetic field (i.e. $h = 0$), the expansion anisotropy drops rapidly. The field preserves the anisotropy so that ζ does not tend to zero. Instead, $\dot{\zeta} \to 0$ at late times and h experiences a slow logarithmic decay

$$h = \frac{h_0}{1 + 2\ln(t/t_0)}\,, \tag{82}$$

in agreement with the 'quasi-static' solution obtained in [27].

For dust, the above agrees with the FRW equation (37) in [10].

Assuming a weak background magnetic field has very little effect on the shear evolution. In fact, for $c_{\rm a}^2, \zeta \ll 1$ equation (72) gives

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - \frac{c_{\rm s}^2}{(1+w)S^2}\Sigma_{ab} - \frac{c_{\rm a}^2}{2S^2}\mathcal{B}_{\langle ab\rangle} + \tfrac{1}{3}c_{\rm a}^2\mathcal{R}_{\langle ab\rangle} + \tfrac{1}{2}\Pi_{ab} + \frac{c_{\rm a}^2}{\mu h S}H^c\tilde{\nabla}_c\mathcal{M}_{\langle ab\rangle} - E_{ab} - \sigma_{c\langle a}\sigma^c{}_{b\rangle} + \frac{c_{\rm s}^2c_{\rm a}^2}{\mu h(1+w)S^2}\Pi_{c\langle a}\Delta^c{}_{b\rangle} - \frac{c_{\rm a}^2}{\mu h}\Pi_{cd}\mathcal{R}_{\langle a}{}^c{}_{b\rangle}{}^d - \frac{4c_{\rm a}^2\Theta}{3\mu h}\Pi^c{}_{\langle a}\varepsilon_{b\rangle cd}\omega^d\,, \tag{88}$$

which is considerably more complicated than equation (38) in [10]. This is not surprising, given that the shear reflects the intrinsic spatial anisotropy of the Bianchi I spacetime.

6.2.2. Dynamics. At the weak field limit, equation (75) that monitors gradients in the fluid energy density, reduces to†

$$\dot{\Delta} = w\Theta\Delta - (1+w)\mathcal{Z} + \tfrac{1}{2}(1+w)c_{\rm a}^2\Theta\mathcal{B} - (1+w)c_{\rm a}^2\Theta\mathcal{K}_{ab}n^an^b\,. \tag{89}$$

Again, all directional dependences in the contribution of Δ have died away. The difference between the above and its FRW counterpart, given by equation (54) in [10], is in their magneto-curvature terms. The two equations coincide when $\mathcal{K}_{ab}n^an^b = \frac{1}{3}\mathcal{K}$.

When the field is relatively weak, $r = 0$ and equation (77), for the propagation of the expansion gradients simplifies to

$$\dot{\mathcal{Z}} = -\tfrac{2}{3}\Theta\mathcal{Z} - \tfrac{1}{2}\mu\Delta + \tfrac{1}{4}(1+w)\mu c_{\rm a}^2\mathcal{B} - \tfrac{3}{2}(1+w)\mu c_{\rm a}^2\mathcal{K}_{ab}n^an^b - \frac{c_{\rm s}^2}{1+w}\tilde{\nabla}^2\Delta - \tfrac{1}{2}c_{\rm a}^2\tilde{\nabla}^2\mathcal{B} + c_{\rm a}^2\tilde{\nabla}^2\left(\mathcal{K}_{ab}n^an^b\right) - 2\zeta\Theta S^2\tilde{\nabla}^2\left(\sigma_{ab}n^an^b\right)\,, \tag{90}$$

since all the directional differences in the $\mathcal{Z}$, $\mathcal{D}$ and $\mathcal{B}$ contributions have vanished. Nevertheless, the anisotropy of the Bianchi I background is still imprinted in equation (90) via $\mathcal{K}_{ab}n^an^b$ and the new shear term, both of which are direction-dependent. Note that the Laplacian contributions become negligible on wavelengths larger than the Hubble horizon. Also, when $\mathcal{K}_{ab}n^an^b$ takes the average value $\frac{1}{3}\mathcal{K}$ all the differences in the spatial curvature terms, between equation (90) above and equation (55) in [10], disappear.

At the weak field limit, almost all the directional effects in equation (78) are removed. The magnetic energy gradients propagate as

$$\dot{\mathcal{B}} = \frac{4}{3(1+w)}\dot{\Delta} + \frac{4\left(c_{\rm s}^2 - w\right)\Theta}{3(1+w)}\Delta + 2S^2\tilde{\nabla}^2\left(\sigma_{ab}n^an^b\right)\,, \tag{91}$$

where terms quadratic in the smallness parameters $c_{\rm a}^2$ and ζ have been left out. Relative to equation (61) in [10], there is an extra input depending on the shear component parallel to H_a. This effect, which results from the background anisotropy, becomes negligible on superhorizon scales.

Finally, a weak background magnetic field means that the all anisotropic effects in equation (80) are smoothed out. The latter reduces to its FRW counterpart

$$\dot{\mathcal{K}} = \frac{4c_{\rm s}^2\Theta}{3(1+w)}\Delta + \tfrac{2}{3}c_{\rm a}^2\Theta\mathcal{B}\,, \tag{92}$$

† Looking at equation (75), it becomes clear that the ζ-contributions along and transverse to the field lines are relatively weak when $w, c_{\rm s}^2 \neq 0$. Even in the dust era, when $w = 0 = c_{\rm s}^2$, the effect of these terms is negligible. Indeed, on using definition (59) one can verify that $\dot{\Delta} \pm \zeta\Theta\Delta \simeq \dot{\Delta}$ for $\zeta \ll 1$. Similarly, one shows that $\dot{\Delta} \pm c_{\rm a}^2\Theta\Delta \simeq \dot{\Delta}$ when $c_{\rm a}^2 \ll 1$.

where 3-curvature perturbations are now represented by the dimensionless scalar $\mathcal{K} \equiv S^2\mathcal{R}$ (compare to equation (57) in [10]).

7. Particular solutions

7.1. Large scales

On wavelengths larger that the Hubble horizon the Laplacian terms in equations (90) and (91) are negligible. On these scales, the evolution of magnetized density gradients is monitored by the following system of ordinary differential equations

$$\dot{\Delta} = w\Theta\Delta - (1+w)\mathcal{Z} + \tfrac{1}{2}(1+w)c_{\mathrm{a}}^2\Theta\mathcal{B} - (1+w)c_{\mathrm{a}}^2\Theta\mathcal{K}_{ab}n^an^b\,, \tag{93}$$

$$\dot{\mathcal{Z}} = -\tfrac{2}{3}\Theta\mathcal{Z} - \tfrac{1}{2}\mu\Delta + \tfrac{1}{4}(1+w)\mu c_{\mathrm{a}}^2\mathcal{B} - \tfrac{3}{2}(1+w)\mu c_{\mathrm{a}}^2\mathcal{K}_{ab}n^an^b\,, \tag{94}$$

$$\dot{\mathcal{B}} = \frac{4}{3(1+w)}\dot{\Delta} + \frac{4\left(c_{\mathrm{s}}^2 - w\right)\Theta}{3(1+w)}\Delta\,, \tag{95}$$

$$\dot{\mathcal{K}} = \frac{4c_{\mathrm{s}}^2\Theta}{3(1+w)}\Delta + \tfrac{2}{3}c_{\mathrm{a}}^2\Theta\mathcal{B}\,. \tag{96}$$

When $\mathcal{K}_{ab}n^an^b = \frac{1}{3}\mathcal{K}$, the above coincide with the equations obtained by the FRW-based analysis of [10]†. On this occasion, which one might call the case of ‘average’ or ‘isotropic’ curvature perturbations, the predictions of the Bianchi I analysis are identical to those of the FRW-based treatment.

Given that the above system closes whenever $\mathcal{K}_{ab}n^an^b \propto \mathcal{K}$, it is worth considering two additional critical cases. The ‘minimum’ curvature case, when $\mathcal{K}_{ab}n^an^b = 0$, and that of ‘maximum’ curvature with $\mathcal{K}_{ab}n^an^b = \mathcal{K}$ (i.e. for $\mathcal{K}_{ab}v^{ab} = 0$). In the former situation, the equations are clear of any magneto-curvature complexities and the evolution of Δ proceeds as predicted by the FRW treatment. In the case of maximum curvature, however, there is a small numerical difference between the Bianchi I equations and those of the almost-FRW analysis. Note that the isotropic and the zero curvature cases have already been addressed in [10] and subsequently refined in [11]. Here, we will reproduce these results for comparison reasons and concentrate on the case of maximum curvature. Despite the fact that all three cases are rather special, they are quite important. As we shall see next, their study illustrates the effect of the magneto-curvature coupling on the growth of linear density perturbations.

7.1.1. Radiation era. During the radiation era $w = c_{\mathrm{s}}^2 = \frac{1}{3}$, the fluid density falls as $\mu \propto S^{-4}$ and the Alfvén speed can be treated as a constant (i.e. $\left(c_{\mathrm{a}}^2\right)^{\cdot} = 0$)‡. Expressed in terms of the scale factor, to facilitate the solution, the system (93)–(96) can be written as

$$S^2\frac{\mathrm{d}^2\Delta^{(\nu)}}{\mathrm{d}S^2} = 2\Delta^{(\nu)} - \tfrac{4}{3}c_{\mathrm{a}}^2\mathcal{B}^{(\nu)} + 8c_{\mathrm{a}}^2\mathcal{K}^{(\nu)}\,, \tag{97}$$

$$\frac{\mathrm{d}\mathcal{B}^{(\nu)}}{\mathrm{d}S} = \frac{\mathrm{d}\Delta^{(\nu)}}{\mathrm{d}S}\,, \tag{98}$$

† Any numerical differences between our equations and solutions and those of [10, 11] are due to differences in the Alfvén speed definitions.

‡ When radiation dominates the evolution of the background Alfvén speed depends on ζ, the shear impact parameter (see equation (57)). At the weak field limit this dependence translates into a non-linear logarithmic decay for c_{a}^2, identical to that of h (see equation (82)). In any case, the coupling between c_{a}^2 and μ or Θ in equations (93), (94) and (96), ensures that the Alfvén speed behaves as if it was time independent.

$$S\frac{\mathcal{K}^{(\nu)}}{\mathrm{d}S} = \Delta^{(\nu)} + 2c_{\rm a}^2\mathcal{B}^{(\nu)}\,, \tag{99}$$

where the expansion gradients have been decoupled from the equations and we have set $\Lambda = 0$. Also, $\Delta^{(\nu)}$ is the νth harmonic component of Δ and similar decompositions have been applied to the rest of the inhomogeneity scalars. The above has the following power-law solution

$$\Delta^{(\nu)}(S) = \Delta_0^{(\nu)} + \sum_\alpha \Delta_\alpha^{(\nu)} S^\alpha\,, \tag{100}$$

where $\Delta_0^{(\nu)}$, $\Delta_\alpha^{(\nu)}$ are arbitrary positive constants and the parameter α satisfies the cubic equation

$$\alpha^3 - \alpha^2 - 2\left(1 - \tfrac{2}{3}c_{\rm a}^2\right)\alpha - 8c_{\rm a}^2 = 0\,, \tag{101}$$

to lowest order in $c_{\rm a}^2$. This has one positive and two negative roots that correspond to one growing and two decaying modes. For small $c_{\rm a}^2$, the roots are perturbatively found to be

$$\alpha = \begin{cases} -1 + \frac{28}{9}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \\ \\ 2 + \frac{8}{9}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \\ \\ -4c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \end{cases} \tag{102}$$

which implies the following evolution

$$\Delta^{(\nu)} = \Delta_0^{(\nu)} + \Delta_+^{(\nu)} S^{2+\frac{8}{9}c_{\rm a}^2} + \Delta_{1-}^{(\nu)} S^{-1+\frac{28}{9}c_{\rm a}^2} + \Delta_{2-}^{(\nu)} S^{-4c_{\rm a}^2} \tag{103}$$

for the density contrast. In the absence of the magnetic field, namely for $c_{\rm a}^2 = 0$, the above reduces to the standard solution of cosmological density perturbations (see [32] for example). Hence, the field presence has led to a slight increase of the growing mode. As we shall see next, this increase results from the coupling between magnetism and geometry. In particular, for minimum curvature perturbations, $\mathcal{K}_{ab}n^an^b = 0$ and

$$\alpha = \begin{cases} -1 + \frac{4}{9}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \\ \\ 2 - \frac{4}{9}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \end{cases} \tag{104}$$

to lowest order in $c_{\rm a}^2$ [10]. Thus, in the absence of any curvature contributions the pure magnetic effect is to reduce the growth rate of the density contrast. After recombination, an analogous reduction of the growing mode was also identified both by the relativistic analysis (see [10]) and by the Newtonian treatment (see [3]). In [11], this particular effect was attributed to the 'frozen in' property of the field. The latter combines with the dilution of the magnetic force lines caused by the expansion to oppose gravitational collapse. Alternatively, for isotropic curvature perturbations, $\mathcal{K}_{ab}n^an^b = \frac{1}{3}\mathcal{R}$ and

$$\alpha = \begin{cases} -1 + \frac{4}{3}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \\ \\ 2 + \mathcal{O}(c_{\rm a}^4)\,, \\ \\ -\frac{4}{3}c_{\rm a}^2 + \mathcal{O}(c_{\rm a}^4)\,, \end{cases} \tag{105}$$

to lowest order in $c_{\rm a}^2$ [11]. On this occasion the density contrast grows exactly as in non-magnetized cosmologies.

Comparing results (102), (104) and (105) we notice that as the magneto-curvature contribution gets stronger, from $\mathcal{K}_{ab}n^an^b = 0$ to $\mathcal{K}_{ab}n^an^b = \frac{1}{3}\mathcal{K}$ and finally to $\mathcal{K}_{ab}n^an^b = \mathcal{K}$, the growing mode of Δ successively increases. In fact, *by the time the curvature input has taken its maximum value, the pure magnetic effect has been completely reversed*. In this case, the density contrast grows faster than its counterpart in non-magnetized cosmologies. Again, the reason behind this effect is the special form of the anisotropic magnetic stress tensor. The latter ensures that there is a negative pressure, a tension, along the direction the field lines. As mentioned in section 5, when combined with geometry, this property of the field opposes the action of the magnetic energy density gradients (see equations (93), (97)).

Undoubtedly, the weakness of the background field means that the aforementioned magnetic effects are small 'corrections' to the standard evolution of density perturbations. Interestingly, however, whether the field boosts or inhibits gravitational collapse may entirely depend on the coupling between magnetism and geometry.

7.2. Dust era and small scales

In the dust era, equations (93), (94) and (96) guarantee that, to lowest order in $c_{\rm a}^2$, 3-curvature has no effect on the propagation of Δ. In other words, after recombination long-wavelength magnetized density inhomogeneities proceed exactly as predicted by the Friedmannian treatment [10, 11]. Thus, it becomes clear that the FRW framework offers a quite accurate treatment of large-scale magnetized cosmological perturbations in almost all situations.

For a general $\mathcal{K}_{ab}n^an^b$, however, equations (89)–(92) are not adequate to describe the evolution of Δ. In order to close the system, one requires the propagation equation for $\mathcal{K}_{ab}n^an^b$ instead of (92). This, in turn, is immediately coupled to the shear and the electric Weyl tensors via the first-order relation

$$\mathcal{K}_{ab}n^an^b = \tfrac{1}{3}\mathcal{K} - \tfrac{1}{3}S^2\mu h - \tfrac{2}{3}S^2\sigma^2 - \tfrac{1}{3}\Theta S^2\sigma_{ab}n^an^b + S^2E_{ab}n^an^b \qquad (106)$$

obtained from equation (86). It follows that when the anisotropy in the curvature perturbation is 'non-critical' the problem becomes very complicated to address analytically. In this case, as well as on subhorizon scales, one has to resort to numerical methods for solutions. Alternatively, one might employ dynamical system techniques (see [31] for an overview) to obtain qualitative answers. This approach has been employed by [25] to analyse an exact Bianci I cosmological model permeated by source-free large-scale magnetic field. Dynamical system methods were also used in [16] to study qualitatively non-magnetized density inhomogeneities within perturbed Bianchi I cosmologies. Note that one may invoke fluid backreaction arguments and ignore, to first order, the small scale shear effects. In this case the system (89)–(92) reduces to its FRW counterpart on every scale provided $\mathcal{K}_{ab}n^an^b = \frac{1}{3}\mathcal{K}$ [10].

8. Conclusion

We have pursued the study of magnetized cosmological perturbations within a perturbed Bianchi I spacetime, in the first attempt of this nature. So far, the subject has been only addressed within the limits of perturbed FRW cosmologies, given the observed high isotropy of the CMB spectrum. Strictly mathematically speaking, however, a spatially isotropic spacetime cannot naturally accommodate large-scale magnetic fields. Even when the background field is treated as random or weak, there are problems related to the generic anisotropy of the magnetic stresses. Although it is physically plausible that weak fields can be adequately treated within the almost-FRW limit, mathematically speaking the Friedmann spacetimes are unsuitable hosts for cosmic magnetism. In this respect, the spatially anisotropic Bianchi I background provides a better framework for studying the magnetic effects on cosmological perturbations. Moreover, such study can be easily applied to primordial magnetic fields of arbitrary strengths, if required.

The search for mathematical consistency was not the only motivation behind this work. The relativistic, but almost-FRW, treatments of [10, 11] suggested a number of unexpected magnetic effects on the kinematics and the dynamics of the universe. All of them seemed to derive from an intriguing coupling between magnetism and geometry. The potential importance of these magneto-curvature effects meant that a new study, this time in a more natural environment, was necessary. The Bianchi I model, which has long been known to be the simplest anisotropic spacetime able to accommodate cosmological magnetic fields, provided just this environment.

We began by assuming that the entire background anisotropy was due to the magnetic presence, in order to isolate the anisotropic effects of the field. Thus, in the background kinematic anisotropies as well as anisotropies in the spacetime geometry are both the result of the magnetic stresses. In mathematical terms this is achieved by demanding that, to zero order, the magnetic vector is an eigenvector of the shear tensor. Although this scheme may not represent the most general case, it is sufficient for studying the lowest-order effects of the magnetic stresses at the onset of structure formation. Any zero-order anisotropies that are not the result of the magnetic presence are irrelevant to our study and therefore ignored. At the linear level, however, the field is no longer assumed to be a shear eigenvector. Thus, to first order, the shear does not entirely depend on the magnetic stresses. Our first step was to linearize the non-linear formulae of [5] about a Bianchi I spacetime, and also provide a couple of new but necessary first-order relations. As in [10, 11], we applied our equations to the case of a barotropic medium and then examined in detail the magnetohydrodynamical effects on the kinematics and the dynamics of the universe. It was at this level that we were first able to identify the implications of the anisotropic background.

In particular, it became clear that some of the effects of the FRW-based treatment were in fact direction-dependent. To be precise, a number of quantities were found to contribute differently parallel to the field lines than normally to them. For example, for a relatively strong magnetic field, the decelerating effect from matter aggregations was no longer isotropic. We found that only density perturbations parallel to the direction of the zero-order field vector can actually affect the average volume expansion. In contrast, density gradients perpendicular to the magnetic force lines get weaker as the strength of the field increases. Nevertheless, the isotropy of this effect was immediately re-instated at the weak field limit to match the predictions of [10].

The role of the background field stresses was further illustrated through the magneto-curvature contribution to the deceleration parameter. In [10], the isotropy of the unperturbed FRW spacetime meant that the curvature effects were direction-independent. Here, however, we found that their contribution was entirely confined along the zero-order field vector. As in [10], the resulting magneto-curvature term adds to the deceleration of disturbances with local negative spatial curvature, but accelerates positively curved regions. This seemed odd at first, given that positive curvature is traditionally associated with gravitational collapse. The explanation came from the directional dependence of the curvature effect. Recall that there is a negative pressure, a tension, in the direction of the magnetic lines of force. This is another way of saying that the field lines are elastic, opposing any attempt to distort their equilibrium pattern. When coupled to geometry, this magnetic property ensures that the field will react to any changes in the spatial curvature by modifying the local expansion rate accordingly. This relativistic effect is closely analogous to the classical curvature stress exerted by field lines with a nonzero curvature radius. Note that, qualitatively speaking, the aforementioned effect does not depend on the field's relative strength.

Similarly, the role of the field as a source of universal rotation is also independent of its strength. As in [10], we find that when the curl of the magnetic vector varies along the direction of the background filed, cosmic magnetism generates vortices in the fluid flow. However, the magnetic effect on the evolution of these vortices differs according to the strength of the field. In particular, the presence of strong magnetic fields can modify the dilution rate of such rotational disturbances.

Our study also facilitated a direct test for the FRW-based treatments of weakly magnetized density perturbations. We can see now the number and the nature of the corrections due to the background anisotropy, and identify the domains they mainly affect. Therefore, we are in a better position to judge the accuracy of the Friedmannian approximation. For an arbitrarily strong magnetic field, we find several directional contributions in the propagation equations of the basic inhomogeneity variables. As before, the key direction is determined by the background magnetic field. At this level, there is a considerable gap separating the Bianchi I equations from those of the FRW approximation. At the weak field limit, however, these differences are reduced dramatically. In particular, we find that only curvature and shear effects propagate along the preferred direction of the background field vector. Moreover, after recombination, density perturbations proceed essentially unaffected by curvature complications. Also, the shear contributions, which result from the kinematical implications of the background field, are confined to sub-horizon scales only. As a result, the large-scale equations of the FRW analysis are identical to those of the Bianchi I treatment when $\mathcal{R}_{ab}n^an^b$, the curvature perturbation along the field lines, takes the 'isotropic' value $\frac{1}{3}\mathcal{R}$. Also, the two approaches are essentially indistinguishable in two additional critical cases, when $\mathcal{R}_{ab}n^an^b = 0$ and for $\mathcal{R}_{ab}n^an^b = \mathcal{R}$. In the former case the curvature perturbation parallel to H_a is said to be 'minimum' and in the latter 'maximum'. Conclusively, we feel justified to argue that, at least on super-horizon scales, density perturbations are accurately treated by the FRW approximation. Even on small scales, one may be able to invoke backreaction arguments and ignore the aforementioned shear effects from the linear perturbation equations.

With regard to density perturbations, the Bianchi I analysis has lead to an additional, this time less trivial, result. In particular, let us recall the results of the isotropic and the minimum curvature cases mentioned above, which were addressed in [10, 11]. There, it was found that in the absence of any curvature input (i.e. for $\mathcal{R}_{ab}n^an^b = 0$) the magnetic presence suppresses the growth rate of matter aggregations. In contrast, when isotropic curvature contributions were allowed (i.e. $\mathcal{R}_{ab}n^an^b = \frac{1}{3}\mathcal{R}$), the overall magnetic effect was reduced and the density gradients grew as fast as in non-magnetized universes. This pattern was also confirmed here. Moreover, for maximum curvature contributions, that is, when $\mathcal{R}_{ab}n^an^b$ increases from $\frac{1}{3}\mathcal{R}$ to $\mathcal{R}$, the density contrast grows even faster than in magnetic-free cosmologies. Thus, the coupling of the field to the geometry can actually reverse the original magnetic impact on gravitational collapse. The reason behind this effect is again the tension properties of the magnetic force lines, which leads to a curvature stress proportional to $c_a^2\mathcal{R}_{ab}n^an^b$. The latter tends to counterbalance the action of the magnetic pressure gradients, thus reversing the pure magnetic impact on linear density condensations. As a result, in the magnetic presence, the density contrast may even grow faster than in non-magnetized cosmologies.

Acknowledgments

CGT was supported by PPARC. The authors would like to thank John Wainwright, John Barrow, Jai-chan Hwang and Marco Bruni for helpful comments.

Appendix. Interpreting $\mathcal{D}_a$

Following [19], the connecting vector δx^a between two neighbouring worldlines (i.e. fundamental observers) satisfies the restriction

$$u^b\nabla_b\delta x_a = \delta x^b\nabla_b u_a \,. \tag{107}$$

Also, the relative position vector, which connects the same two points P, P' on these worldlines at all times, is

$$\delta x_{\langle a\rangle} = h_{ab}\delta x^b \,, \tag{108}$$

and propagates as

$$\left(\delta x_{\langle a\rangle}\right)^{\cdot} = \left(\sigma_{ab} + \varepsilon_{abc}\omega^c + \tfrac{1}{3}\Theta h_{ab}\right)\delta x^{\langle b\rangle} \,. \tag{109}$$

Suppose now that $\mu(x_c)$ and $\mu(x_c + \delta x_{\langle c\rangle})$ is the fluid density at P and P' respectively, relative to a frame at P. Then, to first order,

$$\mu(x_c + \delta x_{\langle c\rangle}) = \mu(x_c) + \left(\tilde{\nabla}^a\mu\right)\delta x_{\langle a\rangle} \Rightarrow$$

$$\delta \equiv \frac{\delta\mu}{\mu} = \mathcal{X}^a\delta x_{\langle a\rangle} \,, \tag{110}$$

where $\delta\mu \equiv \mu(x_c + \delta x_{\langle c\rangle}) - \mu(x_c)$ and $\mathcal{X}_a \equiv \tilde{\nabla}_a\mu/\mu$. Equation (109) ensures that within a FRW spacetime

$$\delta x_{\langle a\rangle} = S\left(\delta x_{\langle a\rangle}\right)_0 \,, \tag{111}$$

where $\left(\delta x_{\langle a\rangle}\right)_0$ is a constant. Therefore, result (110) becomes

$$\delta = \left(\delta x^{\langle a\rangle}\right)_0\mathcal{D}_a \,, \tag{112}$$

where $\mathcal{D}_a \equiv S\mathcal{X}_a$. Thus, in an almost-FRW universe the co-moving fractional density gradient $\mathcal{D}_a$ describes the spatial density variations between two neighbouring fundamental observers. However, in a Bianchi I spacetime the background shear is non-zero and $\delta x_{\langle i\rangle}$ no longer evolves according to (111). Consequently, in a perturbed Bianchi I universe $\mathcal{D}_a$ no longer describes the spatial density changes seen by a pair of neighbouring fundamental observers. Obviously, in the case of weak background shear $\sigma/\Theta \ll 1$ and the aforementioned complication is negligible, and D_a regains the desired interpretation.

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