Magnetized gravitational waves

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We investigate the influence of cosmic magnetic fields on gravitational wave perturbations and find exact solutions on large scales. We show that a large-scale magnetic field can generate large-scale non-decaying gravitational waves. In the general case where gravitational waves are generated by other mechanisms, a large-scale magnetic field introduces a new decaying tensor mode and modifies the non-decaying mode. The direct effect of the magnetic field is to damp the gravitational waves, while an indirect magneto-curvature effect can either damp or boost the waves. A magnetic field also leads to a breaking of statistical isotropy, and the magnetic imprint on the tensor spectrum in principle provides a means of detecting a primordial field.

I. INTRODUCTION

Magnetic fields seem to be everywhere that we can look in the universe, from our own Sun out to high-redshift Lyman-\(\alpha\) systems. The fields we observe (based on synchrotron radiation and Faraday rotation) in galaxies and clusters have been amplified by gravitational collapse and possibly also by dynamo mechanisms. They are either primordial, i.e. originating in the early universe and already present at the onset of structure formation, or they are protogalactic, i.e. generated by battery mechanisms during the initial stages of structure formation. One way to distinguish these possibilities would be to detect or rule out the presence of fields coherent on cosmological scales during recombination via their imprint on the cosmic microwave background (CMB) radiation. Considerable work has been done to investigate the nature of the magnetic imprint on the CMB (see [1] for recent reviews and further references).

We show here that cosmological magnetic fields also leave a characteristic imprint on the cosmological gravitational wave background. Direct detection of this background, especially on large scales, is not likely for a considerable time (see, e.g., [2]), so that our results do not provide a practical means for detecting or limiting a large-scale magnetic field. However, these results are a necessary first step in a theoretical understanding that can be developed in anticipation of gravitational wave detection. Furthermore, our results suggest a similar investigation of the magnetic imprint on gravitational waves generated by compact objects. This astrophysical problem is more complex than the cosmological problem that we discuss, but it is likely to lead to larger effects with stronger prospects for observational detection.

The effects of gravitational waves on electromagnetic fields, in particular the generation of electromagnetic pulses by gravitational waves, has been previously considered, including the implications for gravitational wave detection [3] (see [4] for recent results and further references). Here we consider a different and new effect, i.e. the influence of a large-scale magnetic field on gravitational waves, including the capacity of the field to generate gravitational waves. This latter possibility has been previously investigated on small scales in [5], where tangled magnetic fields during recombination were shown to generate gravitational wave perturbations, which in turn contribute to CMB anisotropies, thus providing a way of limiting the strength of the magnetic field.

We consider in a general cosmological context the problem of how magnetic fields affect gravitational waves, focusing on large-scale fields. A large-scale homogeneous magnetic field is strongly limited by large-angle CMB anisotropies [6]: \(B_{\text{now}} \lesssim 10^{-9}\) G. Although the maximal energy density supported by the field is a very small fraction of the total cosmic energy density, this is also the case for gravitational waves, so that in principle, as we show, the magnetic effect on these waves need not be negligible. We find exact analytic solutions on large scales. These solutions show that the direct effect of the magnetic field is to damp the gravitational waves, while an indirect effect due to the coupling between the field and the spatial curvature can either damp or boost the waves. Qualitatively, the presence of a large-scale cosmic magnetic field is signaled by a breaking of global statistical isotropy in the gravitational wave spectrum, which is in principle detectable (anisotropy detection is discussed in general in [7]).

II. PERTURBATION EQUATIONS

The background is a spatially flat Friedmann spacetime with a non-magnetized perfect fluid, representing non-interacting baryons, radiation and cold dark matter with the same 4-velocity \(u^a\), and Hubble rate \(H = a / a\). The background equations are

\[
\dot{\rho} = -3H(\rho + p),
\]

\[
3H^2 = \rho + \Lambda,
\]

where \(\rho\) and \(p\) are the total energy density and pressure. The perturbed universe is permeated by a weak large-scale magnetic field \(B_a\), whose energy density \(\rho_{\text{mag}} = \frac{1}{2} B^2\) is a first-order quantity (so that \(B_a\) is “half-order” [5]). The magnetic field also has first-order isotropic pressure \(\frac{1}{2} B^2\) and tracefree anisotropic stress

\[
\pi_{ab} = -B_a B_b = -\left[ h_{(a} \dot{h}_{b)} - \frac{1}{2} h^{cd} h_{ab} \right] B_c B_d,
\]
where $h_{ab}=g_{ab}+u_a u_b$ projects into the comoving rest space and the round brackets on indices denote symmetrization. The baryonic fluid is treated as a perfect fluid of infinite conductivity, with energy density $\rho_b$ and isotropic pressure $p_b$. High conductivity ensures that any electric fields that might have been present dissipate quickly. For tensor perturbations, which do not excite relative velocity perturbations, $u^a$ is the 4-velocity of baryons, radiation ($p_r=\frac{4}{3} \rho_r$) and collisionless cold dark matter.

Both isotropic and anisotropic magnetic effects are of the same perturbative order as gravitational waves. The covariant Maxwell equations [8] imply the induction equation, whose nonlinear form is [9]

$$h_a^b \dot{B}_b = (\sigma_{ab} + e_{abc} \omega^c - \frac{2}{3} \Theta h_{ab}) B^b,$$  \hspace{1cm} (4)

where $\sigma_{ab}$, $\omega_a$ and $\Theta$ are respectively the shear, vorticity and expansion. In the background, $\Theta = 3H$ and $\omega_a = 0 = \sigma_{ab}$. The induction equation implies the linearized conservation equations [9,10]

$$ \dot{B}^2 = -4HB^2, $$

$$ \dot{\pi}_{ab} = -4H \pi_{ab}, $$

where the dot denotes $u^a \nabla_a$. It follows that

$$ \pi_{ab} = -B_0 \left( \frac{a_0}{a} \right)^4 n_{(an)b}, \quad n_a = B_a B^b, \quad n^a = 0. $$  \hspace{1cm} (7)

The unit magnetic direction vector $n^a$ is parallel-propagated along each observer world line to first order. Maxwell’s equations also imply the constraint [11]

$$ D^b \pi_{ab} = e_{abc} B^b \text{curl} B^c - \frac{1}{8} D^c B_c ^2, $$

$$ \text{where } D_a = \text{the projected covariant derivative, } D_a S^{bc} \cdots = (\nabla_a S^{bc} \cdots) - (\nabla_a \cdots) S^{bc}, \text{ which leads to covariant spatial curl operators: } \text{curl}_S = e_{abc} D^b S^c, \text{ curl}_S = e_{cd(b} D^c S_{d)} .$$

The 4-acceleration $A_a$ is determined by Euler’s equation [9,10]

$$ \left( \rho_b + p_b \right) A_a = -D_a p_b + \varepsilon_{abc} B^b \text{curl} B^c.$$  \hspace{1cm} (9)

For tensor perturbations, the 4-acceleration shall vanish at first order. The non-magnetized condition for transverse traceless (pure tensor) modes is [12,13]

$$ \omega_a = 0 = D_a \rho = D_a p . $$  \hspace{1cm} (10)

When there is no magnetic field, this ensures that $A_a = 0 = D_a \Theta$ and that all the traceless tensors, such as $\sigma_{ab}$, are transverse. In the magnetized case, two additional constraints need to be imposed:

$$ D_a B^2 = 0 = e_{abc} B^b \text{curl} B^c; $$ \hspace{1cm} (11)

i.e., the magnetic energy density is homogeneous and the field is force free, to first order. Equations (10) and (11) together imply

$$ A_a = 0 = D_a \Theta, $$ \hspace{1cm} (12)

and that all traceless tensors are transverse. (The expansion gradient is seen to vanish from the propagation equation for $D_a p$ [9].) The magnetic field is felt only through its transverse traceless anisotropic stress $\pi_{ab}$.

Gravitational radiation is covariantly described by the electric ($E_{ab} = E_{(ab)}$) and magnetic ($H_{ab} = H_{(ab)}$) parts of the Weyl tensor, which support different polarization states [12] and which obey equations remarkably analogous to Maxwell’s [8,12,13]:

$$ \dot{E}_{ab} = -3HE_{ab} + \frac{3}{2} H \pi_{ab} - \frac{1}{2} (\rho + p) \sigma_{ab} + \text{curl} H_{ab}, $$ \hspace{1cm} (13)

$$ \dot{H}_{ab} = -3HH_{ab} - \text{curl} E_{ab} + \frac{1}{2} \text{curl} \pi_{ab}, $$ \hspace{1cm} (14)

$$ \dot{\sigma}_{ab} = -2H \sigma_{ab} - E_{ab} + \frac{1}{2} \pi_{ab}, $$ \hspace{1cm} (15)

$$ H_{ab} = \text{curl} \sigma_{ab}, $$ \hspace{1cm} (16)

where

$$ D^b E_{ab} = D^b H_{ab} = D^b \sigma_{ab} = D^b \pi_{ab} = 0. $$ \hspace{1cm} (17)

In the magnetized case, $\pi_{ab}$ is given by Eq. (7).

The direct relation between the shear and the magnetic Weyl tensor means that we can describe gravity wave evolution via $E_{ab}$ and $\sigma_{ab}$ alone. Equations (13)–(15) show how the magnetic field is a source of shear and gravitational wave perturbations via its anisotropic pressure. Equation (16) allows us to decouple Eq. (14) from the system, which reduces to Eq. (15) and

$$ \dot{E}_{ab} = -3HE_{ab} + \frac{3}{2} H \pi_{ab} - \frac{1}{2} (\rho + p) \sigma_{ab} - D^2 \sigma_{ab}, $$ \hspace{1cm} (18)

on using the linearized identity $\text{curl} \text{curl} S_{ab} = -D^2 S_{ab}$. It follows via the role of $\pi_{ab}$ in these equations that there is a directional effect at all scales on gravitational waves due to the magnetic field. We will further discuss this effect on large scales below. Note also that to first order, the evolution of the magnetic field is independent of any gravity wave effects. The field affects gravity waves, but there is no back-reaction on the field.

We can further reduce the system of equations to a single covariant wave equation in $\sigma_{ab}$:

$$ D^2 \sigma_{ab} - \sigma_{ab} = 5H \sigma_{ab} + \left[ \frac{1}{2} (\rho - 3p) + 2 \Lambda \right] \sigma_{ab} + 2B_0 \left( \frac{a_0}{a} \right)^4 Hn_{(a} R_{b)}, $$ \hspace{1cm} (19)

on using Eqs. (1), (2), (7), (13) and (16). The solution of this wave equation then gives the gravitoelectromagnetic tensors via Eqs. (15) and (16). In practice, it is better to solve the first order coupled system for $E_{ab}$ and $\sigma_{ab}$ than the single second order wave equation.
The covariant Maxwell-Weyl approach to gravitational wave perturbations may be related to the metric-based approach (see also [13,12,14]). The transverse traceless metric perturbation $f_{ij}$ is defined by

$$ds^2 = -dt^2 + a^2[\delta_{ij} + f_{ij}]dx^idx^j.$$  

In these coordinates, it follows that $E_{0a} = 0$ and [15]

$$E_{ij} = \frac{1}{2}Hf_{ij} + Hf_{ij},$$  

on large scales. The wave equation for $f_{ij}$ then shows that, neglecting anisotropic stresses, $E_{ij} = \frac{1}{2}Hf_{ij}$. Thus the energy density in gravitational waves, $\rho_{gw} = \frac{1}{2}f_{ij}f^{ij}$, is given in the absence of a magnetic field in Maxwell-Weyl form as

$$\rho_{gw} = \frac{1}{H^2}E_{ab}E^{ab},$$  

on large scales. Whether or not there is a magnetic field, the dimensionless contribution to the power spectrum on large scales is determined by $\lambda^4E_{ab}E^{ab}$, where $\lambda$ is the wavelength [12]. Thus a dimensionless measure of amplitude on large scales is given by

$$\Gamma^2 = a^4E_{ab}E^{ab}.$$  

III. LARGE-SCALE SOLUTIONS

On superhorizon scales we can neglect the Laplacian term in Eq. (18). In the radiation era, Eqs. (15) and (18) then have the solution

$$E_{ab} = C_+^{-1} + C_+^{-5/2} - \frac{1}{2}n(a)^bB_0^{-1},$$  

where $C_+ = 0$ and $\tau = t/\sigma$. In the matter era,

$$E_{ab} = C_+^{-4/3} + C_+^{-3} - \frac{1}{2}n(a)^bB_0^{-1} - \frac{2}{3},$$  

and they satisfy $\Sigma_{ab} = n_{ab}$, $\Sigma = aE_{ab}n^{ab}$, $E = a^2E_{ab}n^{ab}$, $\chi = a^3E_{ab}a^{ab}$.

The magnetic field introduces a new mode of gravitational wave perturbations on large scales, which decays less rapidly than the standard, non-magnetized decaying mode, and it modifies the standard non-decaying mode (of $a^2E_{ab}$); i.e., magnetic terms enter $C_+^{ab}$ (see below). The directional influence of the large-scale magnetic field on gravitational waves means that the field breaks the statistical isotropy of the large-scale tensor spectrum, as further discussed below.

The new magnetized modes mean that if there are no gravitational wave perturbations initially present, then these can be generated by a large-scale magnetic field. The generation of tensor perturbations by small-scale tangled magnetic fields is investigated in Ref. [5]. This could happen if a large-scale magnetic field is created at time $t_0$. Large-scale magnetogenesis can occur for example in the recombination era [16] or the preheating era after inflation (see [17] and references cited therein). Since $E_{ab}(t_0) = 0 = E_{ab}(t_0)$, it follows from Eq. (24) that

$$E_{ab} = \left[\frac{1}{2} - \frac{1}{2} - \frac{5/2}{2} - \frac{1}{2}\right]B_0^2\delta_{a}n_{b},$$  

if $t_0$ is in the radiation era, with a similar result if $t_0$ is in a matter-dominated era. These "purely magnetic" gravitational waves have a non-decaying mode

$$\Gamma = \frac{5\sqrt{3}}{18}a_0^2B_0^2,$$  

and they satisfy $E_{ab} = \pi_{ab}$, so that in some sense they are maximally anisotropic (see below). The generation of tensor perturbations via magnetogenesis may be compared with the magnetic generation of density perturbations, which produces a growing mode [18,10].

In the general case, tensor perturbations are generated by other mechanisms and then influenced by the magnetic field. The dimensionless measure of amplitude, $\Gamma^2$, defined in Eq. (23), is seen from Eqs. (24)–(27) to be constant at late times ($t \gg t_0$), i.e. when we neglect the decaying modes. Explicitly, we find the following magnetic influence on the non-decaying mode of large-scale gravitational waves.

In the radiation era,

$$\Gamma^2 = \frac{1}{9} \left[4\Gamma_0^2 + 4a_0^2H_0^2\Sigma_0 - 8a_0H_0\epsilon_0\right]$$

$$+ \frac{2}{27}a_0^2B_0^2\left(a_0^2B_0^2 - 6\epsilon_0 + 6a_0H_0\epsilon_0\right),$$  

where we expressed $C_+^{ab}$ in terms of the dimensionless physical scalars

$$\Sigma = a^2\sigma_{ab}\sigma^{ab}, \quad S = a\sigma_{ab}n^a\sigma^b, \quad \epsilon = a^2E_{ab}n^a\sigma^b, \quad \chi = a^3E_{ab}a^{ab}.$$  

In the matter era,

$$\Gamma^2 = \frac{9}{25} \left[\Gamma_0^2 + a_0^2H_0^2\Sigma_0 - 2a_0H_0\epsilon_0\right]$$

$$+ \frac{3}{50}a_0^2B_0^2\left(a_0^2B_0^2 - 6\epsilon_0 + 6a_0H_0\epsilon_0\right).$$  

The effect of the magnetic field is more clearly brought out if we use the Gauss-Codazzi equation [9]

$$R_{ab}^* = E_{ab} - H\sigma_{ab} + \frac{1}{3}\pi_{ab},$$  

where $R_{ab}^*$ is the Ricci tensor of the spatial hypersurfaces orthogonal to $u^a$. This equation implies

$$\mathcal{R} = \mathcal{E} - aHS - \frac{1}{3}a^2B^2,$$  

where

$$\mathcal{R} = a^2R_{ab}^*n^an^b$$  

is a dimensionless curvature scalar, giving the anisotropic 3 Ricci curvature along the magnetic field, due to the gravitational waves. Using Eq. (35), we can rewrite Eqs. (30) and (33):
in the radiation era, and
\[ \Gamma^2 = \frac{1}{9} \left[ 4 \Gamma_0^2 + 4 a_0^2 H_0^2 \Sigma_0^2 - 8 a_0 H_0 \chi \right] - \frac{2}{27} a_0^2 B_0^2 (a_0^2 B_0^2 + 6 R_0), \] (37)
in the matter era.

These equations show that there are two aspects to the magnetic effect: a "pure" magnetic effect (neglecting curvature) and a magneto-curvature effect. The pure effect, proportional to $-B_0^2$, serves to damp the amplitude relative to the non-magnetized case. It is due to the tension of the magnetic field lines, which means that the magnetic field resists the distortions induced by a gravitational wave [19]. The magneto-curvature effect, proportional to $-B_0^2 R_0$, arises from the geometric coupling between curvature and the field in general relativity, due to the vector nature of the field from the geometric coupling between curvature and the field [9–11,19]. It depends on the sign of $R_0$, i.e. on the sign of the anisotropic 3-Ricci-curvature component along the magnetic direction at time $t_0$. If $R_0 > 0$, then the damping is reinforced; otherwise, the magneto-curvature effect enhances the amplitude, acting opposite to the pure magnetic effect.

**IV. MAGNETIC IMPRINT ON GRAVITATIONAL WAVES**

In the above solutions, the terms in round brackets give the magnetic correction to the non-magnetized model. A rough estimate of the relative magnitude of the magnetic imprint on the large-scale gravitational wave spectrum is therefore

\[ \alpha = \frac{\Gamma^2}{\Gamma^2_{|B=0}} \sim \frac{a_0^4 B_0^4}{\Gamma_0^2}. \] (39)

On using Eqs. (2), (22) and (23), we find

\[ \alpha \sim \left( \frac{\rho_{mag}^4 \rho_{gw}}{H^2 \rho_{gw}} \right) \sim \left( \frac{\rho_{mag}^4 \rho_{gw}^2}{\rho_{gw}^4} \right) \frac{\rho_{gw}}{\rho_{0}}. \] (40)

Now $\rho_{mag} / \rho$ is constant to first order, and large-angle CMB anisotropies at the $10^{-5}$ level imply the rough limit $\langle \rho_{mag} / \rho \rangle_{dec} < 10^{-5}$. The CMB quadrupole also places an upper limit on large-scale tensor perturbations [20]: $\langle \Omega_{gw} / \Omega_0 \rangle_{now} < 10^{-10}$. Clearly $\alpha$ is sensitive to the tensor contribution to CMB anisotropies: the smaller this contribution, the higher $\alpha$ is. However, this increase in $\alpha$ comes at the cost of greater difficulty in detecting the gravitational wave background. In the most favorable case for detection, when the tensor contribution to the CMB quadrupole is of the same order as the scalar contribution and the magnetic contribution to the CMB is maximal, then, with $t_0 = t_{dec}$, it follows that $\alpha \approx (\rho_{gw} / \rho_{dec})^{-1} = 10^{-1}$.

Thus, in principle, observations of the large-scale gravitational wave background can provide quantitative limits on a possible large-scale magnetic field, although in practice detection of such a background presents great difficulties (see [2] and references therein). Thus the CMB remains the best way to limit a magnetic field, on all scales [1,5,6,21,22].

Qualitatively, the presence of a magnetic field is signaled by the breaking of global statistical isotropy in the tensor spectrum. This may be compared with a similar situation regarding the effect of a large-scale magnetic field on CMB polarization anisotropy, where a magnetic presence is signaled by a correlation between $B$-polarization and temperature anisotropies, which is impossible if statistical isotropy holds [21]. A possible measure of global statistical anisotropy is the average over all directions $n^a$ of the quantity

\[ \xi = \frac{\Gamma^2 - 3 \mathcal{E}^2}{\Gamma^2}. \] (41)

In the absence of a magnetic field, statistical isotropy will give $\langle \xi \rangle_n = 0$, since

\[ \langle (E_{ab} n^a n^b) \rangle_n = \frac{1}{2} E_{ab} E^{ab}. \] (42)

A measurable magnetic field will produce a non-zero $\xi$ for the magnetic direction $n^a$. In the case of gravitational waves generated by a magnetic field, Eqs. (28) and (41) show that $|\xi| = 1$.

Although we were only able to find exact solutions on large scales, the small-scale solutions should share the same basic features. Finally, we point out that if the same qualitative physical effects of magnetic fields on gravitational waves occur in the context of compact objects, then since the gravitational and magnetic fields can be very strong, this could have more immediate and important observational implications. This is a topic currently under investigation.

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