The two fixed centers problem revisited

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The study of the motion of a point mass, moving in the gravitational field of two fixed attracting centers, is a problem first posed by Euler in the 18th century, as an intermediate step towards the solution of the famous three-body problem. Euler himself, in a series of three papers (Euler 1766a, 1766b and 1767), was able to integrate the equations of motion for the two-dimensional (2-D) case, i.e. the case where the point mass moves on a plane containing the two attracting centers. Almost a century later Jacobi (1842) showed that the corresponding potential of the full 3-D case is separable in prolate spheroidal coordinates. Another century later Erikson and Hill (1949) found, in explicit form, the third integral of motion of the full three-dimensional (3-D) case (besides the total energy and the angular momentum about the axis passing through the two centers). Since then the problem has been considered as a non-exciting example of a separable potential and it is included, as such, in many textbooks of Theoretical Mechanics.

That this is not the case can be understood by the multitude of interesting applications, which appeared in the literature after the paper by Erikson and Hill. Thus, the problem of the two fixed centers has been used, among others, in the calculation of satellite trajectories in the gravitational field of the Earth (Marchal 1966, 1986), in the quantum mechanical problem of the positive ion of the hydrogen molecule, H_2^+ (Strand and Reihardt, 1979) and in the ac-

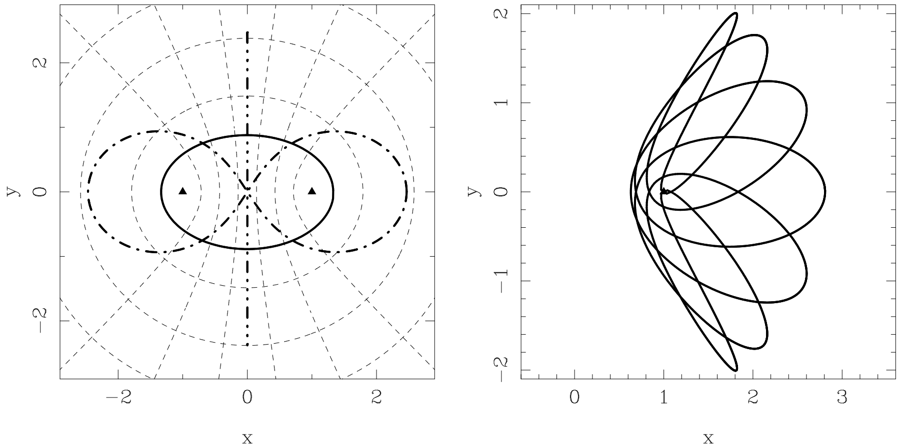

Figure 1. (a) (left) The figure-8 non-isolated periodic orbit (dot-dashed) and the only isolated ones: unstable (3-dot-dashed line) and stable (solid line) orbits on an elliptic, $\xi - \eta$, coordinates grid. (b) (right) A non-periodic trajectory of class $P3$. It fills densely the available configuration space defined by the zero velocity curve and, therefore, approaches arbitrarily close the attracting center.

celeration of electrons in atomic collisions (Jakas, 1995, 1996). Generalized forms of the problem have been considered as well, such as the finite dipole (Howard and Wilkinson, 1995), the repulsive dipole (Kallrath, 1993) and the equally charged two black holes system (Contopoulos and Papadaki, 1993).

Although the corresponding dynamical system is integrable, the qualitative behaviour of its solutions was, up to now, not very well understood, probably due to the fact that the solutions are expressed in the form of elliptic functions. One thorough attempt for the classification of the solutions has been done by Deprit (1962). A similar work, but restricted only to a subclass of the solutions, has been carried out by Strand and Reinhardt (1979). Finally Contopoulos and Papadaki (1993) calculated the initial conditions and the characteristic exponents of many periodic trajectories. However the above authors did not construct a complete phase portrait of the dynamical system.

In the present work we focus our interest on the simplest possible case of the two fixed centers problem, where the masses of the two attracting bodies are equal and the trajectory of the third body lies on a plane. We use elliptic coordinates, $\xi - \eta$. Following Charlier (1902), Deprit (1962) and Strand and Reinhardt (1979), the orbits may be classified, according to the values of the two integrals of motion and the available region of configuration space they define, into three basic classes. In the first class, $P1$, belong the trajectories that lie within an elliptic annulus that encircles the two attracting centers. In the second class, $P2$, belong the trajectories that lie within a simply connected region that contains both attracting centers. Finally in the third class, $P3$, belong the trajectories that lie within two disconnected regions, each one

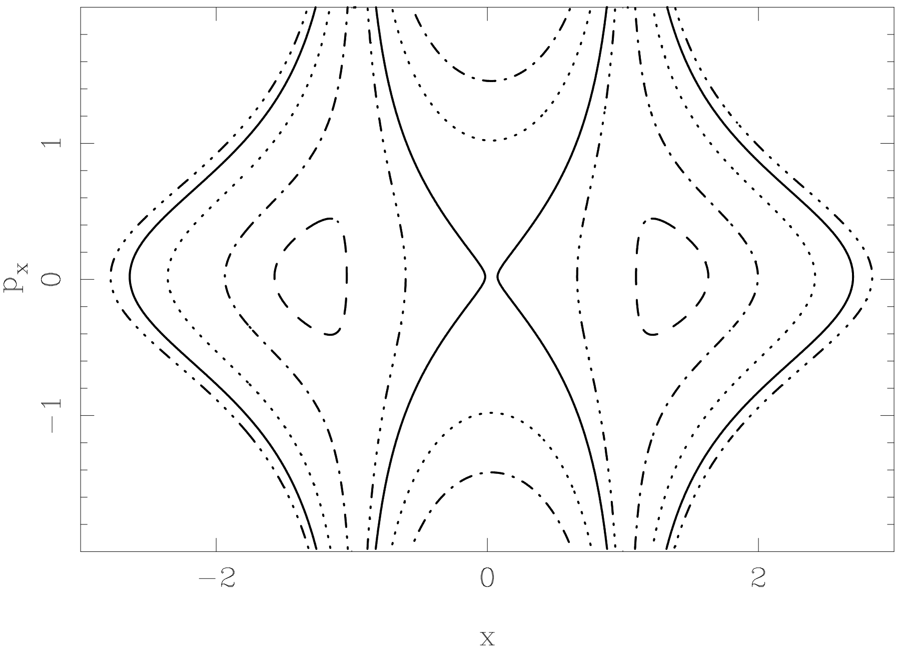

Figure 2. Surface of section plot for several non-periodic trajectories: (a) P1 class, dashed curves, (b) P2 class, dotted and dashed-dotted curves, (c) P3 class, solid and dashed-3-dotted curves

containing one of the attracting centers. The configuration space and the most characteristic periodic orbits are shown in Fig. 1. For any value of the energy integral, one unstable isolated periodic orbit of class $P2$ lies on the y-axis ($\eta = 0$) and a pair of stable periodic orbits of class $P1$ lie on an ellipse ($\xi = const.$), one orbit described clockwise and the other counter-clockwise (e.g. see Broucke, 1980; Meletlidou and Ichtiaroglou, 1999). These orbits may be easily identified on a surface of section, from the hyperbolic and elliptic, correspondingly, structure of the invariant curves in their vicinity. The figure-8 orbit however does not appear explicitly on the surface of section (Fig. 2). The reason is that all periodic orbits, besides those mentioned above, are non-isolated. Therefore the structure of the invariant curves in their vicinity is parabolic and their characteristic exponents are zero. Indications for this general property were found numerically by Contopoulos and Papadaki (1993). It is interesting to note that there do not exist 1:1 periodic orbits belonging to the class $P3$.

In an integrable dynamical system all trajectories are either periodic or quasi-periodic. However in the case of the two fixed centers problem the situation appears more complicated. The reason is that, usually, we tend to consider a dynamical system as integrable, if its Hamiltonian can be expressed in action-angle variables. A necessary condition for this is that the available phase space region is compact. However this is not the case for the two fixed centers problem. In fact we have proved that all non-periodic orbits of class $P1$ and $P2$ are either *true* or *asymptotic* collision orbits. In particular (a) true collision orbits are of zero measure in the 4-D phase space but of complete measure in a 3-D submanifold and (b) asymptotic collision orbits are of complete measure in the 4-D phase space.

On a surface of section this situation is depicted by the fact that the corresponding invariant curves tend to infinity, in the momentum axis, in an unusual way: invariant curves for consecutive values of the second integral are not nested, but they seem to intersect at infinity (Fig. 2). This picture does not agree with the usual one, where the phase space of an integrable system is folliated in nested tori. Drawing a surface of section, however, that does not contain the lines $\xi = 1$ or $\eta = \pm 1$ shows that the invariant curves' segments are joined in such a way, that the folliation picture is recovered. Indeed, a regularisation scheme shows that the points at infinity lie on a circle, so that the invariant curves do not really cross.

The phase portrait in the regularised coordinates, the distribution of periodic orbits and their relation to the rotation curves will be the topic of a future publication.

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