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TRANSPORT IN HAMILTONIAN SYSTEMS AND ITS RELATIONSHIP TO THE LYAPUNOV TIME

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ABSTRACT

The assumption that transport in a Hamiltonian system can be described as a normal diffusion process leads naturally to a power law dependence of the exit time, T_E, on the Lyapunov time, $T_L=1/\lambda$, where by λ we denote the maximal Lyapunov characteristic number, LCN. Since transport in perturbed integrable Hamiltonian systems can be modeled as normal diffusion only in regions where most of the KAM tori are destroyed, the power law dependence appears when the perturbation is strong. In this way the dependence $T_E \sim T_L^c$, found numerically by Murison *et al.* (1994) for the motion of asteroids in the outer belt, can be naturally interpreted, since in this region it is well known that resonances are closely spaced and, therefore, it is expected that KAM tori are mostly destroyed. However there is no theoretical reason why the exponent, c, should have a universal value. © *1996 American Astronomical Society.*

1. INTRODUCTION

In a recent paper Murison *et al.* (1994), continuing earlier work of their research team (Soper *et al.* 1990, Lecar *et al.* 1992a, Lecar *et al.* 1992b, Franklin *et al.* 1993), presented numerical evidence for the existence of a power law concerning the motion of asteroids in the outer asteroidal belt. This law relates the value of the Lyapunov time, $T_L=1/\lambda$ (where by λ we denote the maximal Lyapunov Characteristic Number, LCN, of the asteroid's trajectory) to the time interval ("event time", T_E) needed for this asteroid to become a planet crosser.

In this paper we show that the "power law" behavior can be naturally recovered through a diffusion-equation formalism of the problem. To this purpose we model the transport of a particle in a finite connected region (i.e., a "box") of action space as a Markovian process (i.e., essentially a random walk, corresponding to standard diffusion in action space). In the case of perturbed integrable Hamiltonian systems this is possible if the perturbation is strong. This case is equivalent to the fact that the measure of the set of surviving cantori (for 2-D systems) or higher dimensionality geometrical objects (for n-D systems, $n>2$) in phase space is small (Shlesinger *et al.* 1993). In the rest of this paper we will use the term "quasi-barriers" for this kind of object. In the case of strong perturbation the process can be adequately described by a diffusion equation of the Fokker-Planck type, with a characteristic time-scale representing the "exit" of the particle from a certain "box" in action space. This regime is essentially the "resonance overlap" regime. In applying the above ideas to the problem of asteroidal motion we note that the asteroid's eccentricity may be considered as one of the actions. We recall also that, in the case of the asteroidal belt, the outer region is governed mainly by the interaction of many closely spaced resonances, a fact that is not true for the inner belt.

In the case of the numerical experiments by Murison *et al.*, which lead to the formulation of the power law $\log T_E=a+b\log(T_L)$, the model used was the Elliptic Restricted Three Body Problem, ERTBP, with the mass of the second body taken 10 times the actual mass of Jupiter. Since in the ERTBP the "perturbation" is proportional to the mass of Jupiter, it is easy to understand that this selection, which corresponds to the "strong" perturbation regime discussed above, further reduces the already small set of surviving invariant tori of the real dynamical system. Therefore Murison *et al.* dealt essentially with a dynamical Hamiltonian problem in which the main assumption of our model above (i.e., process corresponding to normal diffusion) is inherent.

2. DIFFUSIVE APPROACH

In the case of transport in Hamiltonian systems, considered at discrete time steps, one should carefully differentiate between Levy flights (i.e., fractal random walks, "ballistic motion") and normal diffusion (i.e., normal random walks, "Brownian motion") (Shlesinger *et al.* 1993). In both cases the second moment ($\langle R^2(t)\rangle$) of the probability density, $p(R)$ (where R is the "jump" distance at every time step), scales with time as

$$\langle R^2(t)\rangle \propto t^\gamma, \tag{1}$$

where γ is a constant, but in the case of normal diffusion $\gamma=1$ while in the Levy flight $\gamma=2$. Now in the case of normal diffusion we know that the evolution of the distribution function in a region of action space is governed by a diffusion equation of the form (Melrose 1980)

$$\frac{\partial N(I,t)}{\partial t}=\frac{\partial}{\partial I}\left(D\,\frac{\partial N(I,t)}{\partial I}\right)-\frac{N(I,t)}{T_E}, \tag{2}$$

where $N(I,t)$ is the distribution function in action space (I), D is the diffusion coefficient, T_E is the escape time, and the term $N(I,t)/T_E$ represents the escape rate from the region of interest.

The above equation can be readily solved if one makes the conjecture that the diffusion coefficient is actually constant (not depending on the actions). This is not a bad assumption, since the use of the Lyapunov exponents as characterizing a whole phase space region already surmises a kind of an "appropriate averaging" all over the phase space region available to the particle's trajectory. In this way one is lead to the natural question: in this limit, is there any relation between the value of the diffusion constant in a phase space region and the value of the maximal Lyapunov exponent in that region? It seems that the answer is affirmative. As shown by Konishi (1989), numerical experiments strongly suggest that there is a relation of the form

$$\log(D)=a+b\ \log(H_{K-S}), \tag{3}$$

where H_{K-S} is the Kolmogorov–Sinai entropy. The value of b may depend on the specific dynamical system considered. Since $H_{K-S}=\Sigma\lambda_+\approx\lambda$, the diffusion coefficient is related to the maximal Lyapunov characteristic number through the relation

$$D=a\lambda^b. \tag{4}$$

Following a standard practice in problems of this form, we seek a solution of Eq. (2) of the form $N(I,t)=F(t)S(I)$. In the Appendix we show that, in our case, this method leads to a result equivalent to the one obtained through the general solution. Then one finds that

$$\frac{1}{DF(t)}\frac{dF(t)}{dt}+\frac{1}{DT_E}=\frac{1}{S(I)}\frac{d^2S(I)}{dI^2}=c. \tag{5}$$

We are interested only for the time evolution of the distribution function $N(I,t)$, that is only for the evolution of the function $F(t)$. Then one has to solve the equation

$$\frac{1}{DF(t)}\frac{dF(t)}{dt}+\frac{1}{DT_E}=c \tag{6}$$

with $c>0$ constant. The solution is of the form

$$F(t)=F_0\ \exp\left[\left(cD-\frac{1}{T_E}\right)t\right] \tag{7}$$

with $F_0=F(t=0)$ constant. If we assume that there are no sinks or sources, where particles are created or removed, we have that $F(t)\simeq F_0$ for any time t, in which case for $t\neq 0$ the relation $cD-1/T_E=0$ must be satisfied. Substituting the diffusion coefficient from Eq. (4) to the above relation, one finds

$$\log T_E=-\log ac+b\ \log T_L. \tag{8}$$

This result shows that a relation of the form found numerically by Murison *et al.* is a natural consequence of the simple assumption that transport in a specific region of action space of a Hamiltonian system is normal diffusion described by a Fokker-Planck type equation. This assumption is actually acknowledged in the paper by Lecar *et al.* (1992b), where it is stated that they work "in the spirit of random walk calculations." It should be noted that there is no theoretical reason why the numerical parameters a, b, and c entering the above relation should take specific values, so that it is possible to find different values in different dynamical systems or, even, in different regions of the same system, depending on the "local" structure of phase space (i.e., islands, cantori, quasi-barriers, etc.). It is interesting to note that the dimensions of the action space do not enter in our solution, as long as the diffusion coefficient is considered a constant, so that in this case the result is valid for any number of degrees of freedom. In the case of asteroidal motion, for example, one may select as actions the eccentricity and/or the semi-major axis, etc.

3. DISCUSSION AND CONCLUSIONS

We should comment on the relation of the above results to the "real" problem of asteroidal motion, i.e., the case where the mass of the second body in the ERTBP is taken equal to the one of Jupiter. In this case the measure of the invariant tori depends on the region of phase space considered. In the outer belt the approximation that transport can be thought of as normal diffusion is not bad, since this region presents closely spaced resonances, which result in transport approaching normal diffusion (the "resonance overlap" regime). In other regions, however, this may not be true. Typical example is the 2:1 resonance region, where surviving quasi-barriers strongly confine asteroidal motion in phase space regions with low eccentricities for very long times (e.g., see Ferraz-Mello 1995). Therefore application of any power law connecting LCN's with exit times in this region (e.g., see Franklin 1994) may lead to erroneous results, since transport there corresponds more to Levy flights than to normal diffusion. In this case one should use the "fractal" Fokker-Planck-Kolmogorov equation, as derived by Zaslavsky (1992, 1994), and an appropriate generalization of the idea of diffusion coefficient. This is a highly non-trivial task, even in the case of simple dynamical systems (e.g., see Shlesinger *et al.* 1993) and is therefore outside the scope of the present work.

Finally we should comment briefly on the phenomenon of "stable chaos" reported recently by Milani & Nobili (1992). These authors have found that the asteroid 522 Helga has a very sort Lyapunov time, T_L, of the order of a few thousand years (6.9×10^3). Subsequently several other asteroids with Lyapunov times of the same order of magnitude were found by Levison & Duncan (1993) as well as by Milani and coworkers (1995). Murison *et al.* (1994) interpreted the observation of "stable chaotic" asteroids as the "tail" of an initial distribution, most of the members of which have already escaped. How does this interpretation relate to the results of the present work? The answer is simple: the Lyapunov time is, by definition, a kind of "correlation" or "mixing" time, after which a trajectory "loses" memory of its initial conditions. In the framework of pure Levy flights (i.e., $\gamma=2$ in Eq. (1)), this time scale is not related to kinetic behavior (transport) in phase space. The latter is described by the self-similar distribution of jumps, which, in turn, is governed by the topology and the structure of surviving quasi-barriers. A large set of quasi-barriers enclosing a phase space region may "penalize" long jumps, producing a long time confinement of a trajectory in that region. The Lyapunov time may be thought of as an escape time scale from that region only in the limit of a "random walk" approximation [i.e., $\gamma \approx 1$ in Eq. (1)], in which case one is lead back to the Murison *et al.* result and the diffusion formalism presented in the present work.

In conclusion we may summarize our results as follows. The value of the Lyapunov number λ depends on local properties of phase space, while that of T_E on both local and global properties (i.e., the diffusion coefficient plus the topology and the physical dimensions of the dynamical system). The value of the diffusion coefficient depends on the LCN only in the case when most of the quasi-barriers of a perturbed integrable dynamical system have been destroyed. In the opposite limit the phenomenon of transport is not a normal diffusion, so that it cannot be considered as a random walk process and cannot be described by an ordinary Fokker-Planck equation. Therefore the functional dependence found by Murison *et al.* (1994) exists only in the limit of large perturbations [non-ballistic motion, $\gamma \approx 1$ in Eq. (1)]. Moreover the values of the coefficients in the relation found by Murison *et al.* are probably model-dependent. Finally the "stable chaotic" behavior of asteroids, found by Milani and Nobili, originates, most probably, from the presence of consecutive layers of quasi-barriers in certain regions of phase space, where transport is governed by Levy flights rather than random walks (Shlesinger *et al.* 1993), so that the Lyapunov time and the exit time are not strongly correlated.

APPENDIX

We observe that Eq. (2) is a linear, irreducible partial differential equation with constant coefficients. It is easy to find that it has particular solutions of the form

$$N_i(I,t) = C_i \exp(k_i I) \exp\left(\left(Dk_i^2 - \frac{1}{T_E}\right)t\right). \tag{9}$$

The general solution is a linear superposition of the above particular solutions

$$N(I,t) = \sum_i C_i \exp(k_i I) \exp\left(\left(Dk_i^2 - \frac{1}{T_E}\right)t\right). \tag{10}$$

If we assume that there are no sinks or sources, where particles are created or removed, the solution has to obey the constraint

$$\int_0^\infty N(I,t)dI = \int_0^\infty N(I,0)dI. \tag{11}$$

From this constraint we find that the relation

$$T_E = (k_i^2 D)^{-1} = (k^2 D)^{-1} \tag{12}$$

must be satisfied. Substituting the diffusion coefficient from Eq. (4) to the above relation, one finds

$$\log T_E = -\log ak^2 + b \log T_L, \tag{13}$$

which is the same as Eq. (8) with $c = k^2 > 0$.

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