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Spectral analysis of asteroidal trajectories in the 2:1 resonance

H. Varvoglis[1], G. Voyatzis[1], and H. Scholl[2]

[1] Section of Astrophysics, Astronomy and Mechanics, Department of Physics, Aristoteleion University of Thessaloniki, GR-54006 Thessaloniki, Greece
[2] Observatoire de Nice, Department G.D. Cassini, B.P. 229, F-06304 Nice Cedex 4, France

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Abstract. The evolution of the eccentricity of particular asteroidal trajectories in the 2:1 resonance is analysed by a spectral scheme. This analysis is an alternative to the well known analysis based on the computation of Liapunov exponents. The method can be used to estimate the long time evolution of a trajectory from a segment considerably shorter than the one needed for a Liapunov exponent calculation.

The particular trajectories analysed by our method may be divided into two classes: (a) those appearing to be unstable, in the sense that their eccentricities become overcritical within the time interval of our numerical integrations, which, in turn, implies close encounters with a planet (Mars, the Earth or Jupiter) and (b) those appearing, in the above sense, to be stable. Trajectories are computed in the frame of four models: (i) the elliptic restricted three body problem (ERTBP) model with a fixed value of 0.048 for Jupiter's eccentricity, e_J, (ii) the ERTBP model with a fixed $e_J \geq 0.055$, (iii) the ERTBP model with an artificially oscillating e_J with amplitudes corresponding to the full problem, including all outer planets, but with a frequency not corresponding to the real problem, and (iv) the six-body problem Sun - outer planets - asteroid.

The main results are: (a) There are trajectories which are stable in model (i) and which are chaotic as well as unstable in models (ii)-(iv). This suggests that chaos sets on when Jupiter's eccentricity exceeds a certain threshold value. This is not universal all over the 2:1 resonance region. (b) Trajectories which are stable in all four models are chaotic in models (ii)-(iv) according to our spectral analysis.

Key words: chaotic phenomena – celestial mechanics, stellar dynamics – minor planets

1. Introduction

It has been proposed by several authors (e.g. see Wisdom 1982, 1983, 1987; Yoshikawa 1989, 1991) that the creation of the Kirkwood gaps in the asteroidal belt is associated with the chaotic increase of asteroids' eccentricities. The mechanism proposed is based on the fact that, if the eccentricity of an asteroid exceeds a critical value (depending on the resonance associated with the gap), then its trajectory crosses the orbit of a major planet (usually Mars or Earth). It is expected that the asteroid will be removed from the resonance region by a close encounter with the planet. Also close encounters with Jupiter are possible. This scenario seems to work in the framework of the elliptic restricted three-body problem (ERTBP) in most of the resonances corresponding to main gaps, except for the case of the 2:1 resonance. In this case, numerical integrations of trajectories in the frame of the ERTBP for Jupiter masses and eccentricities corresponding to the present system (i.e. $e_J = 0.048$ and $m_J/(m_J + M_S) = 0.001$) failed to reach overcritical eccentricities within time intervals of some million years, except for asteroidal trajectories starting at the outer boundary of the resonance region (Yoshikawa 1991). Several authors showed that this problem may be overcome for certain trajectories, if a more realistic model is used instead of the ERTBP: either the general four-body problem (Wisdom 1987; Yoshikawa 1989, 1991), as confirmed by us, or a variant of the ERTBP in which Jupiter's eccentricity is considered to have an artificial sinusoidal variation with period T_{eJ} (Varvoglis 1991, hereafter referred to as Paper I).

A brief inspection shows that the above models differ from the standard ERTBP by introducing three "new" phenomena: (a) the variation of e_J itself, (b) the ensuing increase of e_J beyond the present value of 0.048, and (c) the variation of Jupiter's longitude of perihelion. It is not clear which one of the above three "new" phenomena is the dominant reason for the appearance of overcritical eccentricities in the asteroidal trajectories. The results of Varvoglis (1993, hereafter referred to as Paper II) show that the variation of e_J alone, even between values lower than 0.048, is probably sufficient to cause the appearance of overcritical eccentricities. However this result was based solely on a simple inspection of the evolution of $e(t)$ and not on a "strict" numerical criterion.

One drawback of the direct trajectory integration method is that the integration time needed for detecting the appearance of overcritical eccentricities is usually very long (of the order of

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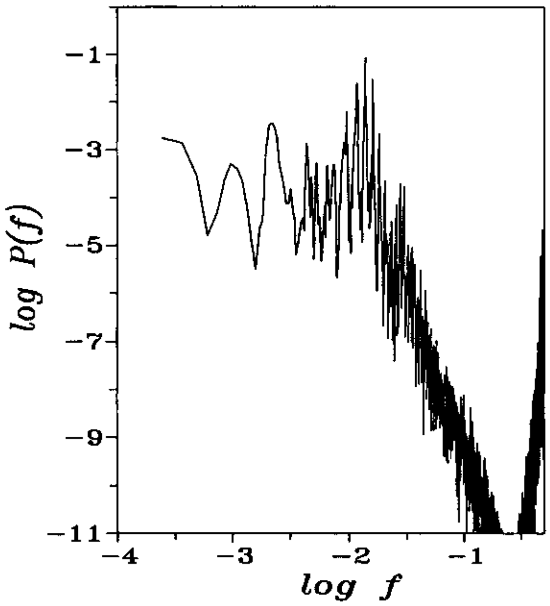

Fig. 1a. Power spectrum of trajectory IVSaM with e_{Jmin} = 0.03, e_{Jmax} = 0.06 and T_{eJ} = 113.23 T_J. In this one, as well as in the rest of the figures, the sampling period, f_0^{-1}, is approximately equal to 6 years

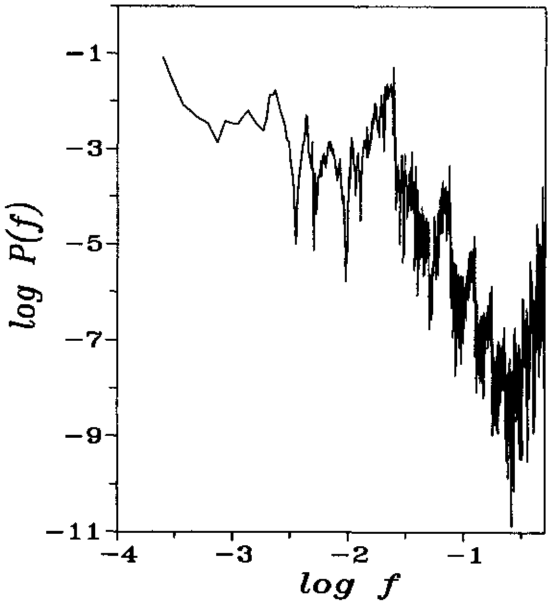

Fig. 1b. Power spectrum of trajectory IUaM with e_{Jmin} = 0.03, e_{Jmax} = 0.06 and T_{eJ} = 113.23 T_J

several million years, at least) and, in any case, unpredictable, making the investigation of this hypothesis rather difficult. Another drawback of the direct integration method is that it does not reveal the dominant mechanism causing the eccentricity increase, at least in the frame of the standard ERTBP or any other many-body problem model. Note, however, that this drawback can be waived if we assume appropriate (artificial) frames instead of the conventional mentioned above.

In this work we perform frequency analysis on time-series of dynamical variables constructed through numerical integration of trajectories. This scheme, which has been proposed by Voyatzis & Ichtiaroglou (1992), is faster than the direct numerical integration in assessing the chaotic or non-chaotic character of orbits, since it requires a, usually, shorter integration time, and thus it enables a more efficient numerical study of the problem. It can give also objective (numerical) answers to the "degree" of chaotic properties of each trajectory. Moreover, this method can answer to the question which of the three "new" phenomena (a)-(c) discussed above is sufficient to change the apparent non-chaotic character of a trajectory in the standard ERTBP. It should be noted that the above method has already been applied with very good results in the case of a mapping describing the motion of asteroids near the 3:1 resonance (Hadjidemetriou & Voyatzis 1993).

2. Description of the method

The method applies to any Hamiltonian dynamical system which can be considered as a perturbed integrable system. It is based on the analysis of time series containing the numerical values of an integral of motion of the integrable system, calculated along the trajectory of the perturbed system. In the case of the three-body problem we may select for this integral the eccentricity of the small body, which is, in the unperturbed case of the two-body problem, an integral of motion. The method is simple and easy to implement. It is based on the behaviour of the power spectral density, $P(f)$, of the time series in the

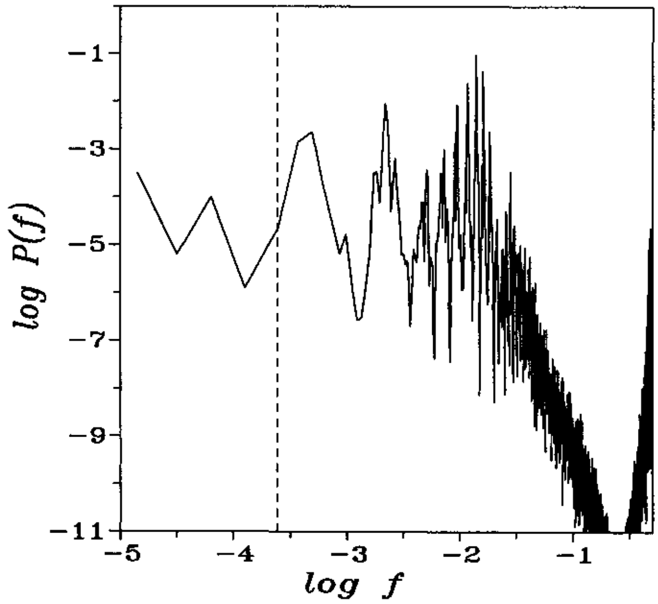

Fig. 1c. Same as Fig. 1a, except that T_{eJ} = 1113.23 T_J. Although the trajectory may be classified as "chaotic" from a segment 40,000 years long, for confirmation reasons the integration was followed over 400,000 years. The part of the trajectory between 40,000 and 400,000 years corresponds to the left of the dashed line

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low frequency domain: If the low band of the spectrum converges almost exponentially to zero as $f \to 0$, then the trajectory considered is classified as ordered. On the other hand, if the spectrum converges to a, usually significant, finite value or if it diverges to infinity (like $1/f^{\alpha}, \alpha > 0$) as $f \to 0$, then the trajectory is classified as chaotic. If one makes the reasonable assumption that, since here we are dealing with a dynamical system with more than two degrees of freedom, a chaotic trajectory will eventually reach the critical eccentricity or will, at least, reach a significantly larger eccentricity than the one at the beginning, then this method may be used to "predict" the long time behaviour of any trajectory by just looking at a modest-size time interval. It should be noted that the "degree of stochasticity" of a trajectory is a local property. On the other hand the eccentricity increase is due to transport of a trajectory in phase space, which is a global property. Therefore different dynamical systems might give similar $P(f)$ curves, but totally different rates of transport. Therefore use of $P(f)$ to compare transport rates might be meaningful only among trajectories of the same dynamical system.

The behaviour of the low frequency band emerges clearer when the lower cut-off frequency, f_{min}, of $P(f)$ is decreased, which corresponds to an increase of the time interval covered by the time series. There is, up to now, no firm theoretical estimate for the optimum value of f_{min}. However practice has shown that, as a rule, a good selection is $(f_{min}/f_0) \approx 10^{-4}$, where f_0 represents the sampling frequency (which is of the order of the basic frequency of the trajectory, in our case Jupiter's period of revolution, T_J). In this paper we use, following this rule, numerical integrations over no more than 10^5 years with very good results. This time interval is considerably less than the million-years numerical integrations required to obtain overcritical eccentricities in some of the "difficult" trajectories of the 2:1 resonance. Numerical errors play a minor role in such short integration times.

Spectral analysis of time series constructed by generalised co-ordinates, $q_i(t)$, or momenta, $p_i(t)$ (or, in general, functions of them, as it is the case with $e(t)$) provides also information on the stability of the trajectory: the presence of a strong line in the spectrum at the low frequency region indicates that the trajectory is near a separatrix (e.g. see Powell & Percival 1979; Beloshapkin & Zaslavskii 1993). The occurrence of the "separatrix index" in a trajectory is not always associated (in our models) with overcritical eccentricities and, hence, with "instability" (in the sense introduced in this paper). This is due to the fact that a trajectory near a separatrix is not necessarily within the chaotic phase space strip containing the separatrix. Alternatively, the trajectory might lie in the chaotic strip, but the Liapunov characteristic exponents may be very close to zero. In this latter case the "drifting" of the trajectory in phase space might be vanishingly slow. This shows that, according to our criterion, asteroid trajectories starting near unstable periodic orbits do not become necessarily Mars (or Jupiter) crossers, at least within time intervals of physical importance.

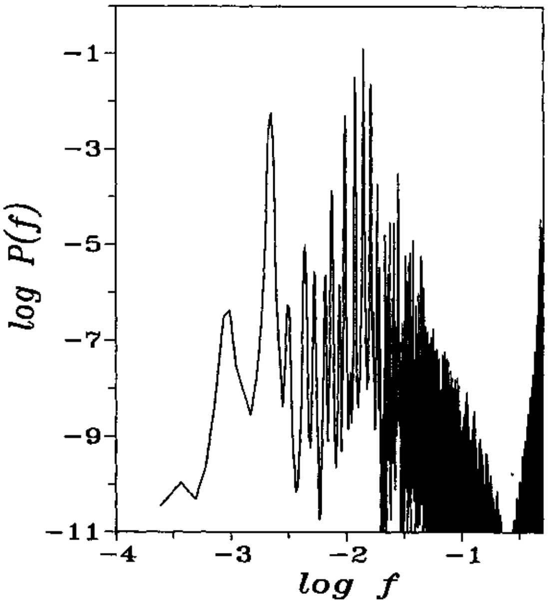

Fig. 2a. Power spectrum of trajectory IVSaM of Paper II with e_J constant and equal to 0.03

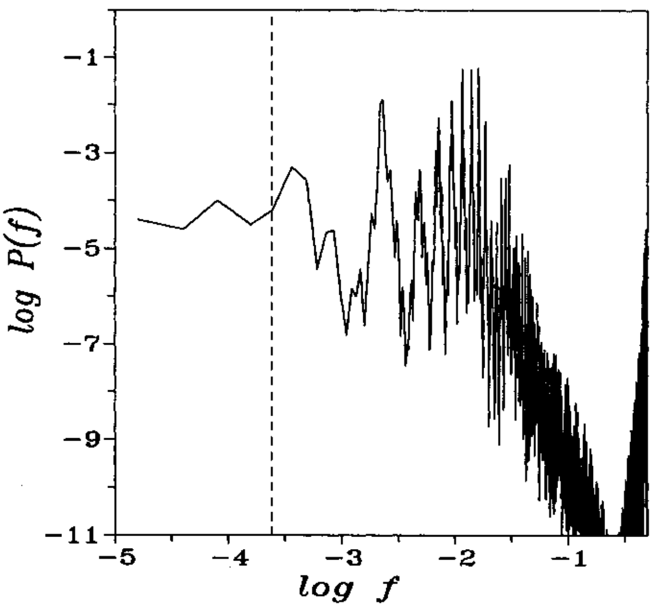

Fig. 2b. Power spectrum of trajectory IVSaM of Paper II with e_J constant and equal to 0.06. For the meaning of the dashed line see Fig. 1c

3. Results and discussion

Before presenting typical results of our method, we would like to give here the basic conclusions that may be drawn from them. By examining trajectories of the variant-ERTBP model introduced in Paper I (which can differentiate phenomenon (a) from phenomenon (b)) we have arrived at the conclusion that *either one* of these phenomena may be sufficient to cause the required increase of the asteroid's eccentricity. Note that phenomenon (c) has not been examined in the framework of the variant-

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ERTBP model (iii), since the artificially low T_{eJ} used in the model changes in this case the dynamics of the problem by introducing unreasonably high additional non-inertial "forces". However, we applied our method to trajectories calculated in a six-body model (Sun, four outer planets and the asteroid, model iv), where the behaviour of the trajectories is expected to approach the "real" problem. One would expect that these trajectories should show a more pronounced chaotic behaviour, since the perturbation is more complex, what we exactly have observed.

The power spectral density was calculated with the subroutines FFTB2 and FFT87B2 of Borland's "Numerical Toolbox" and a Hann filter (in order to suppress the continuum and any sidelobes). Applying our method to the variant ERTBP model we used the trajectory families IVS and IU of Paper II, which were integrated numerically for various values of e_{Jmin}, e_{Jmax} and T_{eJ} and for a time interval of 8192 π dimensionless units, which corresponds to approximately 40,000 years. In Figs. 1 we give typical power spectra for the behaviour of the eccentricity of trajectories IUaM (Figs. 1a and 1c) and IVSaM (Fig. 1b). All three trajectories were calculated for e_{Jmin} = 0.03, e_{Jmax} = 0.06 while T_{eJ} = 113.23T_J for trajectories 1a and 1b and T_{eJ} = 1113.23T_J for trajectory 1c. In Figs. 1, as well as in the rest of the figures, frequency in the x-axis is measured in units of the sampling frequency, f_0, while the power spectral density, $P(f)$, in the y-axis is measured in arbitrary FFT units. In Fig. 1a we can see that enough quasiperiodic features remain in the time interval examined, but the spectrum seems to converge to a non-zero (but relatively low, of the order 10^{-3}) value, which shows that the trajectory is mildly chaotic. In Fig. 1b we see that at the low band the spectrum diverges, probably as $1/f^{\alpha}$, an indication of complete chaos. Figure 1c is qualitatively similar to Fig. 1a, except that the spectrum shows at the low frequency limit a convergence to a lower value. This difference may be attributed to the fact that the period of variation of Jupiter's eccentricity in Fig. 1c is taken ten times longer than in Fig. 1a. It is interesting to note that these results are consistent with those of Tables 1 and 2 of Paper II, where we see that trajectory IVSaM (Fig. 1a) attains overcritical eccentricities within 5 10^6 years, while trajectory IUaM (Fig. 1b) within only 2 10^4 years. This is an indication that the "degree of stochasticity" of a trajectory, estimated through the shape of $P(f)$ in the present method, is related to the time interval needed for this trajectory to attain overcritical eccentricities.

Besides the above result, it is interesting to show that the present method could be used to differentiate between the effects of phenomena (a) and (b) concerning the appearance of overcritical eccentricities. In Figs. 2a and 2b we show the power spectrum of trajectory IVSaM for a constant value of e_J (i.e. for the standard ERTBP model) equal to 0.03 (2a) and 0.06 (2b). In Fig. 2a we see that the power at the low band converges to zero, showing a clearly non-chaotic behaviour. However Fig. 2b shows a clear convergence of $P(f)$ to a value of the order 10^{-4}, which indicates that at $e_J = 0.06$ the trajectory IVSaM is chaotic and may, therefore, reach overcritical eccentricities. By repeated numerical experiments, using the set of trajectories IVS and VS

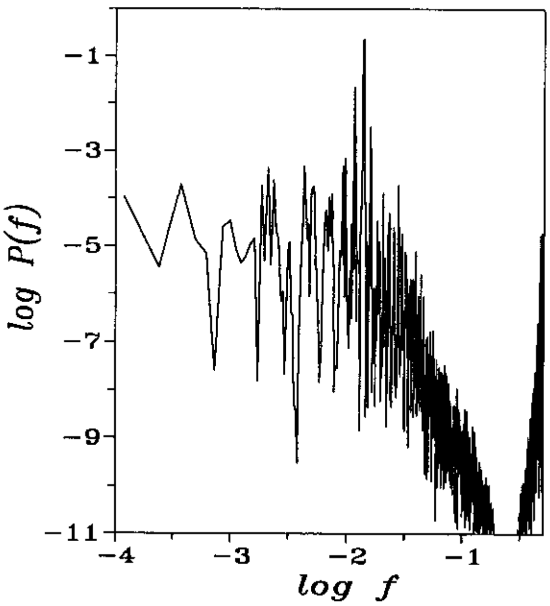

Fig. 3a. Power spectrum of trajectory IVSaM with e_{Jmin} = 0.01, e_{Jmax} = 0.04 and T_{eJ} = 113.23 T_J

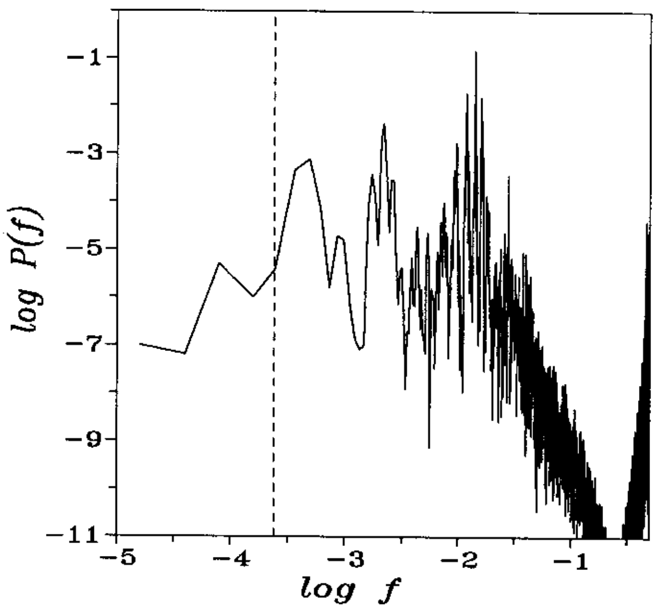

Fig. 3b. Same as Fig. 3a, except that T_{eJ} = 1113.23 T_J. For the meaning of the dashed line see Fig. 1c

of Paper II, we found that the "critical" value of e_J is close to 0.055. Now $e_J \geq 0.055$ within an interval of 20,000 years in a complete period of 54,000 years. During the high-e_J time intervals, the trajectory becomes chaotic and its eccentricity may increase by a small amount, while during the low-e_J intervals the eccentricity stays, on the average, constant. The accumulation of the small eccentricity increases during the high-e_J intervals might total an overall eccentricity increase beyond the critical value. Therefore this mechanism, alone, might be sufficient for

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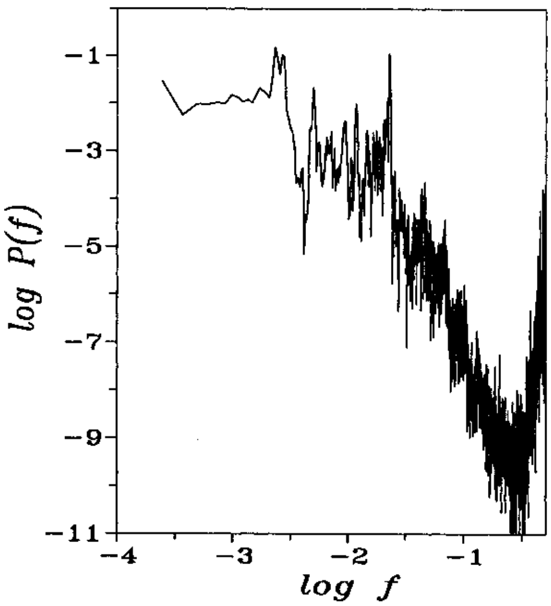

Fig. 4a. Power spectrum of trajectory of the full six body model with starting values a = 3.27 AU, e = 0.20, i = 5°, σ = 151°)

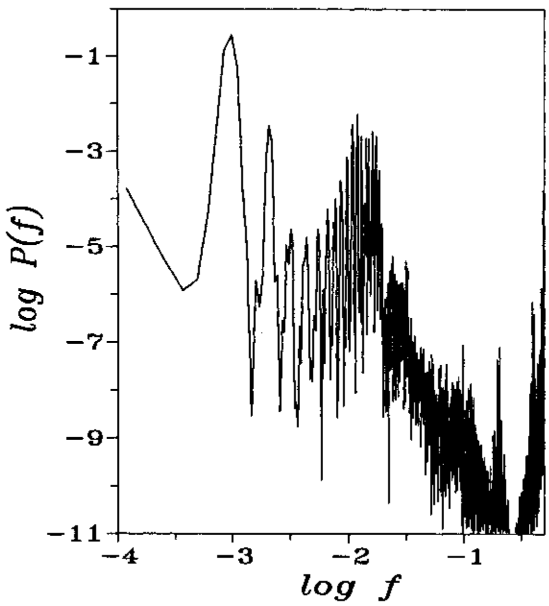

Fig. 4c. Power spectrum of trajectory of the full six body model with starting values a = 3.28 AU, e = 0.25, i = 5°, σ = −30°)

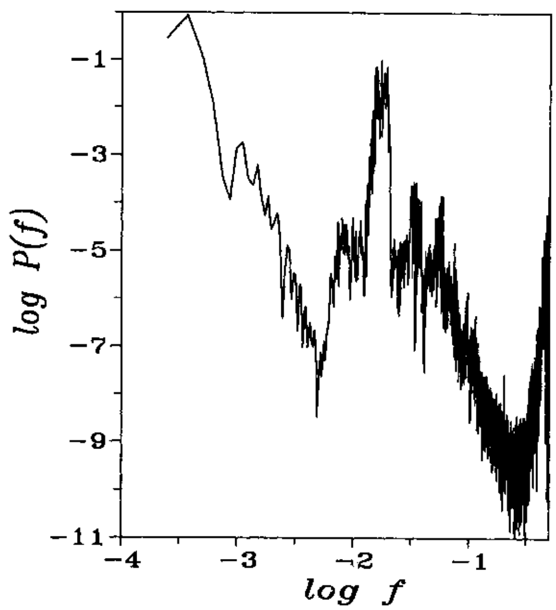

Fig. 4b. Power spectrum of trajectory of the full six body model with starting values a = 3.28 AU, e = 0.25, i = 5°, σ = 151°

the appearance of high eccentricities in the trajectories at the 2:1 resonance.

In Figs. 3 we show the power spectrum of trajectory IVSaM for e_{Jmin} = 0.01, e_{Jmax} = 0.04 and T_{eJ} = 113.23 T_J (Fig. 3a) and T_{eJ} = 1113.23 T_J (Fig. 3b). The values of e_{Jmin} and e_{Jmax} were selected in such a way, as to have the same amplitude with the actual case (0.03-0.06) but with an e_{Jmax} value well below 0.055, where chaotic behaviour appears for constant e_J. $P(f)$ in these figures as well presents an obvious convergence to non-zero values in the low band, which is more pronounced in the case with the lower period of variation of e_J (Fig. 3a). Therefore we may conclude that the variation of e_J solely may induce chaotic behaviour into a trajectory, leading to an increase of its eccentricity, even when e_{Jmax} does not exceed the critical value 0.055.

Finally in Figs. 4a, 4b and 4c we give power spectra of three trajectories calculated with the more realistic model of the full six-body problem. The power spectrum in Fig. 4a indicates homogeneous chaos, with a possible peak at the very low frequencies, in complete agreement with the fact that this trajectory attained overcritical eccentricities within 2.5 10^6 years. The trajectory in Fig. 4b shows a clear strong peak at a very low frequency, an indication of the presence of a separatrix in the close neighbourhood, together with a $1/f^{\alpha}$ - like behaviour, an indication of strong chaos. This trajectory attained overcritical eccentricities in a time interval of 14.5 10^6 years. It is very interesting to note that the power spectrum of the trajectory of Fig. 4c, which differs from that of Fig. 4b only in the initial value of the critical argument, shows a strong peak at a low (but higher than that of Fig. 4b) frequency, together with a clear convergence to a non-zero value ($\approx 10^{-4}$), an indication of weak chaos. This trajectory did not attain an overcritical eccentricity within a time interval of nearly 10^8 years. A qualitatively very similar (to that of Fig. 4c) power spectrum was found for a trajectory with the same initial conditions as those of Fig. 4a but with e = 0.30. We speculate that the last two trajectories, too, will eventually become Mars/Earth crossers, unless there is a barrier which separates the now known "unstable chaotic (overcritical eccentricities)" from the "stable chaotic" region. Since, as a rule, we have found that trajectories in the "stable chaotic" region are close to separatrices (Fig. 4c), we may conjecture that surviving invariant tori and/or cantori in the neighbourhood of the separatrix play the role of such a barrier. An asteroid very

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close to the separatrix lies, quite probably, in the corresponding "chaotic strip" and might, therefore, escape quickly to regions of high eccentricities. On the other hand an asteroid not so close to the separatrix would not be in the chaotic strip and should, therefore, migrate to regions of high eccentricities through Arnold diffusion (i.e.*along* resonances rather than *across* them), a process that might be longer than the presently available numerical integration results (or, even, the age of the Solar System). If the above scenario on how a trajectory may reach overcritical eccentricities is correct, then spectral analysis of consecutive segments of a trajectory, possessing a peak at a low frequency and attaining overcritical eccentricities, should show a, more or less, monotonous "drift" of this peak towards even lower frequencies. This kind of analysis of trajectories in Figs. 4 gave results which seem to be consistent with the above scenario, as described in the next section.

4. Conclusions

In concluding we may summarise as follows. The main effort in this paper was to investigate the effect of the variation of Jupiter's eccentricity, e_J, on the degree of stochasticity of asteroidal trajectories near the 2:1 resonance as well as on the transport of these trajectories in phase space towards regions of high eccentricities. The main result is that trajectories, which appear ordered in the frame of the classical planar elliptical restricted three body problem for the present value of e_J, become chaotic when anyone of the "new phenomena" introduced by the addition of the outer planets to the above model is taken into account separately. Since the corresponding dynamical system has more than two degrees of freedom, any chaotic trajectory is expected to migrate, eventually, to phase space regions of high eccentricity. However it is not known whether the time scale of this transport, which is expected to depend on the structure of phase space and the width of the chaotic strips along separatrices, is less than the age of the Solar System. In order to answer this question we applied our spectral analysis method to asteroidal trajectories calculated in the frame of a six-body model, where all the outer planets are included. We have found that the power spectral density of trajectories, which attain overcritical eccentricities in a fairly short time interval, show a pronounced peak at the low frequency regime, an indication that the trajectory lie close to a separatrix (note that Fig. 4a might be interpreted as possessing more than one peaks, smoothed by the filtering technique used in the analysis). Spectral analysis of segments of the trajectory of Fig. 4b shows a drifting with time of the low frequency peak to lower frequencies, which is consistent with the above presented scenario. On the other hand the trajectory in Fig. 4c, has a power spectral density with a peak at the low frequency band which presents a slow drift to lower frequencies, but it did not attain overcritical eccentricities within a time interval of a hundred million years. It should be noted that the initial conditions of this trajectory were selected in a phase space region where asteroidal appear to be stable in the sense introduced in the present paper (e.g. Moons & Morbidelli 1993). It is obvious that the problem, which might be of central importance to the creation and existence of the 2:1 Kirkwood gap, needs further investigation.

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References

Beloshapkin V.V., Zaslavskii G.M., 1993, Phys. Lett. 97A, 121
Hadjidemetriou J., Voyatzis G., 1993, Celest. Mech. 57, 95
Moons M., Morbidelli, A., 1993, Celest. Mech. 57, 99
Powell G.E., Percival I.C., 1979, J. Phys. A: Math. Gen. 12, 2053
Yoshikawa M., 1989, A&A 213, 436
Yoshikawa M., 1991, Icarus, 92, 94
Varvoglis H., 1991, Earth, Moon and Planets, 54, 257
Varvoglis H., 1993, A&A 275, 301
Voyatzis G., Ichtiaroglou S., 1993, J. Phys. A: Math. Gen. 25, 5931
Wisdom J., 1982, AJ 87, 577
Wisdom J., 1983, Icarus, 56, 51
Wisdom J., 1987, Icarus, 72, 241

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