On a Fokker-Planck approach to asteroidal transport

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Recent studies show that chaotic motion should be considered as the rule rather than the exception in the asteroid belt, if the perturbations of many planets are taken into account. Assuming that asteroids are constantly diffusing away from the main belt, we may model their transport, i.e. the evolution of a distribution of initial conditions in the action space of the respective dynamical system, through a kinetic equation of the Fokker-Planck type. This consists in a two-step procedure: (i) calculation of the transport coefficients and (ii) solution of the diffusion equation, which depends critically upon the functional form of the transport coefficients. Recent results on this subject can be found in Varvoglis & Anastasiadis (1996) and Murray & Holman (1997) (hereafter M&H). The latter authors performed analytical estimates for the diffusion coefficients in mean motion resonances of the planar elliptic restricted three body problem (ERTBP). In this note, we present preliminary results on a numerical calculation of 'local' (action-dependent) diffusion coefficients. We selected for this study four mean motion resonances: the 5/3, 7/4, 9/5 and 12/7 resonances of the ERTBP.

The ERTBP has a 2-D action space. However, diffusion through a mean motion resonance can be studied as a 1-D process, since resonant orbits may reach planet-crossing eccentricities while, at the same time, the semi-major axis remains practically unchanged. The actions are defined by $L = \sqrt{(1-\mu)a}$ and $I = L(1-\sqrt{1-e^2}) \approx \frac{1}{2}Le^2$, where a denotes the semi-major axis and e the eccentricity of the asteroid. The *diffusion coefficient* in the action I (similarly for L) is defined by

$$D(I_0) = \lim_{t\to\infty} \frac{\langle (\Delta I)_0{}^2 \rangle}{t} = \lim_{t\to\infty} \frac{\langle (I(t)-I_0)^2 \rangle}{t} \tag{1}$$

the average being taken over all possible realizations of $I = I_0$. In general D is a function of I. Assuming that after a relatively short time, τ_C, phase randomization is established, one obtains the quasi-linear approximation of the diffusion coefficient, $D_{QL}(I) = \langle (\Delta I)^2 \rangle / \tau_C \simeq D_\infty(I)$. Note that τ_C need not be the same, and in general it is not, for every I_0. The numerical calculation of $D(I)$ for a given resonance was done as follows: setting $a = a_{res}$, the eccentricity range (0.01-0.25) was split into a number of bins and a distribution of N_p test-particles with the same initial value of e was set

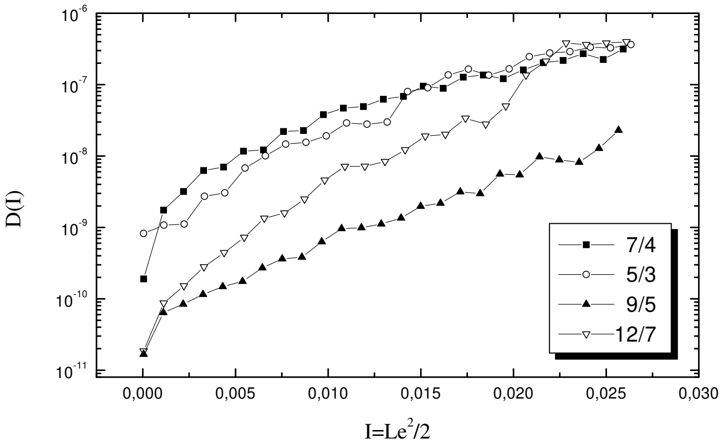

Figure 1. The $D(I)$ curves as a function of I for the four resonances studied.

on each bin. The angles, ϖ(=perihelion longitude) and l(=mean anomaly), were chosen at random. Jupiter was set at $a' = 5.2\mathrm{AU}$ with $e' = 0.048$, $\varpi' = 0$ and $l' = 0$. These distributions were then integrated up to $t = \tau_c$, which was selected within the range $10^4 - 10^5$ years. The integrations were performed with the SWIFT integrator (Levison & Duncan 1994). Figure 1 shows the resulting values of $D(I)$ as a function of the initial I, for all the resonances tested. The distributions used for these graphs consist of $N_p = 500$ particles each and the eccentricity range was split into 25 bins. We repeated the calculations of $D(I)$ for $50 < N_p < 1000$ particles. The values of $D(I)$ are not affected much by N_p, at least for small eccentricities, and any value $N_p > 100 - 200$ is enough. The choice of the integration time is a more important subject. For $e \sim e' = 0.048$ the time needed for $D(I_0; t)$ to saturate may exceed 10^5 years, owing to long-time correlated chaotic motion and even regular orbits, while for $e > 0.15$ the time needed is typically $\leq 10^4$ years. The functional form of $D(I)$ shows that an average (i.e. constant) coefficient cannot be used to characterize the transport process in the whole eccentricity range, even for the most chaotic resonance (7/4). As the order of the resonance increases the slope of $D(I)$ becomes steeper. Note that while the $D(I)$'s of different resonances vary by orders of magnitude in the small-e_0 region, they tend to converge for $e > 0.2$ (0.3 for the 9/5 case). Thus, at high eccentricities, particles diffuse at a very similar, and almost constant, speed, which is independent of the initial semi-major axis. Figure 2 shows the ratio $D(L)/D(I)$ as a function of initial I (left) and the diffusion coefficient in apocentric distance $D(Q)$ as a function of initial $Q = a(1 + e)$ (right).

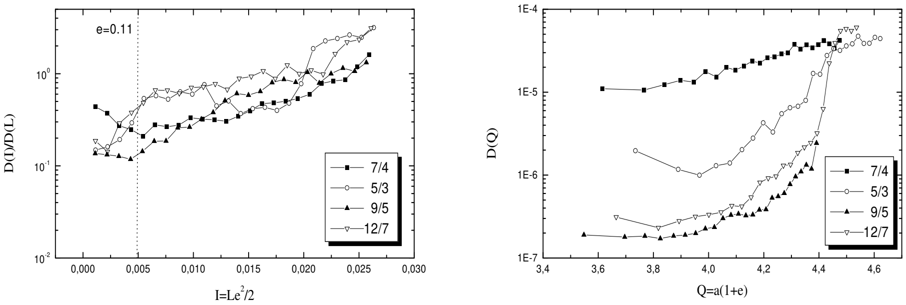

Figure 2. (left) The ratio $D(L)/D(I)$ as a function of I. (right) The diffusion coefficient $D(Q)$ in apocentric distance.

At small eccentricities ($e < 0.1$) $D(I)$ is about an order of magnitude larger than $D(L)$, but for $e > 0.1$ $D(L)$ becomes comparable to $D(I)$, in all the resonances tested. This can be thought of as the limit of validity of the 1-D approximation. Also, it is clear that for $Q > 4.3$AU a 'jump' in $D(Q)$ occurs and the values tend to converge, independent of the initial value of a. This value of Q marks the border of the 'resonance-overlap' regime and is in agreement with the value predicted by the '$\mu^{2/7}$-law' (Wisdom, 1980). Combining these results we come to the conclusion that, at large eccentricities, transport can be regarded as 2D standard diffusion (Brownian motion) with a constant coefficient, $D = D_{R-O}$.

M&H have shown that $D(I)$ may be approximated by a power-law. Fitting our data by power-laws ($D(I) \propto I^b$) the resulting exponents, b, are those given in Table 1. The exponent indeed increases with the order of the resonance but the uncertainties are rather large, at least for the 12/7 and the 9/5 cases. Using the solution of the Fokker-Planck equation given in M&H, the removal time of an asteroid, T_R, can be estimated. These results are given in Table 1. Note however that, due to the power-law dependence of $D(I)$ on I, T_R depends critically on the fit. Therefore the values in Table 1 are indicative of the different paces of the escaping process among the four resonances studied. The T_R estimates are compared to the results of a long-term (10-50 Myrs) numerical integration of 300 particles at each resonance, with $a = a_{res}$, $0.08 < e < 0.1$ and ϖ, l chosen at random. In this experiment the escape time is the time at which the particle approaches Jupiter within Hill's radius. The median of the histogram of the escape times, $\bar{T}_E$, is chosen as a characteristic escape time for each resonance. A comparison between the values of $\bar{T}_E$ and T_R (Table 1) shows that T_R's tend to be smaller than the corresponding $\bar{T}_E$'s. Note that for the 12/7 ($q = 5$) resonance T_R (and $\bar{T}_E$) is smaller than for the 9/5 case ($q = 4$). A detailed study of chaotic motion in this resonance was given in Tsiganis et al. (2000).

Table I. The exponent, b, of the power-law fit for $D(I)$ and the T_R estimates for $e = 0.05$ and $e = 0.1$ are given along with $\bar{T}_E$ (from the integrations)

Resonance	$b \pm \delta b$	$T_R(0.05)$ (Myrs)	$T_R(0.1)$ (Myrs)	$\bar{T}_E$ (Myrs)
5/3	2.6 ± 0.2	0.11	0.09	0.26
7/4	2.3 ± 0.1	0.06	0.05	0.16
9/5	5.3 ± 0.6	32.08	4.57	21.12
12/7	4.9 ± 0.6	17.68	2.81	7.29

These preliminary results show that we can reproduce, qualitatively, the properties of transport within different resonances. Quantitative agreement with numerical integration is not achieved so far and a detailed comparison with the analytic work of M&H cannot be made. This is most probably due to the fact that our calculations are based on *osculating* elements. At the moment new calculations are performed, addressing diffusion in the space of *proper* elements.

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