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ORIGINAL ARTICLE

Quasi-critical orbits for artificial lunar satellites

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Abstract We study the problem of critical inclination orbits for artificial lunar satellites, when in the lunar potential we include, besides the Keplerian term, the J_2 and C_{22} terms and lunar rotation. We show that, at the fixed points of the 1-D averaged Hamiltonian, the inclination and the argument of pericenter do not remain both constant at the same time, as is the case when only the J_2 term is taken into account. Instead, there exist *quasi-critical* solutions, for which the argument of pericenter librates around a constant value. These solutions are represented by smooth curves in phase space, which determine the dependence of the quasi-critical inclination on the initial nodal phase. The amplitude of libration of both argument of pericenter and inclination would be quite large for a non-rotating Moon, but is reduced to $<0^\circ.1$ for both quantities, when a uniform rotation of the Moon is taken into account. The values of J_2, C_{22} and the rotation rate strongly affect the quasi-critical inclination and the libration amplitude of the argument of pericenter. Examples for other celestial bodies are given, showing the dependence of the results on J_2, C_{22} and rotation rate.

Keywords Lunar artificial satellite · Critical inclination · C_{22}

1 Introduction

The motion of an artificial satellite around an oblate body has been extensively studied during the past decades, by several authors. The interest was mainly focused on the effect of the J_2 term of the expansion of the potential in spherical harmonics which leads to a critical value of the inclination, equal to $63^\circ.43$ (Allan 1970; Hughes 1981; Jupp 1988). The critical orbits have constant inclination and argument of pericenter. This, combined with a resonance in mean motion, results in a repeat ground track orbit. Molniya and Tundra type orbits, widely used to cover the communications of high latitude areas, take advantage of that feature

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(Liu and Innanen 1986; Delhaise and Henrard 1993). Apart from these critical orbits, near-polar periodic orbits would be of interest (Ferrer et al. 2007).

In the case of the Earth, this approximation describes quite well the actual case, as the small value of C_{22} does not cause significant perturbations to the critical inclination. The Moon, however, exhibits a different behavior. The parameter C_{22} cannot be neglected, since its value is only nine times smaller than J_2. (De Saedeleer and Henrard, 2006,hereafter DSH06) studied the lunar case, showing that the critical inclination depends on the initial phase of the ascending node. They also showed that, when only the effect of the C_{22} term is considered, the critical inclination becomes $39°14'$, no matter what the initial nodal phase is. In their derivation the time variation of the averaged inclination was neglected, most likely being considered small on the assumed time scale.

As will be shown in the following, under the combined effect of J_2 and C_{22} terms (or C_{22} alone), the averaged inclination does not remain constant, but performs long periodic oscillations, whose characteristics depend on the model. The only exceptions are the stationary points on the plane of inclination—nodal longitude which do not correspond to a constant value of the argument of pericenter (contrary to the 'J_2' model). The time evolution of the inclination causes variations in the argument of pericenter of similar order of magnitude. Under those circumstances, the term 'critical inclination' becomes meaningless. At the same time, a strong dependence of the 'critical inclination' on the longitude of the ascending node was found in DSH06. We will show that, when the rotation of the Moon is taken into account, the previously described phenomena are considerably weakened.

It is also interesting to study what the situation is for other celestial bodies, whose physical parameters may be quite different from those of the Moon, as can be found e.g. in Bertotti et al. (2003) for Mars, in Anderson et al. (1997) for Europa and in Anderson et al. (2001) for Callisto. The asteroid 433 Eros and the Earth (parameters taken from DSH06) are also considered as limiting cases, as the former has a very large value of C_{22}, compared to J_2, while the latter a very small one (see Table 1). Moreover, these bodies have widely different rotational periods, varying from some hours (Eros) to almost a month (Moon). This characteristic helps us to appreciate the contribution of the rotation rate to the properties of the orbits.

2 Hamiltonian of the problem and equations of motion

We consider an artificial satellite in orbit around the Moon. We use a rotating frame whose origin is at the center of the Moon, the x axis passes through the longest lunar meridian

Table 1 The values of the parameters J_2, C_{22} and the angular velocity expressed in normalized units[†], for 433 Eros, Callisto, Europa, Moon, Mars and Earth

	J_2	C_{22}	n_M	J_2/C_{22}	J_2/n_M	C_{22}/n_M
433 Eros	0.117344	0.0533278	0.270416	2.20075	0.433777	0.197104
Callisto	32.7×10^{-6}	10.2×10^{-6}	0.001682	3.20588	0.019445	0.006065
Europa	389×10^{-6}	117×10^{-6}	0.006234	3.32479	0.062398	0.018767
Moon	202×10^{-6}	22.271×10^{-6}	0.000769	9.07009	0.262719	0.028965
Mars	1959×10^{-6}	63.17×10^{-6}	0.018940	31.0116	0.103432	0.003335
Earth	1082.6×10^{-6}	1.57×10^{-6}	0.016380	689.554	0.066092	0.000096

[†] The gravitation constant $\mathcal{G}$, the mass of the primary and its radius are all equal to 1. See Section 4.

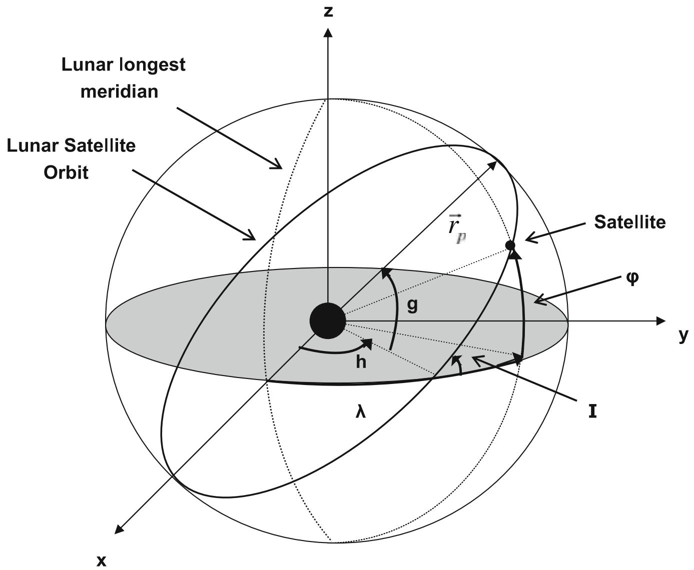

Fig. 1 Orientation of the orbit in space. The *arrows* indicate the direction of the angles

and the $x - y$ plane coincides with the lunar equatorial plane (see De Saedeleer 2006 for a detailed discussion). This frame rotates at the rate of the Moon's mean synchronous rotation n_M (see Fig. 1). The potential, written in selenographic coordinates, has the general form (Vallado 2001; Sidi 2002; Bertotti et al. 2003)

$$V = -\frac{\mu}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)\,[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda] \tag{1}$$

where $\mu = \mathcal{GM}$, R is the mean equatorial radius of the Moon, (λ, ϕ) are the selenographic longitude and latitude, respectively, r the selenocentric distance of the satellite, C_{nm} and S_{nm} the non-normalized gravity coefficients and P_{nm} the Legendre Polynomials of degree n and order m. For $m = 0$, we get the zonal harmonic coefficients, for which we will use the notation $J_n = -C_{n0}$. Considering only the effects of the J_2 and C_{22} terms, and including the lunar rotation, the Hamiltonian of the problem has the following form (DSH06):

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{J_2} + \mathcal{H}_{C_{22}} + \mathcal{H}_{n_M} \tag{2}$$

$\mathcal{H}_0$ is the Keplerian term, while $\mathcal{H}_{J_2}$ and $\mathcal{H}_{C_{22}}$ express the perturbing terms. $\mathcal{H}_{n_M}$ describes the rotation of the Moon. In selenographic coordinates, the previous equation can be written as

$$\mathcal{H} = \left(\frac{u^2}{2} - \frac{\mu}{r}\right) + \frac{\epsilon\mu}{r^3} P_{20}\,(\sin\phi) + \frac{\delta\mu}{r^3} P_{22}\,(\sin\phi)\cos 2\lambda - n_M\, p_\lambda, \tag{3}$$

where the term $u^2/2 - \mu/r$ is the unperturbed (Keplerian) Hamiltonian of the problem, $\epsilon = J_2 R^2$, $\delta = -C_{22} R^2$, $P_{20}(\sin\phi) = (3\sin^2\phi - 1)/2$ and $P_{22}(\sin\phi) = 3\cos^2\phi$. The angles ϕ, λ and the distance r can be written as functions of the semi major axis a, the eccentricity e, the inclination I of the orbit relative to the equatorial plane, the longitude of the ascending node Ω, the argument of pericenter ω and the mean anomaly M. Reverting to the canonical set of Delaunay variables $[l, g, h, L, G, H]$, which are defined here as follows:

$$l = M, \quad g = \omega, \quad h = \Omega - n_M t$$
$$L = \sqrt{\mu a}, \quad G = \sqrt{\mu a \left(1 - e^2\right)}, \quad H = \sqrt{\mu a \left(1 - e^2\right)} \cos I$$

and averaging over the fast angle, l, the following first order averaged Hamiltonian was obtained by DSH06:

$$\bar{\mathcal{H}} = -\frac{\mu^2}{2L^2} + \frac{\epsilon\mu^4}{4G^3L^3} - \frac{3\epsilon\mu^4 H^2}{4G^5L^3} + \frac{3\delta\mu^4}{2G^3L^3}\cos 2h - \frac{3\delta\mu^4 H^2}{2G^5L^3}\cos 2h - n_M H \quad (4)$$

We have checked that the same first order result can be obtained using the simpler method described in Roy (1982). Moreover, we checked that the second order terms, with respect to the perturbations, are indeed negligible in the absence of nearby commensurabilities between the mean motion of the satellite and the rotation rate of the Moon. We did so writing Eq. 3 in Delaunay elements, expanding up to degree 4 in eccentricity (see Murray and Dermott 1999) and applying the Lie transform method, as presented by Morbidelli (2002). Note that, here, the orbital elements $[a, e, I, \Omega, \omega, M]$ are defined in an inertial frame, while the Delaunay variables are defined in the rotating frame described above. However, the transformation only affects the definition of the nodal longitude h.

The first-order averaged equations of motion are

$$\dot{l} = \partial\bar{\mathcal{H}}/\partial L \quad (5)$$
$$\dot{L} = -\partial\bar{\mathcal{H}}/\partial l = 0 \quad (6)$$
$$\dot{g} = \partial\bar{\mathcal{H}}/\partial G = -\frac{3\epsilon\mu^4}{4G^4L^3}\left(1 - 5\frac{H^2}{G^2}\right) + \frac{3\delta\mu^4}{2G^4L^3}\cos 2h\left(-3 + 5\frac{H^2}{G^2}\right) \quad (7)$$
$$\dot{G} = -\partial\bar{\mathcal{H}}/\partial g = 0 \quad (8)$$
$$\dot{h} = \partial\bar{\mathcal{H}}/\partial H = -n_M - \frac{3\epsilon\mu^4 H}{2G^5L^3} - \frac{3\delta\mu^4}{G^5L^3}H\cos 2h \quad (9)$$
$$\dot{H} = -\partial\bar{\mathcal{H}}/\partial h = \frac{3\delta\mu^4 H}{G^3L^3}\sin 2h - \frac{3\delta\mu^4}{G^5L^3}H^2\sin 2h \quad (10)$$

We are going to study four different models: The '$J_2 + C_{22}$', 'C_{22}', '$C_{22} + n_M$' and '$J_2 + C_{22} + n_M$' models, each of them taking into account the terms indicated by its label.

2.1 Definition of the quasi-critical inclination

The critical inclination in the 'J_2' model is defined as the constant value of I for which the time derivative of g is zero. This value (if there is one) corresponds to a fixed point in the (h, H) phase diagram, i.e. a periodic orbit ($\dot{H} = \dot{h} = 0$). If $\dot{g} = 0$, but $\dot{H} \neq 0$ and $\dot{h} \neq 0$, which is the case in more complicated physical models, we cannot use the term 'critical' value of inclination, as the time evolution of H (or, equivalently I) and h will lead to changing values of I and $\dot{g}$. Having in mind applications of constant g, we can search for solutions, for which the mean value of the argument of pericenter remains constant ($\langle \dot{g}(t) \rangle_t = 0$) and, preferably, the amplitude of libration is small. As will be shown later, these solutions correspond to smooth curves in the phase diagram (see Figs. 2, 3). We define them as *quasi-critical orbits*. We draw attention to the fact that the initial values (h_0, H_0) that give quasi-critical orbits belong to the same curve, a trajectory of the averaged system. Thus, there exists in fact a single *critical* value of the action $\mathcal{A}$ (i.e. the area enclosed by the curve)

$$\mathcal{A} = \oint H dh$$

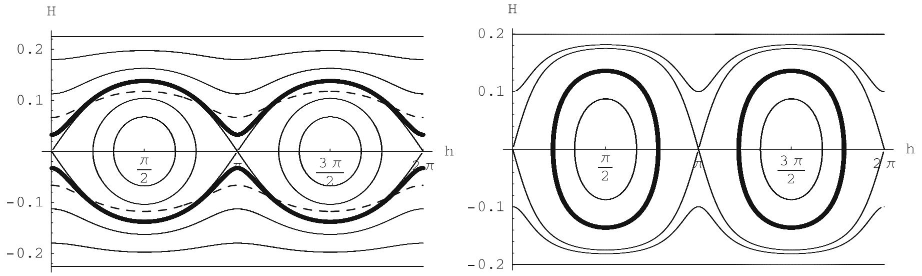

Fig. 2 Phase diagrams for the 'J_2+C_{22}' model. The two diagrams show the cases of Moon (*left*) and Callisto (*right*), for which the critical curves (*bold lines*) are outside and inside the separatrix, respectively. The dashed line (*left panel*) corresponds to the solution of DSH06

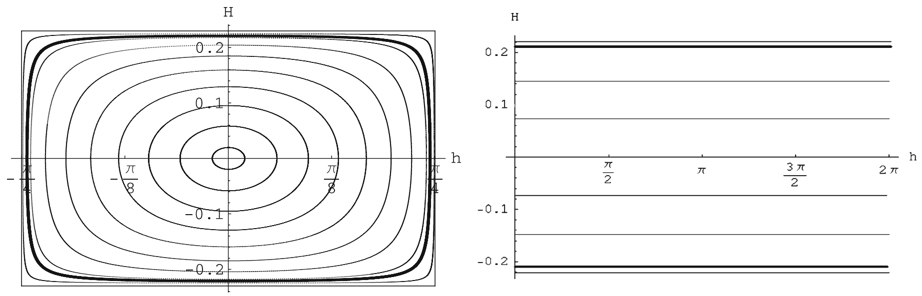

Fig. 3 Phase diagrams for the 'C_{22}' (*left*) and '$C_{22}+n_M$' model (*right*) for the case of Moon. The phase diagram for the '$J_2+C_{22}+n_M$' model is very similar to that of the '$C_{22}+n_M$' model

It is not an easy task to solve the system of differential equations 7–10 for the functions $g(t)$, $G(t)$, $h(t)$, $H(t)$. Nevertheless, it is separable, since g is an ignorable coordinate, and integrable. In this case it can be found an expression for $H(h)$, by solving the differential equation obtained by dividing Eqs. 9 and 10. The solution is

$$H(h) = \frac{-C \pm \sqrt{C^2 + (C_1 - A\cos 2h)\,(D + B\cos 2h)}}{D + B\cos 2h} \tag{11}$$

The parameters A, B, C and D are defined as follows

$$A = \frac{3\delta\mu^4}{G^3L^3} \tag{12}$$

$$B = -\frac{3\delta\mu^4}{G^5L^3} \tag{13}$$

$$C = -n_M \tag{14}$$

$$D = -\frac{3\epsilon\mu^4}{2G^5L^3} \tag{15}$$

The expression that gives the integration constant C_1 as a function of the initial conditions H_0 and h_0 is

$$C_1 = 2CH_o + DH_o^2 + A\cos 2h_o + BH_o^2\cos 2h_o \tag{16}$$

Substituting $H(h)$ in Eq. 7, we get an expression for $\dot{g}(h)$. Dividing $\dot{g}(h)$ by (9), we have an expression for $dg(h)/dh$, which has the form

$$g'(h) = \frac{dg(h)}{dh} = \frac{\dot{g}(h)}{\dot{h}(t)} = \Big[-40G^{10}L^6{n_M}^2 - 60G^5H_0L^3n_M\epsilon\}\,\mu^4 + 9G^2\epsilon^2\mu^2$$
$$- 45{H_0}^2\epsilon^2\mu^8 - 72G^2\delta^2\mu^8\cos^2 2h + 90G^2\delta\epsilon\mu^8\cos 2h_0 - 90{H_0}^2\delta\epsilon\mu^8\cos 2h_0$$
$$- 6\delta\mu^4\cos 2h[20G^5H_0L^3n_M + 3G^2\epsilon\mu^4 + 15{H_o}^2\epsilon\mu^4 - 30(G^2 - {H_0}^2)\delta\mu^4\cos 2h_0]$$
$$+ 20G^{10}L^6n_MW\Big]/\big[6G^6L^3\mu^4W(\epsilon + 2\delta\cos 2h)\big] \tag{17}$$

where

$$W = \Big\{(2G^5L^3n_M + 3H_0\epsilon\mu^4)^2 + 36G^2\delta^2\mu^8\cos^2 2h - 18(G^2 - {H_0}^2)\delta\epsilon\mu^8\cos 2h_0$$
$$+ 6\delta\mu^4\cos 2h[4G^5H_0L^3n_M + 3G^2\epsilon\mu^4 - 6(G^2 - {H_0}^2)\delta\mu^4\cos 2h]\Big\}^{1/2} G^{-5}L^{-3} \tag{18}$$

In all models, the quasi-critical orbit is defined by the condition

$$\left\langle g'(h)\right\rangle_h = \int_{h1}^{h2} g'(h)dh = 0$$

The mean value of $g'(h)$ can be found by integrating (17) along the corresponding invariant curve. The appropriate limits (h_1, h_2) are found by solving the equation $H(h) = 0$. If the critical curve in phase space represents a libration, then $0 \leq h_1, h_2 \leq 2\pi$. If the critical curve belongs to the region of rotations, the equation $H(h) = 0$ has no real solution and (h_1, h_2)=$(0, 2\pi)$, respectively. The expression (17) is complicated enough and it cannot be integrated easily. On the other hand, the numerical calculation of the integral gives satisfactory and precise results. Thus, we calculated numerically the value of $\left\langle g'(h)\right\rangle_h$ for initial conditions $0 \leq h_0 \leq 360°$ and H_0 such that $0 \leq I \leq 90°$. For a given value of h, the value of I for which the integral is equal to zero is the quasi-critical inclination (I_{qc}). For that value, the argument of pericenter performs oscillations with period and amplitude that vary from model to model, but its mean value is constant with time.

3 Analysis and results

We restrict ourselves to orbits close to the surface of the Moon, as the effect of the Earth cannot be ignored at distances greater than $3R$ from the center of the Moon (De Saedeleer 2006).

All models mentioned above have a common characteristic: I_{qc} depends on h_0 (to be precise, on $\cos 2h_0$). This is not surprising, since h is present in Eq. 7 and, as we already said, the critical quantity is the action $\mathcal{A}$. When we ignore the rotation of the Moon, this dependence is quite strong. Specifically, in the '$J_2 + C_{22}$' model, the I_{qc} varies from 52° to almost 82°, for $0 \leq h \leq 90°$ (see Fig. 4—left panel). The 'C_{22}' model exhibits even stronger dependence, since $I_{qc} \approx 26°.44$ for $h = 0$ and reaches 90° for $h \approx 39°.3$. There is no I_{qc} for $39°.3 < h < 50°.7$, while for $50°.7 < h < 90°$ the value of I_{qc} decreases from 90° to 26°.44 (see Fig. 4—right panel). It is interesting that, in the 'C_{22}' model, for $h = \pm 45°, \pm 90°$ the argument of pericenter freezes, no matter what the initial I. However, this does not mean that

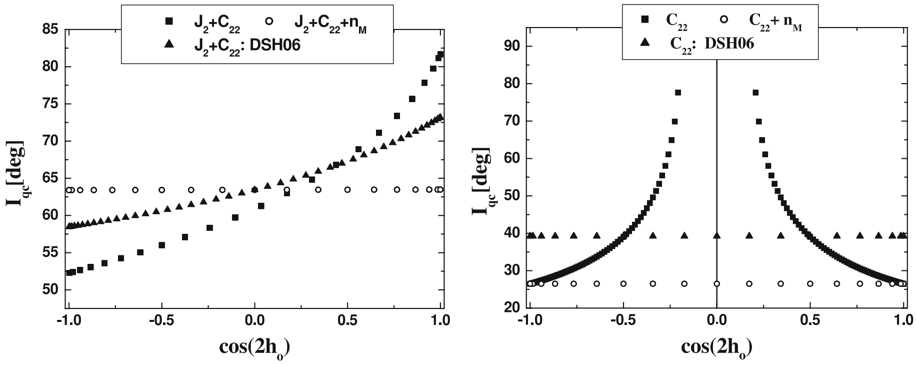

Fig. 4 Quasi-critical inclination as a function of $\cos 2h_0$ for the case of the Moon

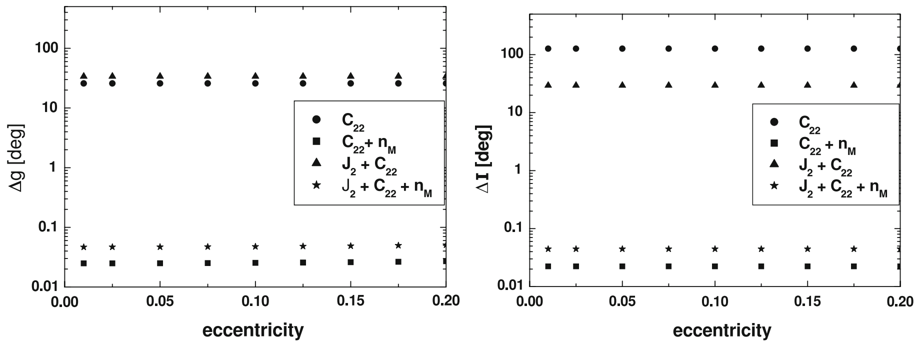

Fig. 5 The libration amplitude of g and I for the different models (Moon)

the inclination remains constant as time goes by. Actually, it varies, approaching asymptotically 180° (when $h_o = 45°$) or 0° (when $h_o = -45°$). This behavior has no effect on g, since the right hand side of Eq. 7 is now equal to zero. When the lunar rotation is included, it clearly dominates over the perturbations, in such a way that $I_{qc} \approx 63°.4$ for the '$J_2 + C_{22} + n_M$' model (i.e. nearly the same as in the simple 'J_2' model) and $I_{qc} \approx 26°.5$ for the '$C_{22} + n_M$' model (see Fig. 4).

As regards the libration amplitude of the argument of pericenter (Δg) and the inclination (ΔI) of a quasi-critical orbit, it is simply defined here as the difference between the minimum and the maximum value of g, or I, respectively. Again, ignoring the rotation, the libration amplitude is quite large. Specifically, for the '$J_2 + C_{22}$' model $\Delta g \approx 33°.7$ and $\Delta I \approx 29°.4$, while for the 'C_{22}' model $\Delta g \approx 25°.7$ and $\Delta I \approx 127°.1$. Taking into account the rotation, Δg and ΔI become at least two orders of magnitude smaller (see Fig. 5), i.e. $<0°.1$.

The analytic expression of $g'(h)$ reveals that I_{qc} and Δg depend on the semi major axis and the eccentricity of the orbit, both of which are constant in the averaged model. The full expression (17), including all the parameters ϵ, δ and n_M, is too long and complicated, thus the exact dependence on a and e is not so clear. In any case, the variations of I_{qc} and Δg caused by changes in the initial value of a and e are limited to the second decimal of their value expressed in degrees. To get a feeling of this, we present a couple of simplified cases.

Assuming $n_M = \epsilon = 0$, the expression for $g'(h)$ takes the following simple form

$$g'(h) = \frac{-2G^2 \cos 2h + 5(G^5 - H_0^2) \cos 2h_0}{2G\sqrt{\cos 2h[G^2 \cos 2h + (H_0^2 - G^2) \cos 2h_0]}} \tag{19}$$

while when only $n_M = 0$, it becomes

$$g'(h) = \frac{(G^2 - 5H_0^2)\frac{\epsilon}{\delta} - 4G^2 \cos 2h + 10(G^2 - H_0^2) \cos 2h_0}{2G\sqrt{(\frac{\epsilon}{\delta} + \cos 2h)\left[H_0^2 \frac{\epsilon}{\delta} + 2G^2 \cos 2h + 2(H_0^2 - G^2) \cos 2h_0\right]}} \tag{20}$$

According to these equations, which are used to obtain I_{qc} and Δg for given initial conditions, when $n_M = 0$ there is no dependence on L whereas a dependence on G still exists.

To confirm that the effect of short-period terms on I_{qc} is negligible, we numerically integrated the full problem (Eq. 3). We use a multi-step numerical integration method, based on the Adams PECE formulas and local extrapolation. The step-size and order are adjusted at each step, such that the local error remains smaller than a predefined small parameter (10^{-14} in our case). This algorithm is thoroughly explained in Shampine and Gordon (1975). For each model, we integrated a number of orbits with h_0 from 0 to 90°, (step of 5°), taking 30 different values of H_0 in the quasi-critical range. In all cases studied, it was clear that short-period terms have a minimal effect on the value of the quasi-critical inclination. This happens since $\dot{l} >> n_M$, which means that no low-order resonance between the mean motion of the satellite and the rotation period of the Moon are possible (see examples in Figs. 6, 7).

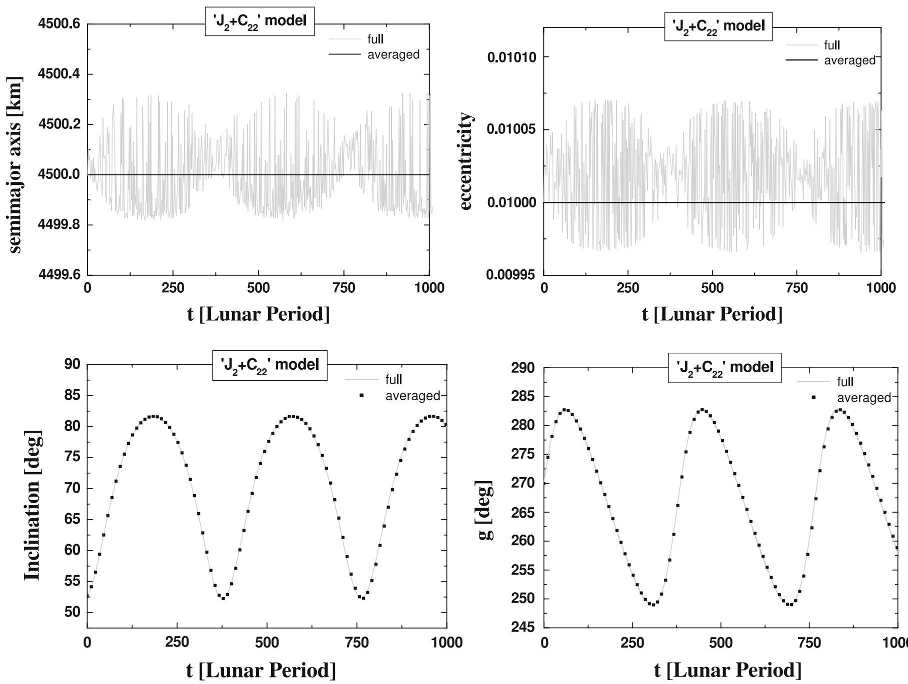

Fig. 6 The time evolution of a, e, I and g for the averaged and the full 'J_2+C_{22}' model. The initial conditions considered are $a = 4500$ km, $e = 0.01$, $I = 52°.6609$, $g = 270°$

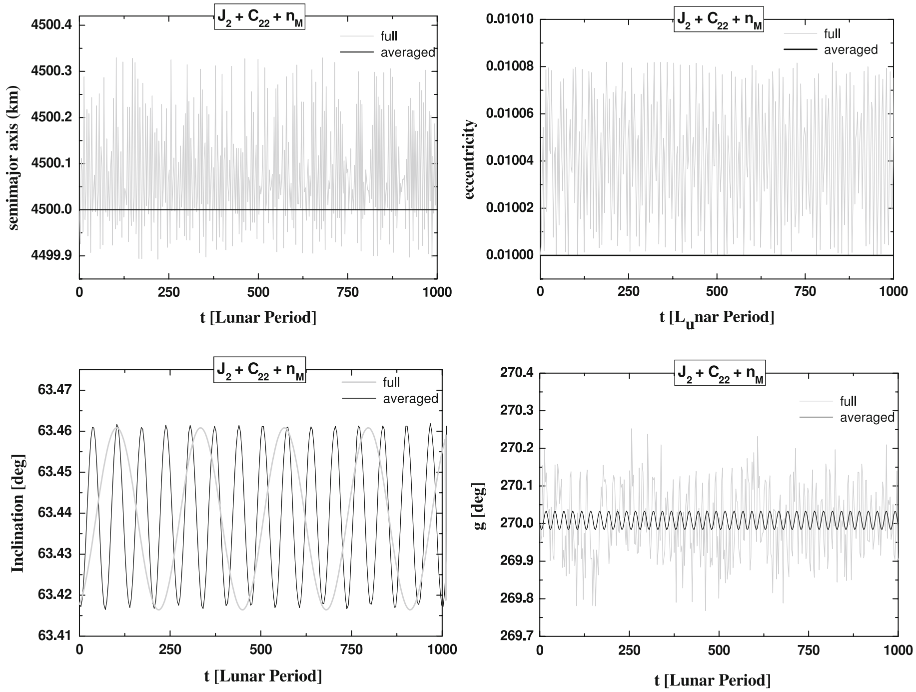

Fig. 7 The time evolution of a, e, I and g for the averaged and the full '$J_2 + C_{22} + n_M$' model. The initial conditions considered are $a = 4500$ km, $e = 0.01$, $I = 63°.4178$, $g = 270°$

4 Parametric study for different values of the perturbing parameters

Despite the Moon's shape, which is characterized by a rather large value of C_{22} and a relatively slow rotation rate that give rise to the phenomena described above, it is interesting to examine what happens for other planets or moons, with different values of J_2, C_{22} and n_M. The celestial bodies chosen here are Mars and the Jovian moons Europa and Callisto. In some plots, Earth and the asteroid Eros are also included. Each of these bodies has different characteristics and they were selected in order to cover a sizeable part of the parameter space. These bodies are used as examples, so the values of their disturbing parameters may not represent the most accurate values today. As regards Eros, its use is just indicative, as its physical structure and rotation period are such that it is difficult to choose an orbit far from resonances and, at the same time, quite close to its surface. To be able to compare the results for each body on the same basis, we chose orbits for which $R/a = 0.35$. This corresponds to a value of a such that the orbit is unaffected by low-order resonances with the rotation period of the central object (which is quite long for all bodies tested here, except Eros) and also allows large values of eccentricity, without physical collision. The system of units we use is defined as follows: we consider that the gravitation constant $\mathcal{G}$, the mass of the primary and its radius are all equal to 1. Then, the period of a particle that orbits around the primary with $a = R = 1$ is $T = 2\pi$. The computations are made for two different eccentricities, $e = 0.1$ and 0.6.

4.1 '$J_2 + C_{22}$' model

When the rotation is ignored, $g'(h)$ can be written as a function of the ratio ϵ/δ (Eq. 20). This means that, celestial bodies with different mass distributions, but with the same ϵ/δ, support the same quasi-critical orbits. The value of ϵ/δ determines the form of the phase diagram, extending the region of librations, as it decreases. So, for some ϵ/δ values the quasi-critical curve is located inside the separatrix (see Fig. 2 - right panel). As a result, for these ϵ/δ values there do not exist quasi-critical values of inclination for every h, as shown in Fig. 8 (left panel). Performing a parametric study for $1 \leq \epsilon/\delta \leq 100$, for a fixed $h_0 = 50°$, we found that the libration amplitude of g has for $\epsilon/\delta \approx 7.9$ a maximum, $\Delta g = 44°$. For the same ϵ/δ value, I_{qc} has a minimum (Fig. 9— left panel). Looking at the phase diagrams again, this behavior indicates that we are very close to the separatrix.

4.2 'C_{22}' model

In this case, the parameter δ drops out from Eq. 19. So, the exact value of C_{22} does not play any role in that model, for what concerns $\langle g'(h) \rangle$. On these grounds, I_{qc} for all bodies exhibits exactly the same dependence on the initial value of h.

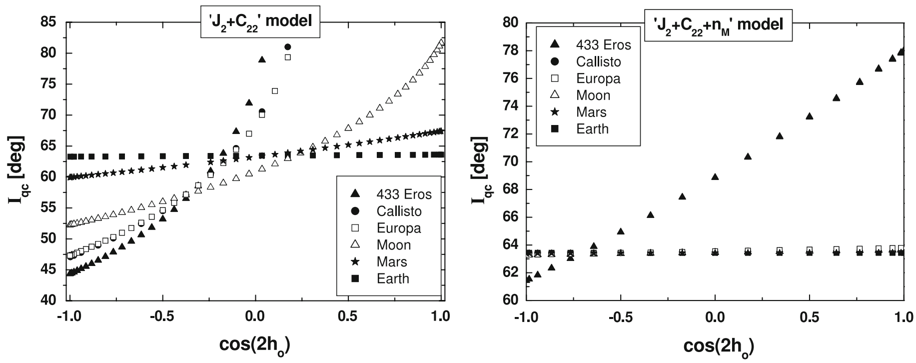

Fig. 8 Quasi-critical inclination as a function of $\cos 2h_0$ for the celestial bodies studied

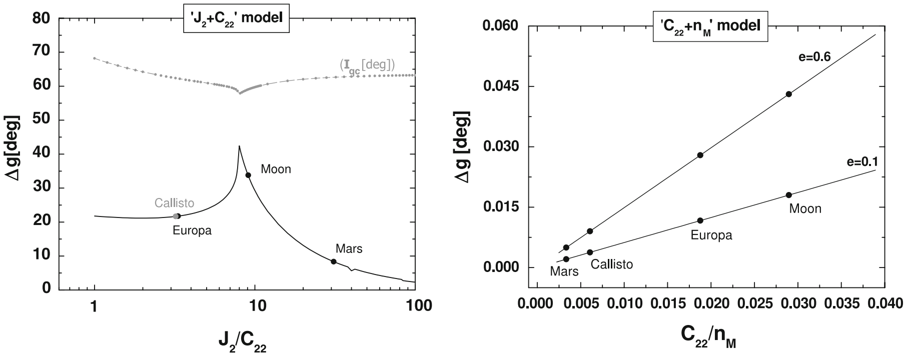

Fig. 9 The libration amplitude of g as a function of the ratio of the considered perturbations. In the left panel the value of the quasi-critical inclination as function of J_2/C_{22} is also plotted (*dotted line*)

4.3 '$C_{22} + n_M$' model

Δg—although very small—seems to depend linearly on δ/n_M (see Fig. 9—right panel). The changes induced to I_{qc} are too small to be worth mentioning. More remarkable changes appear in the case of very small values of C_{22} (i.e. Earth) or very slow rotation (small n_M), both leading to larger values of δ/n_M.

4.4 '$J_2 + C_{22} + n_M$' model

This model contains all three parameters and for that it is the most complicated of all, but also the most realistic one. Here, the ratios that must be considered are three: ϵ/δ, δ/n_M and ϵ/n_M, as all of them are present in the expression for $g'(h)$. However, one of them appears to be the most crucial, δ/n_M. Keeping constant the value of J_2 and changing the value of δ/n_M, we obtain the diagrams of Figs. 10 and 11.

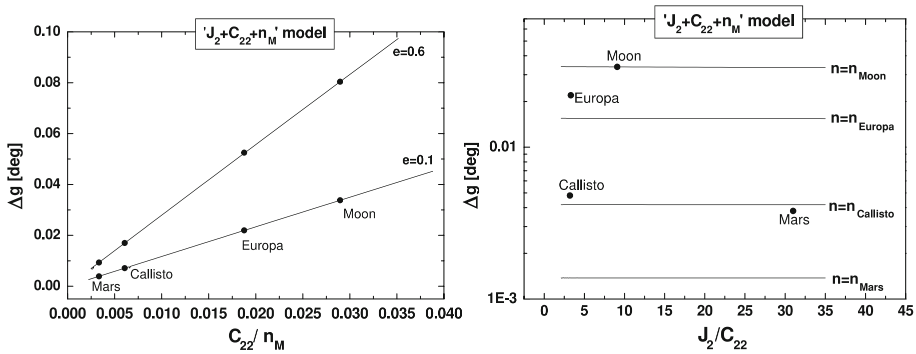

Fig. 10 Δg and I_{qc} as a function of C_{22}/n_M (*left panel*) and J_2/C_{22} (*right panel*). In the left diagram the values of J_2 and n_M are kept constant, while in the right diagram the constant parameter is C_{22}. The values of the constant parameters in both cases correspond to the Moon

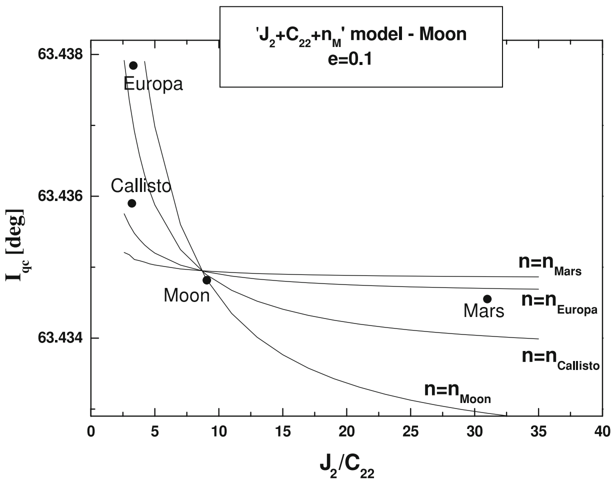

Fig. 11 The quasi-critical inclination as a function of the ratio J_2/C_{22}

Note that the eccentricity, even though present in the equations of all models, has noticeable effects only when combined with rotation (see Fig. 9—right panel, 10). Otherwise, its contribution is negligible, as e.g. in Fig. 9 (left panel), where the curves for $e = 0.1, 0.6$ are nearly one on top of the other.

5 Conclusions

We have shown that when the C_{22} spherical harmonic is included in the gravitational field of the Moon, the problem of the 'critical inclination' of satellite orbits is quite different from the simpler 'J_2' model. In all models, the orbits of the averaged system, for which the argument of pericenter librates around a mean value (i.e. $\langle g'(h)\rangle_h = 0$), are defined here as *quasi-critical*. The initial conditions (h_0, H_0) on a quasi-critical orbit define a curve in phase space, which corresponds to a *critical* value of the action. Hence, the quasi-critical value of the inclination is not unique, but depends on h_0. Therefore, this dependence is quite different from the one found by DSH06, where the variations in (h, H) were not considered. The dependence is strong when the rotation is ignored. The initial values of the semi major axis and eccentricity also play a minor role in the values of I_{qc}. However, rotation smooths out these effects (at least for values of n_M considered here), so that the quasi-critical solutions of the '$J_2 + C_{22} + n_M$' model are nearly identical to the critical inclination solution of the 'J_2' model.

The parametric study for different perturbation strengths and rotation rates revealed that in the '$J_2 + C_{22}$' model, there is a value of the ratio J_2/C_{22} for which Δg has a maximum and I_{qc} a minimum. This value corresponds to the separatrix of the averaged system. The critical curve of the 'C_{22}' model does not depend on δ. In the '$C_{22} + n_M$' model, Δg increases linearly as a function of C_{22}/n_M. In the '$J_2 + C_{22} + n_M$' model the most significant parameter is the ratio C_{22}/n_M, but the values of J_2/n_M and J_2/C_{22} must also be considered.

For lunar satellites, we intend to study in the future the effect of the Earth as a third body perturbation on inclined satellite orbits (see Jefferys and Moser 1966), which is significant at distances greater than $3R$ (De Saedeleer 2006). The resonance between the rotational and the orbital period of the Moon is expected to change the secular dynamics of the satellite. In the general case of an artificial satellite orbiting a central body, additional perturbing terms (e.g. Palacián 2007) and resonances, between the mean motion of the satellite and the rotational period of the primary, have to be considered. In that case, the degrees of freedom of the averaged system will be increased and the long-term stability of the orbits might be affected by the perturbations, especially for a non-rotating central body. This future work may lead to more efficient design of artificial satellite orbits, which could also be used as boundary conditions for Earth-Moon satellite transfers (Perozzi and Di Salvo 2008).

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