STABLE CHAOS AND LOCAL INTEGRALS OF MOTION

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ABSTRACT

In previous publications we suggested that *stable chaos* is related to the existence of local integrals of motion in the phase-space region occupied by the trajectory. In order to test this hypothesis we estimated, using a numerical method, the topological dimension, d_3, of several trajectories in the 3-D embedding space of proper elements. Our first results show that, while chaotic orbits may have $1 \leq d_3 \leq 3$ (regular orbits have $d_3 = 0$), stable-chaotic orbits typically have $d_3 = 1$. Thus stable-chaos can be associated with the near-conservation of 2 local integrals of motion.

Key words: asteroids; stable chaos; local integrals.

1. INTRODUCTION

The notion of *stable chaos* [1] refers to asteroidal trajectories which are characterized by two controversial properties: (a) very short Lyapunov times ($T_L \sim 10,000$ yr) and (b) stable proper elements for $t >> T_L$ (~ 100 Myr). In a series of previous publications [2]-[4], we have shown that this type of motion is quite common for asteroids residing in high-order mean motion resonances with Jupiter. The basic topological feature, that is common among these resonances, is the non-existence of simple resonant periodic orbits. As a result, large (intermittent) eccentricity jumps cannot occur, in contrast to e.g. the 3/1 resonance. The evolution of chaotic trajectories is, in general, nearly diffusive, while there exist orbits with ostensibly stable proper elements, besides their small value of T_L (i.e. stable chaos). Moreover, stable chaotic orbits also have a characteristic spectrum of autocorrelation times. The autocorrelation function of the semi-major axis decays exponentially with time, on a time scale comparable to T_L. On the other hand, for the eccentricity and inclination time series, the autocorrelation functions are almost quasi-periodic functions of time (for $t \sim 1000 \cdot T_L$). As described in [3], this behavior is consistent with the trajectory spanning a lower-than-full dimensional manifold in phase space. This lead us to conjecture the existence of local (non-isolating) integrals of motion, in the phase-space region occupied by stable-chaotic trajectories.

From the above it is easy to conclude that the number of quantities that are conserved for a given orbit can be determined by measuring the topological dimension of the set generated in phase space. In the present paper, we explain the method used for this purpose (a variation of the one used by [5]) and present our first results. We study the trajectories of fictitious asteroids, initially placed in the vicinity of the 12/7 resonance with Jupiter, where the first example of stable chaos was discovered [1] (see also [2]). For simplicity reasons, the dynamical model used in this first study is the 3-D elliptic three-body problem (E3BP). We note however that the method can be extended for more realistic models.

2. METHODOLOGY

The extended phase space of the E3BP is 8-dimensional. Any trajectory will evolve on a 7-D subspace, determined by the value of the "energy" constant. Moreover, one may define a Poincaré map, taken at integer multiples of the revolution period of Jupiter, e.g. at $\lambda' = 0$, where λ' is the mean longitude of Jupiter (i.e. time). Each trajectory will then explore a 6-D manifold, the relevant variables being functions of the semi-major axis, a, the eccentricity, e, the inclination, I, the mean longitude, λ, the perihelion longitude, ϖ and the nodal longitude, Ω. Dimension estimate algorithms (like the one proposed by [5]) are based on splitting each axis of the $n-$D space into a large number, $M(l)$, of bins (l being the length of the bin) and measuring the number of $n-$D cells, $N(l)$, occupied by the N points of the trajectory. Going to smaller and smaller scales (reducing l), the number of occupied cells is found to obey a power-law behavior

$$N(l) = \alpha\, M(l)^d \tag{1}$$

d being the topological dimension of the set generated in phase space by the trajectory consequents. Normally one has to go to very small values of l, in order to be able to estimate d with decent accuracy. This, however, requires a large number of data points, which increases drastically with the number, n, of the dimensions of the embedding space. For $n = 6$, as in our case, the number of computations required becomes so large that the algorithm is practically useless.

One way to reduce the number of effective dimensions (and computations) is to resort to the computation of *proper elements*, which are quasi-integrals of motion. A regular trajectory is characterized by three constant proper elements (a_P, e_P and I_P) and three proper angles that evolve linearly with time. Since the three angles cover densely the available space, we may focus our attention only to the 3-D space defined by a_P, e_P and I_P. The topological dimension of the trajectory will then be $d = d_3 + 3$, where d_3 is the dimension of the set in the (a_P, e_P, I_P) space. For regular trajectories, we expect that $d_3 = 0$. On the other hand, for chaotic trajectories the proper elements are not constant. Depending on the local properties of the phase space region under study, the "diffusion" of the proper elements can lead to the generation of a set of points that has dimension $d_3 = 1$, 2 or 3. There are, however, several computational problems that have to be taken care of (e.g. the accuracy of computation of the proper elements, the ratio between N and $M(l)$, etc.), in order to achieve accurate estimates of d_3.

3. RESULTS

We integrated a set of 30 trajectories, initially placed on a 5×6 grid in the (a, e) plane, with $a \approx a_{12/7} \approx 3.63\,\mathrm{AU}$ and $e < 0.2$, for 10 Myrs. Jupiter was set at $a' = 5.2\,\mathrm{AU}$, with $e' = 0.048$, $I' = 0^0$ and $\varpi' = \Omega' = \lambda' = 0^0$. For the fictitious asteroids, the rest of the elements were set equal to $I = 5^0$ and $\varpi = \Omega = \lambda = 0^0$. The values of a were chosen (after some test integrations spanning 200,000 yrs) so that the critical argument of the resonance, $\sigma = 7\lambda - 12\lambda_J + 5\varpi$, evolved in a typical chaotic manner for most of these orbits (23/30). After the integration was finished, we derived mean elements, by taking the Fourier transform of a, $h = e \sin \varpi$ and $p = I \sin \Omega$, eliminating short-periodic variations and taking the inverse transform. Then, using a running window of 4000 points (corresponding to $\sim 50,000$ yrs), we computed the time series of the proper elements. The proper value of a was calculated within each window as the mean value of the filtered a. The other two elements (e_P and I_P) were computed according to the results of the linear secular theory of the E3BP, i.e. by taking $e_P = (h_{\max} - h_{\min})/2$ and $I_P = (p_{\max} - p_{\min})/2$ within each window.

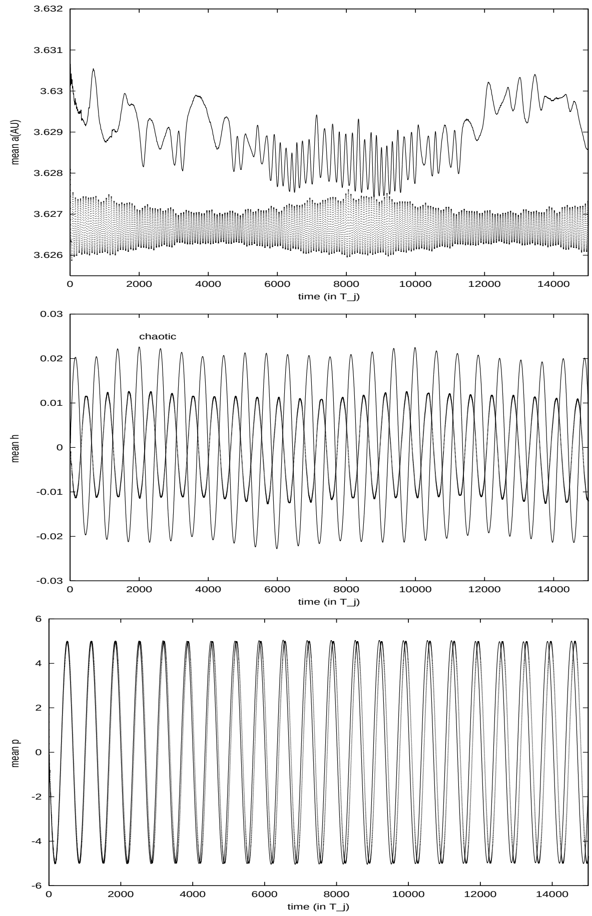

Figure 1. The mean elements a (top), h (middle) and p (bottom) are given, as functions of time, for one regular and one stable-chaotic orbit. The unit of time is the revolution period of Jupiter, $T_J \approx 11.86$ yr. It is easy to realize the different character of the motion between these two orbits, by monitoring the behavior of a. On the other hand, we cannot judge whether the orbit is regular or chaotic by the graphs of h or p. We note that all trajectories in our sample have $I_P \approx 5^0$ but different values of e_P, due to our choice of initial conditions.

All trajectories remained in the vicinity of the 12/7 resonance for the whole integration time span. Moreover, the proper elements were very stable in all cases, the standard deviations taking values $\sigma(a_P/a') < 0.0007$, $\sigma(e_P) < 0.0025$ and $\sigma(\sin I_P) < 0.0038$. For all the chaotic orbits we found $T_L \sim 10,000$ yrs, and thus all these orbits show the typical characteristics of stable chaos. We note that the difference between regular and stable-chaotic orbits is better seen in the behavior of the mean semi-major axis, than of the mean eccentricity and inclination (see Fig. 1). This property is exactly what can be quantified, by computing the autocorrelation functions of these three elements.

The time series of the proper elements had a length of $N = 250,000$ points. In order to calculate d_3 from Eq. (1), we divided the sub-space of the (a_P, e_P, $\sin I_P$) space spanned by each trajectory into $M(l)^3$ cells, with $M(l)$ ranging from 1 to 50. Applying the algorithm to all 30 cases we found that regular trajectories (7/30) have $d_3 = 0$, as expected. For the stable-chaotic trajectories, the value of d_3 was between 0.9 and 1.2 (see Fig. 2). Non-integer (fractal) values of d_3 are a computational effect, related to the finite length of the time series. Thus, the typical value of d_3 for stable chaotic trajectories is $d_3 = 1$. This means that there are two quantities (local integrals), i.e. functions of the proper elements e_P and $\sin I_P$, that are conserved by stable-chaotic motion. However, since stable chaos does not imply stability of the elements for infinite times, it would be better to say that these quantities are nearly conserved (quasi-integrals) or, in other words, that stable-chaotic orbits move around a "sticky" 1-D curve, in the 3-D space of proper elements.

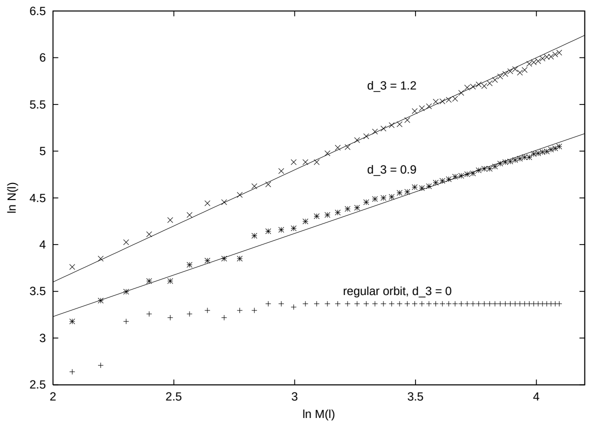

Figure 2. Estimate of d_3 for one regular and two stable-chaotic orbits. The slope of the curve is the value of d_3, indicated on the graph. As expected, the regular orbit yields $d_3 = 0$. Stable-chaotic orbits have $d_3 \approx 1$.

4. CONCLUSIONS

In this paper we presented the methodology and some preliminary results on the determination of the number of local integrals of motion conserved by regular and chaotic orbits. In the framework of the E3BP we find that there is a clear-cut difference between regular and chaotic trajectories, the former giving $d_3 = 0$ and the latter $1 \leq d_3 \leq 3$, in the 3-D embedding space of proper elements.

In our sample of stable-chaotic orbits we found that $d_3 = 1$ is the typical value of the topological dimension. This means that there are 2 local integrals which are (nearly) conserved by stable-chaotic orbits. This behavior, however, cannot last for infinitely long times. Loosely speaking the space has "holes", through which a stable-chaotic orbit will eventually escape, evolving in the 3-D space in a space-filling manner.

Finally, we note that we did not find any chaotic orbits having $d_3 = 2$ or 3, within our sample of 30 orbits. It is not easy to find trajectories which (a) have all three proper elements wildly diffusing and (b) do not cross the orbit of Jupiter, for times long enough for the dimension-estimate algorithm to be efficient. In the 12/7 resonance we studied, orbits starting with $e > 0.2$ escape quite fast ($\sim$ 1 Myr). Evidently a more detailed study of the relationship between unbounded chaos, stable chaos and the number of local integrals of motion is needed. Such an analysis is currently under way.

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