# STABLE CHAOS IN MEAN MOTION RESONANCES

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## 1. Introduction

One of the most interesting, newly discovered, phenomena in solar system dynamics is a type of asteroidal motion commonly referred to as *stable chaos*. An asteroid on stable chaotic motion (ASC) follows a strongly chaotic trajectory (as implied by the fact that it is characterized by a short Lyapunov time, $T_L = 1/\lambda$, where $\lambda$ is the Maximal Lyapunov Characteristic Number, LCN) and yet its orbital elements remain almost constant for thousands of Lyapunov times. The first example of an ASC, asteroid (522) Helga, was discovered by Milani & Nobili (1992) in a 12/7 orbital resonance with Jupiter.

Since then, many more ASCs have been identified (e.g. see Milani *et al.*, 1997; Šidlichovský, 1999), in various orbital resonances with Jupiter. Therefore stable chaos seems to be a relatively common phenomenon in the solar system, usually associated with an orbital resonance with Jupiter. Since the discovery of this new phenomenon, many attempts were made in order to interpret the apparently controversial properties of chaotic motion, on the one hand, and stable orbital elements on the other. The first one was made by Milani & Nobili, who proposed that stable chaos is due to a secular protection mechanism, not related to the chaos-generating orbital resonance. Other mechanisms proposed since then include a diffusive scenario (Murison *et al.*, 1994, Murray & Holman, 1997) and the constructive superposition of critical terms in the disturbing function (Lemaitre, 1997). However all the above interpretations have some weak points.

Varvoglis & Anastasiadis (1996), in trying to put the problem on a firm basis, stressed the fact that the LCN is related to the local properties of a dynamical system, while transport is related to both local and global (the latter related to the overall structure of phase space). As stated in the conclusions of this paper, "... the stable chaotic behavior of asteroids ... originates, most probably, from the presence of consecutive layers of quasi-barriers in certain regions of phase space, where transport is governed by Levy flights rather than random walks".

## 2. What is stable chaos

In order to examine the above conjecture, our group in Thessaloniki studied numerically the time evolution of the trajectories of 5 known ASCs (besides 522 Helga, the other four, 50 Virginia, 6 Semele, 2 Pallas and 33 Polyhymnia, were identified by Šidlichovský (1999) to lie, respectively, in the 11/4, 13/6, 18/7 and 22/9 orbital resonance with Jupiter), as well as a large number of their clones, using initially the 3D elliptic restricted three-body model (Sun-Jupiter-asteroid). Among other

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quantities we calculated, for every object (asteroid or clone), the LCN and the autocorrelation functions of the Delaunay actions. Using the value of the LCN and the decay time of the autocorrelation functions we were able to classify the objects into three classes. In the first class belong the objects which are characterized by an effectively zero LCN and non-decaying autocorrelation functions. For obvious reasons we denote these trajectories as "ordered". In the second class belong the objects characterized by a non-zero LCN and exponentially decaying autocorrelation functions. The decay time for $L = \sqrt{a}$ is of the same order of magnitude as the Lyapunov time, while for $G = L\sqrt{1-e^2}$ and $H = G\cos i$ it is typically $< 10\,T_L$. (Fig. 1a). For obvious, as well, reasons we consider these trajectories as "chaotic". According to the relatively high order of these resonances, this is typically a "slow chaos", which we associate with the slow random-walk diffusion surmised in the theory of Murray and Holman.

The interesting class, however, is the third one, which is characterized by a non-zero LCN but with only one autocorrelation function (the one corresponding to the semi-major axis) decaying on a time scale of the order of the Lyapunov time, while the other two vary quasi-periodically for hundreds of Lyapunov times (Fig.1b). We found that all five ASCs, as well as a considerable number of their clones, belong to the third class. Further tests, described in detail in Tsiganis *et al.* (2000, 2002), lead us to associate the phenomenon of stable chaos to this class (i.e.one decaying autocorrelation function and two quasi-periodic), which is consistent with the existence of an additional, local, integral of motion, besides the "energy" integral. Therefore, according to our interpretation, stable chaos is nothing more than the manifestation of an already known phenomenon in autonomous Hamiltonian systems with more than two degrees offreedom, namely the near-conservation of one more quantity besides the value of the Hamiltonian function (e.g. see Contopoulos *et al.*, 1978; Magnenat, 1985; Malagoli *et al.*, 1986).

In the case of the solar system, stable chaos corresponds to the chaotic evolution of the semi-major axis, on the one hand, but to an almost quasi-periodic evolution of the eccentricity and the inclination, on the other. From this picture the stability of the last two orbital elements of an ASC may be directly interpreted. The macroscopic stability of the third action-like element, namely the semi-major axis, is guaranteed by the resonance condition. Thus, the trajectories of ASCs are sticky to a four-dimensional submanifold, i.e. the cartesian product of a 2-torus by an annulus (Tsiganis *et al.*, 2002), the motion on the annulus resembling that of a driven pendulum (see Tsiganis *et al.*, 1999). Therefore the above four-dimensional hyper-surface acts as a quasi-barrier to the migration of the trajectories in the full-dimensional phase space. Since sticky trajectories are characterized by Levy-like rather than classical random-walk diffusion (e.g. see Shlesinger *et al.*, 1993), it seems that the conjecture of Varvoglis & Anastasiadis was on the correct side.

It should be pointed out that a classification of an asteroid or clone as an ASC is independent of the integration model used. We arrived at this conclusion because we have repeated the numerical integration of all calculated trajectories

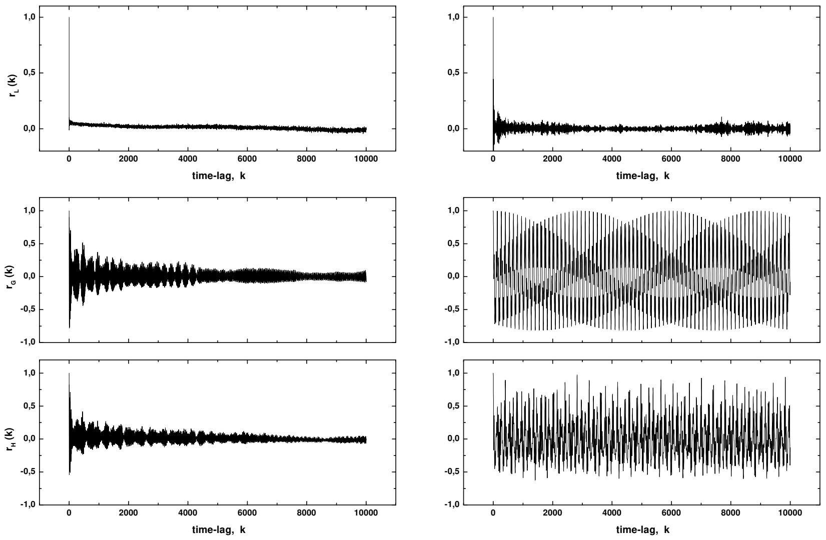


*Figure 1.* The autocorrelation functions $r_I(t)$ of the three Delaunay actions (from top to bottom), $L$, $G$ and $H$.(a) Left frames: a typical chaotic orbit, for which all three decay functions exponentially with time on a time-scale comparable to $T_{\mathrm{L}} \approx 3,241$ years, is shown. (b) Right frames: the same quantities are plotted but for the trajectory of a clone ASC. $r_G(t)$ and $r_H(t)$ are almost quasi-periodic functions of time, while $r_L(t)$ decays exponentially on a time-scale comparable to $T_L \approx 68,445$ years. The orbits were integrated using the OSS model. The time-lag is equal to $10^2$ Jovian periods, i.e. approximately 1,200 years, and the total time-span is $\approx$12.5Myrs.

in a more involved model, namely the Outer Solar System model (Sun-Jupiter-Saturn-Uranus-Neptune-asteroid, OSS) and we arrived essentially at the same classification, i.e. as a rule an asteroid or a clone found to be an ASC in the three body model was found to be stable chaotic in the OSS model too. Therefore the behavior of an ASC seems to be ultimately controlled by the local structure of the phase space, as the latter is dictated by the three-body problem.

## 3. Stable chaos and resonant periodic orbits

As mentioned in the previous section, stable chaos is associated with a "smooth" structure of phase space, which is due to the existence of a local integral of motion, besides the "energy" integral. This smooth structure, in turn, is responsible for the existence of higher dimensional quasi-barriers, to which the trajectory of an ASC is sticky. The existence of a periodic trajectory corresponding to this resonance, on the other hand, shapes a characteristic topological structure in phase space. On a surface of section of the averaged problem, this structure appears as a pair of fixed points, with the homoclinic loop of the unstable fixed point encircling the island of libration around the stable one. The chaotic strip, developing along the unstable manifold and around the island of libration, leads to a "fast" escape of a trajectory from the neighborhood of the corresponding commensurability. Therefore it is natural to conjecture that the phenomenon of stable chaos should appear in the neighborhood of resonances where the corresponding periodic orbit, which always exists in the framework of the planar circular restricted three-body problem, *does not continue* either in the planar elliptic restricted problem or the 3-D circular restricted problem. Indeed, a thorough numerical search showed that the five resonances associated to the ASCs studied in the present paper have exactly this property. Since the increase of the complexity of the model, through the introduction of more perturbing planets, does not change this situation, we can be fairly confident in stating that the phenomenon of stable chaos in the neighborhood of certain mean-motion resonances with Jupiter is strongly associated to the non-existence of a periodic trajectory of the corresponding commensurability.

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