Solar System dynamics, beyond the two-body-problem approach

Harry Varvoglis

Department of Physics, University of Thessaloniki, GR-541 24 Thessaloniki, Greece

Abstract. When one thinks of the solar system, he has usually in mind the picture based on the solution of the two-body problem approximation presented by Newton, namely the ordered clockwork motion of planets on fixed, non-intersecting orbits around the Sun. However, already by the end of the 18th century this picture was proven to be wrong. As discussed by Laplace and Lagrange (for a modern approach see [3] or [2]), the interaction between the various planets leads to secular changes in their orbits, which nevertheless were believed to be corrections of higher order to the Keplerian elliptical motion.

This idea has changed completely the last decades. Now it is well know that the solar system was created from a state of chaotic interactions of planetesimals, primordial bodies the size of a small asteroid, and that since this time many episodes of cataclysmic collisions have shaken all major planets, due to the pronounced chaotic motion of the minor bodies. A new discipline has emerged out of the above new ideas, which is based on the statistical approach to chaotic motion of bodies, in particular those in the asteroid belt. At the same time it has been understood that non-gravitational forces, in particular the Yarkovsky effect, may play an important role on the long-time evolution of the trajectories of kilometer-sized bodies.

Keywords: Secular theory, three-body problem, chaos, Fokker-Planck equation, Yarkovsky effect
PACS: 05.10.Gg, 95.10.Ce, 95.10.Fh, 96.30.Ys

1. KEPLERIAN MOTION (2 BODIES)

1.1. Newtonian approach

The equations of motion of one planet in orbit around the Sun are given by substituting the force of gravitational attraction between the two bodies

$$\vec{F} = \mathscr{G}\frac{\mathscr{M}m}{r^3}\vec{r} \tag{1}$$

into Newton's second law of motion

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \tag{2}$$

where $\mathscr{M}$ and m define, respectively, the masses of the Sun and the planet, $\mathscr{G}$ the gravitational constant and r the distance between the bodies.

It was known since the time of Newton that the above differential equation can be solved exactly. The bounded solutions of a *test particle* moving around the Sun are *ellipses*, with semi-major axis

$$a = -\frac{\mathscr{G}\mathscr{M}}{2E} \tag{3}$$

CP848, *Recent Advances in Astronomy and Astrophysics,*
7th *International Conference of the Hellenic Astronomical Society,* edited by N. H. Solomos

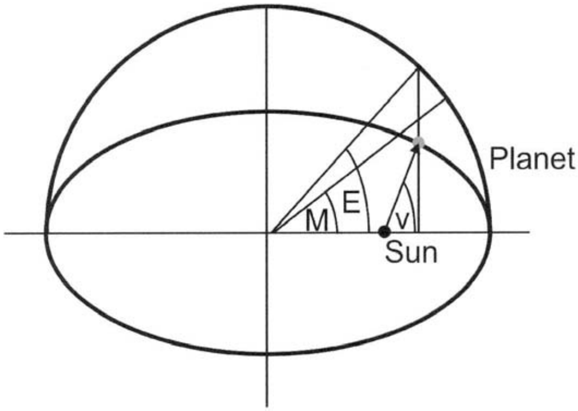

FIGURE 1. Definition of true (v), eccentric (E) and mean (M) anomaly.

and eccentricity

$$e = \sqrt{1 + \frac{2Eh^2}{\mathscr{G}^2 \mathscr{M}^2}} \tag{4}$$

where h denotes the angular momentum and E the energy (with $E < 0$). The orbit on the plane of the ellipse is given, in planar polar co-ordinates, by the function

$$r = \frac{a(1-e^2)}{1+e\cos(v)} \tag{5}$$

where the polar angle v is the *true anomaly*. The position of the planet on its orbit is given by the function $v(t)$, which is known in an *indirect way*. The true anomaly, v, is related to the *eccentric anomaly*, u, through the relation (see Fig. 1)

$$\cos(v) = \frac{\cos(u) - e}{1 - e\cos(u)} \tag{6}$$

The function $u(t)$, in turn, is found by solving *Kepler's equation*

$$u - e\sin(u) = n(t - t_0) \tag{7}$$

where t_0 is the *time of perihelion passage* and the *mean orbital frequency* (or *mean motion*) of the planet is given by Kepler's third law

$$n = \sqrt{\mathscr{G}\mathscr{M}}\, a^{-3/2}. \tag{8}$$

The study of an integrable conservative dynamical system is greatly facilitated if one goes to Hamiltonian formalism and writes the corresponding Hamiltonian in a special set of canonical co-ordinates (I_i, θ_i), known as *action-angle variables*. Then the "new"

Hamiltonian is in *normal form*, i.e. it depends only on the actions, I_i. In this case, as it is evident from Hamilton's equations of motion,

$$\frac{dI_i}{dt} = -\frac{\partial \mathscr{H}}{\partial \theta_i}, \frac{d\theta_i}{dt} = \frac{\partial \mathscr{H}}{\partial I_i} \tag{9}$$

the actions, I_i, are constants of the motion and the angles, θ_i, are linear functions of time. A simple inspection shows that neither v nor u are increasing linearly with time. The only angle with this property is the *mean anomaly*,

$$M = n(t - t_0) \tag{10}$$

and, as we will see in what follows, it is indeed the conjugate angle co-ordinate of an action-angle pair that transforms the Hamiltonian of the problem to a normal form. The "physical" interpretation of this angle is that it measures the percentage of the area of the ellipse spanned by the position vector within time $(t - t_0)$ (as the motion takes place at constant area velocity).

1.2. Hamiltonian approach

The Hamiltonian of the two-body problem in a heliocentric[1] co-ordinate system and in spherical co-ordinates (ρ, θ, ϕ) is:

$$\mathscr{H} = \frac{1}{2}\left(p_\rho^2 + \frac{{p_\theta}^2}{\rho^2} + \frac{p_\phi^2}{\rho^2 \sin^2\theta}\right) - \frac{\mathscr{G}\mathscr{M}}{\rho} \tag{11}$$

where, as in the previous section, we treat the planet as a test particle. The planet follows an elliptical orbit on the x-y plane (usually referred to as *invariant plane*), with the x-axis pointing from the Sun to the vernal equinox, γ, and the z-axis pointing to the north pole of the celestial sphere (see Fig. 2). The position of the planet, at each time, is fully defined by 6 constants, the so-called *orbit elements*: the *semi-major axis*, a, the *eccentricity*, e, the *inclination*, i, the *longitude of the ascending node*, Ω, the *argument of perihelion*, ω, and the *epoch of perihelion passage*, t_0. The first two constants give the shape of the ellipse, the next three are the Euler angles and give the *orientation* of the ellipse in space (Fig. 2) and the last one, in connection with eq.(8), gives the mean anomaly of the planet.

The complete set of canonical action-angle variables for the two-body problem are the Delaunay actions L, G, H and angles l, g, h, which are defined by the relations

$$L = \sqrt{\mathscr{G}\mathscr{M}a} \quad l = M \tag{12}$$

$$G = \sqrt{\mathscr{G}\mathscr{M}a\,(1 - e^2)} \quad g = \omega \tag{13}$$

[1] Note that if we had selected a *barycentric* system instead, we should change its origin every time we would like to "add" a new body in our model!

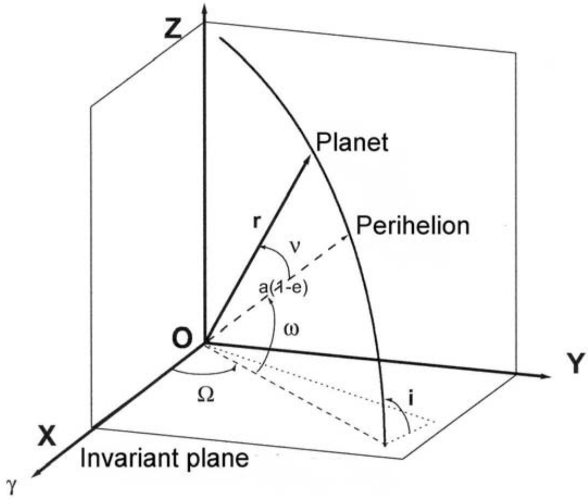

FIGURE 2. Definition of the six orbital elements.

$$H = \sqrt{\mathcal{G}\,\mathcal{M}\,a\,(1-e^2)}\cos i \qquad h = \Omega \tag{14}$$

The Hamiltonian, in these variables, becomes simply

$$\mathcal{H}^* = -\frac{\mathcal{G}^2\mathcal{M}^2}{2L^2} \tag{15}$$

The angle $l = n(t-t_0)$, which is increasing linearly with time, gives the *position* of the body on the ellipse, while the other two angles are constants. The Delaunay variables, however, have a weak point, since the angle ω is not defined for $i = 0$ and the angle M is not defined for $e = 0$. This problem is solved if we go to *modified* Delaunay variables:

$$\Lambda = \sqrt{\mathcal{G}\,\mathcal{M}\,a} \qquad l = \varpi + M \tag{16}$$

$$\Gamma = \Lambda(1-\sqrt{1-e^2}) \qquad \gamma = -\varpi \tag{17}$$

$$Z = \Lambda\sqrt{1-e^2}(1-\cos i) \qquad \zeta = -\Omega \tag{18}$$

which are, as well, action-angle variables for the two-body problem. The two new angles are the *longitude of perihelion*, $\varpi = \Omega + \omega$, and the *mean longitude*, $\lambda = \varpi + l$.

In these variables the Hamiltonian has the same functional form as eq. (15)

$$\mathcal{H}^* = -\frac{\mathcal{G}^2\mathcal{M}^2}{2\Lambda^2} \tag{19}$$

In this form becomes obvious the most important property of the two-body problem, the *degeneracy*, since the Hamiltonian depends only on one (out of three) action. The frequencies $\dot{\varpi}$ and $\dot{\Omega}$ are equal to 0 and, therefore, *besides* the three actions, there exist two more integrals of motion: the orientation angles ϖ and Ω. As a result of the degeneracy, there is only *one* non-zero frequency,

$$\dot{\lambda} = \frac{\partial \mathscr{H}^*}{\partial \Lambda} \tag{20}$$

which is independent of Γ and Z. That is why all bounded orbits of the two body problem are periodic and the period of the orbit does not depend on eccentricity or inclination.

2. THE RESTRICTED THREE-BODY PROBLEM

The next, just *slightly* more complicated than the two body problem, is the model of the *restricted three-body* problem. In this model our planetary system is approximated by three bodies: one large (the Sun), one small (Jupiter) in a *fixed* (circular or elliptic) orbit around the Sun and a massless body (a small planet or an asteroid)[2]. The problem posed is to find the trajectory of the third, massless body. The corresponding Hamiltonian (again in a heliocentric frame of reference) is:

$$\mathscr{H} = \left(\frac{\vec{p}^2}{2} - \frac{\mathscr{G}\mathscr{M}}{r}\right) - \mathscr{G}m' \left(\frac{1}{\| \vec{r} - \vec{r'} \|} - \frac{\vec{r}\cdot\vec{r'}}{r'^3}\right) = \mathscr{H}^* - \mathscr{G}m'\mathscr{R} \tag{21}$$

where p and r are the momentum and position of the third body, $\mathscr{M}$ is the mass of the Sun, m' and r' the mass and position of Jupiter and $\mathscr{R}$ is the *disturbing function*, which gives the "perturbation" of Jupiter to the orbit of the asteroid. We note that, apart from the disturbing function, the Hamiltonian of the three-body problem is identical to eq.(11).

Depending on whether Jupiter is moving on a circular or elliptic orbit and on whether the third body is restricted to move on the invariant plane or not, the restricted three-body problem may have from two degrees of freedom (circular-planar) to three plus time (elliptic-3D). In all cases we use a "particular" system of units, where $\mathscr{M} + m' = 1$, $a' = 1$, $\mathscr{G} = 1$, $\mu = \frac{m'}{M+m'}$, $\mu_1 = 1 - \mu$ and $T' = 2\pi$ or $n' = 1$, so that $\lambda' = n't + \lambda'_0$ (where by primes we denote Jupiter's elements). The modified Delaunay variables for the Hamiltonian (21) in these units become

$$\Lambda = \sqrt{\mu_1 a} \qquad l = \varpi + M \tag{22}$$

$$\Gamma = \Lambda(1 - \sqrt{1-e^2}) \qquad \gamma = -\varpi \tag{23}$$

$$Z = \Lambda\sqrt{1-e^2}(1 - \cos i) \qquad \zeta = -\Omega \tag{24}$$

and the Hamiltonian in this co-ordinate system is written as

$$\mathscr{H} = -\frac{\mu_1^2}{2\Lambda^2} - \mu\mathscr{R}(\lambda, \gamma, \zeta, \Lambda, \Gamma, Z; \lambda'(t)) \tag{25}$$

---

[2] The massive bodies are usually named *primaries*.

The equations of motion of the third, massless, body are

$$\dot{\Lambda} = \mu \frac{\partial \mathscr{R}}{\partial \lambda} \quad \dot{\lambda} = \frac{\mu_1^2}{\Lambda^3} - \mu \frac{\partial \mathscr{R}}{\partial \Lambda} \tag{26}$$

$$\dot{\Gamma} = \mu \frac{\partial \mathscr{R}}{\partial \gamma} \quad \dot{\gamma} = -\mu \frac{\partial \mathscr{R}}{\partial \Gamma} \tag{27}$$

$$\dot{Z} = \mu \frac{\partial \mathscr{R}}{\partial \zeta} \quad \dot{\zeta} = -\mu \frac{\partial \mathscr{R}}{\partial Z} \tag{28}$$

First conclusion: We see that the "elements" of the trajectory of the third body are not constants any more, but their variations are of order $\mathscr{O}(\mu) \approx 0.001$. That is why we introduce the notion of *osculating* elements, i.e. elements of an ellipse which is *tangent* to the real trajectory of the third body and which would be followed by the body *if* the second primary would suddenly disappear.

Second conclusion: We see that in the three-body problem there exist two sets of frequencies, which differ by more than one order of magnitude. High frequency (*short period*)

$$\dot{\lambda}^{-1} = \mathscr{O}(1) \approx \text{3-8 yrs}$$

(as it can be calculated from Kepler's law in the region of the asteroid belt) and low frequency (*long period*)

$$g^{-1} = \dot{\gamma}^{-1} \approx s^{-1} = \dot{\zeta}^{-1} \approx \mathscr{O}(\mu) \approx \text{1,000 - 10,000 yrs.}$$

Therefore the perihelion of the osculating ellipse and the ascending node *are varying* with a period $\geq$ 1,000 years.

Third conclusion The problem possesses *resonances*. As we will see, the Fourier series expansion of $\mathscr{R}$ contains trigonometric terms, whose derivatives are of the form

$$\sin(\sum_i k_i \dot{\theta}_i) \tag{29}$$

where k_i are integers (positive, negative or zero) and $\dot{\theta}_i$ are frequencies. Whenever this sum is zero, a *resonance* exists. Resonances, which are *dense* in phase space, are the cause of the famous *small divisors problem*, which introduces *chaos* in the solutions of the problem, as it was shown by Poincaré. As a result, the solution of eqs. (27-29) is *non-analytic* with respect to the initial conditions and, therefore, it *cannot* be written as *power series* with respect to μ. But there was always the need to calculate the motion of the planets. That is why we still use the *secular theory* of the three-body problem developed by Laplace and Lagrange at the end of the eighteenth century, although we know that it leads to non-convergent series solutions, which are valid only for a limited time interval.

3. SECULAR THEORY

3.1. Secular Hamiltonian

The expansion of $\mathscr{R}$ is one of the older problems in Celestial Mechanics[3]. It has the general form:

$$\mathscr{R} = \sum_{p,q,k_i} A_{p,q,k_i}(a,e,i,a',e',i')\cos(p\lambda + q\lambda' + k_1\varpi + k_2\varpi' + k_3\Omega + k_4\Omega') \tag{30}$$

$$= \sum_{p,q,k_i} A_{p,q,k_i}(a,e,i,a',e',i')\cos\psi_{p,q,k_i}$$

where *each* coefficient A_{p,q,k_i} is a power series of $a/a', e, e', sn, sn'$, with $sn = \sin(i/2)$. For every combination of angles in eq.(30), the coefficient of the *leading* term of the corresponding Taylor series has the form

$$A^{\max}_{p,q,k_i} = f_{p,q,k_i}(a/a')e^{|k_1|}e'^{|k_2|}sn^{|k_3|}sn'^{|k_4|} \tag{31}$$

By keeping *only* the terms $A^{\max}_{p,q,k_i}$, the disturbing function, $\mathscr{R}$, takes the form

$$\mathscr{R} = \sum_{p,q,k_i} f_{p,q,k_i}(a/a')e^{|k_1|}e'^{|k_2|}sn^{|k_3|}sn'^{|k_4|}\cos\psi_{p,q,k_i} \tag{32}$$

The secular theory is a method of solving approximately the equations of motion originating from the above Hamiltonian. The first step is to average $\mathscr{H}$ over the "fast" angles λ' and λ. The "averaged" Hamiltonian is

$$\langle\mathscr{H}\rangle_{\lambda,\lambda'} = \mathscr{H}^* - \frac{\mu}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\sum_{p,q,k_i} A^{\max}_{p,q,k_i}\cos\psi_{p,q,k_i}\right)d\lambda\, d\lambda' = \tag{33}$$

$$-\frac{\mu_1}{2\Lambda^2} - \mu\sum_{0,0,k_i} A^{\max}_{0,0,k_i}\cos(k_1\varpi + k_2\varpi' + k_3\Omega + k_4\Omega').$$

Then λ is an ignorable co-ordinate, Λ becomes an integral of motion and, as a consequence, the osculating semi-major axis, a, of the orbit of the third body is *constant*. Therefore we may drop it from the Hamiltonian. Moreover, since the orbit of Jupiter is fixed, Ω' and ω' are constants and we may take them equal to zero as well. Then the Hamiltonian becomes

$$H_{\rm sec} = \mu\sum_{k_i} A^{\max}_{k_i}\cos(k_1\varpi + k_3\Omega) \tag{34}$$

The appearance of μ as a factor in front of the Hamiltonian has the following consequence. If we *divide* the Hamiltonian by μ, in order to simplify it, we have to *multiply* time by μ, in order to keep the canonical form of the equations of motion. This implies that the "unit" of time is much longer in the secular solution, in agreement with the fact that the corresponding time scale, as we already showed, is of the order of μ^{-1}.

[3] J. Pierce, Astron. J. **1**, 1, 1849.

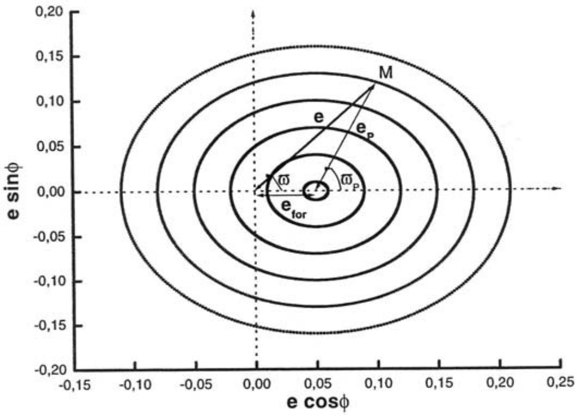

FIGURE 3. The linear solution of the secular theory for the osculating eccentricity

3.2. Linear Secular Theory

In the new Hamiltonian (35), μ is not anymore a "small" parameter. Therefore in order make expansions, we have to use other small parameters, such as the eccentricities and inclinations of the planets which, as a rule, are small. We expand $H_{\rm sec}$ and keep terms A_{ki} up to *second* order in the small quantities e, e', i, i'. This "truncated" Hamiltonian defines a *linear* system of differential equations of motion, whose solution is

$$e\cos\varpi = e_{for} + e_P\cos(gt+\beta) \tag{35}$$

$$e\sin\varpi = e_P\sin(gt+\beta) \tag{36}$$

$$i\cos\Omega = i_{for} + i_P\cos(st+\delta) \tag{37}$$

$$i\sin\Omega = i_P\sin(st+\delta) \tag{38}$$

The quantities e_{for} and i_{for} are called *forced elements* of the trajectory and

- e_f and i_f (named *forced eccentricity* and *inclination*, respectively) depend only (a) on the *motion of primaries* (mass - eccentricity) and (b) on the ratio, a/a' of the semi-major axis of the third body to that of Jupiter,
- e_P and i_P (named *proper eccentricity* and *inclination*, respectively) depend only on the motion of the third body.

The values of e_{for} and i_{for} are proportional, correspondingly, to e' and i'. I.e. in the *planar circular* three body problem we have $e_{for} = i_{for} = 0$.

The solution is better understood graphically (Fig. 3): the osculating eccentricity varies periodically with time and in the plane $(h = e\cos\varpi, k = e\sin\varpi)$ describes a cycle:

$$e^2 = e_f^2 + e_P^2 + 2e_f e_P\cos(gt+\beta) \tag{39}$$

A similar relation holds for the osculating inclination. In this way we see that, in this approximation, eccentricity and inclination are evolving independently. They are *decoupled*.

The quantities e_P and i_P are *integrals of motion* and are called *free* or *proper elements*. The variables Γ, Z are *not* actions for the *secular problem*. But we can define, through a series of canonical transformations, *new* actions

$$\Gamma = \Gamma(e_P), \qquad Z = Z(i_P) \tag{40}$$

which, together with the angles

$$\gamma_P = -\varpi_P = -(gt+\beta) \tag{41}$$

$$\zeta_P = -\Omega_P = -(st+\delta) \tag{42}$$

define a set of action-angle variables. The quantities (g,s) are the *proper frequencies* of the motion of the third body. The values of g and s depend *only* on a/a' and μ.

The method of *averaging* is the first step of an algorithm, which calculates the *normal form* of the Hamiltonian of the three-body problem. With this algorithm we construct, step by step, a canonical transformation, such that in the new variables the "new" Hamiltonian is a function of the "new" actions only. Therefore the new canonical variables are *action-angle variables* of the new Hamiltonian and the equations of motion are solved trivially. The "bad news" is that, in this transformation, which involves a power series in μ, e, e', i, i', appear *small divisors*, i.e. resonant combinations of the frequencies

$$\sum_i k_i \dot{\theta}_i = 0. \tag{43}$$

Therefore the transformation is only *formal*, in that the series does not converge!

4. THE MULTI-PLANETARY PROBLEM

In this model we have $N-2$ "planets" moving on *known* orbits around the Sun and we try to calculate the trajectory of a small "massless" body. By applying the same, as above, method, we find the solution for the massless body as

$$e\cos\varpi = e_P\cos(gt+\beta) + \sum_{j=1}^{N-2} B_j e_j cos(g_j t+\beta_j) \tag{44}$$

$$e\sin\varpi = e_P\sin(gt+\beta) + \sum_{j=1}^{N-2} B_j e_j sin(g_j t+\beta_j) \tag{45}$$

$$i\cos\Omega = i_P\cos(st+\delta) + \sum_{j=1}^{N-2} \Delta_j i_j cos(s_j t+\delta_j) \tag{46}$$

$$i\sin\Omega = i_P\sin(st+\delta) + \sum_{j=1}^{N-2} \Delta_j i_j sin(s_j t+\delta_j) \tag{47}$$

where the constants (B_j, Δ_j) depend on a/a' and the constants (e_j, i_j) are the components of the eccentricities and inclinations of the planets (which, in this case are considered as *osculating elements*, i.e. they are allowed to *vary* with time, but in a known way). The frequencies g_j and s_j are the *fundamental frequencies* of the planetary system and describe the "precession" of the planets and β_j, δ_j are the phases of the planets for the considered epoch. These $4j-2$ constants[4] are known to a high precision.

In sum, the results of the secular theory of our planetary system are as follows:

- The *semi-major axes* of the planets are *constants* of the motion, away from resonances and in the absence of non-gravitational forces
- The osculating eccentricities and the inclinations are varying with very long *secular periods* (of the order of $10^4 - 10^5$ years).

5. RESONANCES

5.1. Types of resonances

Due to the order of magnitude differences in the values of the frequencies involved, we distinguish two *basic* types of resonances:

- Orbital resonances appear in both the three-body and the planetary problem, between the *mean longitude* of a planet, λ', and that of the massless body, λ. We say that an asteroid is in the region of an orbital resonance $p/(p+q)$ with a planet if we have that

$$p\dot{\lambda} - (p+q)\dot{\lambda}' \approx 0. \tag{48}$$

  The location of orbital resonances, to zero-th order, is given by Kepler's law:

$$a_{res} = a'[p/(p+q)]^{2/3} \tag{49}$$

  and is independent of e and i (because of the degeneracy).
- Secular resonances appear only in the multi-planetary model, between the frequencies (g,s) and the proper frequencies of our planetary system

$$p\dot{\varpi}_P + q\dot{\Omega}_P + \sum_j (k_j\dot{\varpi}_j + l_j\dot{\Omega}_j) = 0 \tag{50}$$

  In particular the resonances $(g = g_k)$ or $(s = s_k)$ are symbolized as ν_k and ν_{1k}, respectively and are called *first-order apsidal* (ν_k) or *nodal* (ν_{1k}) resonances.

5.2. Resonant dynamics

The linear secular theory, presented so far, assumes that there exist no resonances, either orbital of secular. However we can find a similar solution if the motion takes place

[4] Because the angular momentum is a constant of the motion, one (s_i, δ_i) pair is equal to zero.

in a region where exists *just one* resonant condition, through the following algorithm.

- Make the resonant combination a new angle co-ordinate, say, θ_1, through a canonical transformation
- Perform a *normalization* for all the other actions, *except* the one whose conjugate angle is the *resonant angle*.

It is easy to show that the "new" Hamiltonian is integrable. It does not depend on the angles $\theta_2, \theta_3, ...$, so that the corresponding actions $I_2, I_3, ...$ are integrals of motion. Therefore they are constants and can be treated as parameters. The Hamiltonian contains *only* one pair of conjugate variables, θ_1, I_1, so that it has *only* one degree of freedom. Then it is integrable by definition and we can, *in principle*, write the corresponding last integral of motion. The topology of the phase space of a 1-D Hamiltonian is, locally, similar to that of the pendulum. Therefore we can use the analytical solution of the pendulum, to describe the evolution of (θ_1, I_1).

6. PROPER ELEMENTS AND ASTEROID FAMILIES

The secular Hamiltonian (35) is *integrable* and, therefore, is has as many first integrals of motion as degrees of freedom. Among all possible sets of integrals of motion the one of physical interest is the set $\{a_P, e_P, i_P\}$. These are the *proper elements* of the asteroid's orbit. As a result, we can classify asteroids through their proper elements, provided that their orbit is neither resonant nor chaotic. In practice these two restrictions are not really strict. In cases where the orbit is close to *a single* resonance, we can construct a secular integrable Hamiltonian as well, as discussed in the previous section, so that *resonant* proper elements may be defined. Finally in cases where the orbit is close to two or more resonances, it is chaotic. In this case the secular Hamiltonian gives a very bad approximation of the real motion, but proper elements may be of some use for time intervals less than the Lyapunov time (i.e. the time after which a chaotic trajectory "forgets" its initial conditions).

If the asteroids are plotted in proper elements space, we can distinguish that some of them are clearly "clustered" in groups, the *asteroid families*. Today we believe that asteroid families consist of the breakup pieces of catastrophic collisions between large parent bodies.

7. BEYOND "CLASSICAL" SOLAR SYSTEM DYNAMICS

7.1. Chaos

Many of the solar system bodies follow chaotic trajectories. This is due to the fact that they are under the influence of more than one resonances. The secular theory and the proper elements presented here hold for ordered trajectories. What about the chaotic ones? One possible solution of this problem, that appeared recently in the literature, is a *statistical approach*. In this method we assume that asteroids are diffusing in action (equivalently, proper element) space and we follow the evolution of an ensemble of

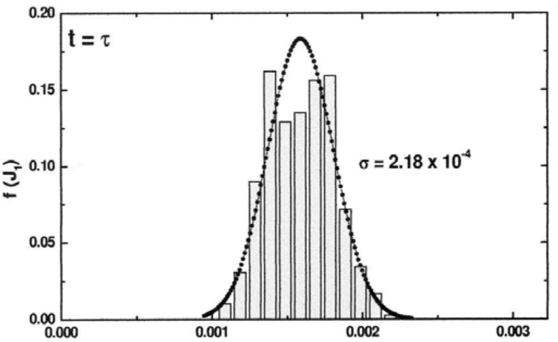

FIGURE 4. Chaotic chronology of the Veritas family. The actual distribution of asteroids is fitted to a Gaussian with a σ corresponding to an age of 8.9 Myrs.

asteroids in this space by using a diffusion equation of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial I}\left(\frac{D}{2}\frac{\partial P}{\partial I}\right) \tag{51}$$

in which enters only one parameter, the *diffusion coefficient* $D(I)$. If we can calculate $D(I)$, either analytically [4] or numerically [9], we can then solve the equation and find the evolution of the initial distribution $P(I_0, 0)$. In the simplest possible case, where the diffusion coefficient is constant and the space where the asteroids are diffusing is considered infinite, the solution is the well known *Fick's law*, a Gaussian centered at I_0 with a dispersion $\sigma^2 = Dt$.

In recent years the diffusion equation method has been used to attack some problems hard to solve through traditional approaches. One such example is the estimation of the age of the Veritas asteroid family through chaotic chronology, by assuming that the members of the family on chaotic orbits originated from the breakup of a parent body after a catastrophic collision [8]. The result, 8.9 Myrs, (Fig. 4) agrees very well with the estimate of 8.3 Myrs made by back integration of the members of the family on ordered orbits [5].

7.2. Non-gravitational forces

As mentioned above, in the case where the orbit of an asteroid is non-resonant or it is affected by a single resonance, the secular theory implies that the semi-major axis is a constant of the motion and the proper eccentricity and inclination remain constant. But even if the orbit is affected by two or more resonances, in which case it is chaotic, numerical integrations have shown that, although eccentricity and inclination are varying in proper elements space, the semi-major axis is hard to change.[5] The only way to "break" effectively the invariance of the semi-major axis is through non-gravitational forces. The most important mechanism for transport in semi-major axis space, for kilometer-size bodies, is the *Yarkovsky effect*, the recoil force acting on a spinning asteroid by the emission of infrared radiation.

The last years the Yarkovsky effect has been invoked in order to interpret otherwise difficult to understand phenomena. One such application was the accurate modelling of the motion of the asteroid Golevka [6] and the shape of the 7:3 Kirkwood gap [10]. In the first example it was shown that the very accurate measurements of the position and velocity of Golevka were not consistent with pure gravitational motion and could be understood only by assuming the action of the Yarkovsky effect. In the second example it was shown that the existence of asteroids with dynamical life-time of 10^7 years in the 7:3 Kirkwood gap can be understood only by assuming that what we observe is just a dynamical equilibrium between the transport of asteroids out of the resonant region and the injection of new asteroids from the adjacent Eos and Koronis asteroid families through the action of the Yarkovsky effect.

7.3. Simulations

The two approaches of studying the evolution of our planetary system under the influence of only gravitational forces (secular theory or statistical description of chaotic motion) are only the two "extreme" cases. For all "intermediate" cases a satisfactory appropriate theory does not exist today and, therefore, these cases have to be studied numerically. Indeed, the motion of a large number of bodies can be studied consistently, including the "hard to model" collisions and/or close encounters, provided that one has the appropriate computer power to follow their evolution for time intervals of the order of the age of the solar system. The last years this approach has gained momentum, through the continuous increase of speed and memory of "classical" computers or the use of dedicated machines. Interesting examples of this new approach are (a) the initial evolution of the outer solar system [7], (b) the simulation of the creation of the Moon through the impact of a Mars-sized body on Earth [1] and (c) the triggering of planetary formation, through the interaction of planetesimals of a protoplanetary disk with a close approaching star [11].

In the first example it has been shown, through a simulation on "traditional" computers, that the outer planets were created in a far more "compact" configuration than today's and that they were transported to their present places through a process called *migration*, caused by their interaction with a disk of planetesimals. In this way one may understand not only the present values of eccentricities and inclinations of the outer planets, but as well the phenomenon of the *Late Heavy Bombardment* of the inner planets by a swarm of Kuiper belt objects, ~ 700 Myrs after the creation of the solar system

[5] This result stems from the degeneracy of the Kepler's problem: the resonances are at $a \simeq a_{res}$ for all values of e, i in the asteroid belt. The semi-major axis, a, undergoes non-periodic but zero-average variations around a_{res}, unless e increases so much, that overlapping with a nearby resonance is possible.

(i.e. 3.9 Gyrs ago).

The other two simulations were performed using a GRAPE machine, a dedicated computer accelerator which calculates the gravitational forces between N bodies. In the first of them it has been shown that most of the debris of such an impact are not ejected on hyperbolic orbits but remain in orbit around the Earth, forming a disk of "satellesimals". It was shown that the such a disk will end up in the creation of a satellite, in the same way that planetesimals create finally the planets, through successive disrupting and sticking collisions.

In the second we have shown that parabolic encounters between gas-free protoplanetary disks result into (i) the exchange of material between the disks, and (ii) the restructuring of the initial disk, into a "core" of nearly circular and co-planar orbits and an extended 3-D cloud of eccentric and inclined bodies. These processes may prove to be particularly important for the formation of planets and small-body belts.

ACKNOWLEDGMENTS

This work was supported by the research program EPEAEK II / Pythagoras I of the Greek Ministry of Education. I would like to thank Empeirikio Foundation of Greece and Aristotle University of Thessaloniki for funding the purchase of the Grape 6/Pro8 system and Dr. K. Tsiganis for reading the first draft of this article.

REFERENCES

1. S. Ida, R.M. Kanup, and G.R. Stewart, *Nature*, **389**, 353–357 (1997).
2. A. Morbidelli, *Modern Celestial Mechanics, Aspects of Solar System Dynamics*, Taylor & Francis, London, 2002, pp. 11–38, 127-230.
3. C.D. Murray and S.F. Dermott, *Solar System Dynamics*, Cambridge University Press, Cambridge, 1999, pp. 22–128, 225–318.
4. N. Murray and M. Holman, *Astron. J.*, **114**, 1246–1259 (1997).
5. D. Nesvorný, W.F. Bottke, H.F. Levison, and L. Dones, *A*strophys. J., **591**, 486–497 (2003).
6. S.R. Chesley, S.J. Ostro, D. Vokrouhlický, D. Čapek, J.D. Giorgini, M.C. Nolan, J.L. Margot, A.A. Hine, L.A.M. Benner, and A.B. Chamberlin, *Science*, **302**, 1739–1742 (2003).
7. K. Tsiganis, R. Gomes, A. Morbidelli, and H.F. Levison, *Nature*, **435**, 459–461 (2005).
8. K. Tsiganis, Z. Knežević, and H. Varvoglis, *Icarus*, submitted (2006).
9. K. Tsiganis, H. Varvoglis, and A. Anastasiadis, "Diffusion of asteroids in the outer belt," in *Modern Celestial Mechanics: from Theory to Applications*, edited by A. Celletti, S. Ferraz-Mello, and J. Henrard D. F., Kluwer, Dordrecht, 2003, pp. 451–455.
10. K. Tsiganis, H. Varvoglis, and A. Morbidelli, *Icarus*, **166**, 131–140 (2003).
11. H. Varvoglis, K. Tsiganis, and Ch. Vozikis, "Large-scale simulations of gravitational Astrophysical systems," in *International Workshop: Rotation of Celestial Bodies*, edited by A. Lemaitre, in press, 2006.