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Dynamics and stability of the Gödel universe

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Abstract
We use covariant techniques to describe the properties of the Gödel universe and then consider its linear response to a variety of perturbations. Against matter aggregations, we find that the stability of the Gödel model depends primarily upon the presence of gradients in the centrifugal energy, and secondarily on the equation of state of the fluid. The latter dictates the behaviour of the model when dealing with homogeneous perturbations. The vorticity of the perturbed Gödel model is found to evolve as in almost-FRW spacetimes, with some additional directional effects due to shape distortions. We also consider gravitational-wave perturbations by investigating the evolution of the magnetic Weyl component. This tensor obeys a simple plane-wave equation, which argues for the neutral stability of the Gödel model against linear gravity-wave distortions. The implications of the background rotation for scalar-field Gödel cosmologies are also discussed.

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1. Introduction

The Gödel universe is an exact solution of the Einstein field equations, which is both stationary and spatially homogeneous [1]. The model, which is of Petrov type D, is also rotationally symmetric about each point and contains a perfect-fluid matter source whose 4-velocity is a Killing vector [2]. Gödel's universe is known for its unusual global properties. The most intriguing among them is the existence of closed timelike curves, which violates global causality and makes time travel theoretically possible in this spacetime. Hermann Weyl had first suggested in 1921 that time travel might occur in general relativity [3] as a consequence of 'very considerable fluctuations in the spacetime metric' but he believed that the fluctuations 'necessary to produce this effect do not occur in the region of the world in which we live'. It was clear that this causal anomaly could arise for *some* distorted geometry and hence for some distribution of mass and energy. Kurt Gödel showed that the required distribution could be extremely simple and his discovery initiated the study of the global properties and causal structure of Einstein's equations [4]. At first, attention was almost entirely focused upon the properties of the equations of motion in the Gödel universe [5–7] and an interesting overview of these investigations has recently been given by Ozsváth and Schücking [8]. Subsequently, Gödel's solution has also triggered a considerable amount of work on rotating solutions of the Einstein field equations and rotational cosmological models (see [9–13] for a representative list). For further details and an extensive discussion the reader is referred to the recent review articles by Krasiński [14–16] and Obukhov [17]. Although the Gödel spacetime is not a realistic model for our universe, it is an important theoretical laboratory for investigating a range of global properties of the spacetime structure in different gravity theories. Recent studies of quantum computation [18, 19] have shown that the presence of closed timelike curves in spacetime provides for a new physical model for quantum computation in compact regions. A quantum computer with access to closed timelike curves can solve NP-complete problems with only a polynomial number of quantum logic gates. A series of studies on the possibility of closed timelike paths in string theories [20–26] has also drawn on the insights gained from the detailed study of the Gödel spacetime and its generalizations to arbitrary space dimensions [27, 28], considering the constraints of possible holographic principles [29] and investigating the accessibility of information in different parts of the spacetime [30].

The Gödel universe is a member of the family of homogeneous spacetimes together with the Einstein static and de Sitter universes. In recent years close attention has been paid to the stability of the de Sitter universe and its physical significance for the consequences of inflation in the early stages of our universe. Under physically realistic conditions on the energy–momentum tensor of matter, all perturbations to isotropy and homogeneity will be seen to fall off exponentially rapidly within the event horizon of a geodesically moving observer in the de Sitter universe [31–33]. This result, in its different technical expressions, is known as the cosmic no hair theorem. The stability of the Einstein static universe has also been examined in a covariant fashion and reveals a more intricate dependence on material content than is conventionally reported [34, 35]. We will complement these detailed studies of the stability of the de Sitter and Einstein static homogeneous spacetimes with a covariant study of the stability of the Gödel universe. Despite the past interest in the causal structure of the Gödel model there have been few perturbative studies of its properties and these have been confined to the investigation of the effects of the global rotation on the evolution of scalar density perturbations in the papers of Silk [36, 37]. Here, we investigate the stability with respect to scalar, vector and tensor perturbation modes using the gauge covariant formalism of Ellis and Bruni [38]. In the process we show that the background vortical energy contributes to the gravitational pull of the matter, while its gradients add to the pressure support. The balance between these two agents effectively determines the stability of the Gödel universe against matter aggregations. By examining the rotational behaviour of the perturbed model, we explain how shape distortions can increase or decrease its overall rotation. We also introduce a set of linear covariant constraints, which isolate the pure tensor perturbations, and analyse their propagation on the rotating Gödel background. Our results argue for the neutral stability of these gravitational-wave distortions. Finally, we consider some of the effects of spatially homogeneous perturbations and discuss the compatibility of scalar fields with the symmetries of the Gödel spacetime.

2. Covariant characterization of the Gödel universe

2.1. The irreducible kinematical variables

The covariant description of the Gödel universe, with respect to its irreducible kinematical quantities, has been given in [39] (see also [4, 40]). Here, we will briefly introduce and extend this description and also present the associated constraints.

Relative to a timelike 4-velocity field u_a (normalized so that $u_a u^a = -1$) that is tangent to the worldlines of the fundamental observers, the Gödel spacetime is covariantly described by [39]

$$\Theta = 0 = \dot{u}_a = \sigma_{ab}, \qquad \text{and} \qquad \omega_a \neq 0. \tag{1}$$

Therefore, with the exception of the vorticity (ω_a), the rest of the kinematical variables, namely the volume expansion (Θ), the shear (σ_{ab}) and the acceleration ($\dot{u}_a$), vanish identically. We note that $\nabla_b \omega_a = 0$, so ensuring that the vorticity vector associated with the 4-velocity field u_a is covariantly constant.

2.2. The twice-contracted Bianchi identities

The stationary nature of the Gödel solution means that there are no propagation equations: they have all been transformed into constraints. For example, on using the twice-contracted Bianchi identities we obtain the standard conservation laws for the energy and momentum densities. When applied to the Gödel spacetime, the latter yield the constraints

$$\dot{\mu} = 0 \qquad \text{and} \qquad \mathrm{D}_a p = 0, \tag{2}$$

where μ and p are the matter density and isotropic pressure, respectively[3]. Here, an overdot indicates differentiation along u_a (e.g. $\dot{\mu} = u^a \nabla_a \mu$). Also, $\mathrm{D}_a = h_a{}^b \nabla_b$ is the covariant derivative operator orthogonal to u_a, and $h_{ab} = g_{ab} + u_a u_b$ is the associated projection tensor.

2.3. The Ricci identities

Covariantly, the kinematic evolution is determined by a set of three propagation equations and three constraints, all of which are derived from the Ricci identities $\nabla_{[a}\nabla_{b]} u_c = R_{dcba} u^d$, where R_{abcd} is the spacetime Riemann tensor. In Gödel's universe, the Raychaudhuri equation, which describes the volume evolution of a fluid element, reduces to

$$\tfrac{1}{2}\kappa(\mu + 3p) - 2\omega^2 - \Lambda = 0, \tag{3}$$

with $\kappa = 8\pi G$ and $\omega^2 = \omega_a \omega^a$. Note how the rotation balances the gravitational attraction of the matter, as well as that of the (negative, see section 2.5) cosmological constant. Thus, in the Gödel model the vorticity has assumed the role played by the (positive) cosmological constant in the Einstein static universe. Of the two remaining propagation equations, the shear evolution formula takes the form

$$E_{ab} + \omega_{\langle a}\omega_{b\rangle} = 0, \tag{4}$$

where E_{ab} is the electric component of the Weyl tensor[4]. The vorticity propagation equation, on the other hand, is trivially satisfied. Two of the three kinematical constraints, the shear

[3] Originally, Gödel's solution was given for dust (i.e. $\mu \neq 0$, $p = 0$, $\Lambda \neq 0$). However, through the transformation $\mu \to \mu' = \mu + p$ and $\Lambda \to \Lambda' = \Lambda + \kappa p$, the Gödel spacetime can be reinterpreted as a perfect-fluid model.

[4] Angle brackets are used to indicate orthogonally projected vectors and the projected, symmetric and trace-free part of second rank tensors.

divergence and the vorticity divergence, are also trivially satisfied, while the gravito-magnetic constraint leads to

$$H_{ab} = 0, \tag{5}$$

with H_{ab} being the magnetic counterpart of E_{ab}. It should be emphasized that the latter is not covariantly constant (i.e. $\nabla_c E_{ab} \neq 0$). Instead, one can use equation (4) to show that

$$\dot{E}_{ab} = 0 = \mathrm{D}_c E_{ab}, \tag{6}$$

in agreement with the stationary nature and spatial homogeneity of the Gödel spacetime. Clearly, result (6b) also guarantees that $\mathrm{D}^b E_{ab} = 0 = \mathrm{curl}\, E_{ab}$.

2.4. The Bianchi identities

The Bianchi identities provide two pairs of propagation and constraint equations for the conformal curvature, which is monitored through the electric and the magnetic parts of the Weyl tensor. When applied to the Gödel spacetime, the $\dot{E}$-equation gives

$$E_{\langle a}{}^{d}\epsilon_{b\rangle cd}\omega^c = 0, \tag{7}$$

while the $\dot{H}$-equation is trivially satisfied. Note that, when (4) is taken into account, the above constraint also becomes trivial. On the other hand, the $\mathrm{div}E_{ab}$ and $\mathrm{div}H_{ab}$ constraints associated with the Bianchi identities lead to

$$\mathrm{D}_a\mu = 0 \tag{8}$$

and

$$\kappa(\mu + p)\omega_a + 3E_{ab}\omega^b = 0, \tag{9}$$

respectively. Results (2a) and (8) combine to ensure that $\nabla_a\mu = 0$, while from equation (3) we obtain $\dot{p} = 0$. In other words, both the energy density and the isotropic pressure of the matter that fill Gödel's universe are covariantly constant quantities.

2.5. Further constraints

Additional constraints are obtained by contracting (4) along ω_a and substituting the result into equation (9). Then one finds that

$$\kappa(\mu + p) = 2\omega^2, \tag{10}$$

which also measures the total inertial mass of the Gödel universe. On using this result, we can recast equation (3) as

$$\kappa(\mu - p) = -2\Lambda, \tag{11}$$

to guarantee that $\Lambda \leqslant 0$ as long as $p \leqslant \mu$. The cosmological constant vanishes only when the fluid has a maximally stiff equation of state, with $p = \mu$. Thus, for conventional matter sources, the Gödel spacetime has non-positive Λ. This restriction is relaxed when dealing with the Newtonian analogue of the Gödel universe (see below). From (10) and (11) it becomes clear that, given the equation of state of the matter, only one of μ, ω or Λ is needed to determine the other two. Finally, constraints (10) and (11) combine to give

$$\kappa\mu + \Lambda - \omega^2 = 0, \tag{12}$$

which is the Gödel analogue of the Friedmann equation.

The generic rotation of the Gödel universe means that the fluid flow lines are not hypersurface orthogonal and, therefore, there are no integrable spatial sections. Nevertheless, one can still employ the generalized Gauss–Codacci equation to evaluate the orthogonally projected Ricci tensor. Applied to the Gödel model, the latter reads

$$\mathcal{R}_{ab} = \tfrac{2}{3}(\kappa\mu + \Lambda - \omega^2)h_{ab}, \tag{13}$$

which by means of constraint (12) ensures that $\mathcal{R}_{ab}$ vanishes.

2.6. The Newtonian analogue

There is a simple Newtonian counterpart to the rotating Gödel universe in the case of zero pressure that has been explored by Ozsváth and Schücking [41] (see also [42]) following Gödel [43]. If we consider a pressureless fluid of constant density μ rotating rigidly with constant angular velocity ω about the z-axis so $\vec{v} = (-\omega y, \omega x, 0)$ then, by integrating the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla \Phi, \tag{14}$$

we find that the Newtonian gravitational potential is

$$\Phi = \tfrac{1}{2}\omega^2(x^2 + y^2) \tag{15}$$

where now ∇ indicates ordinary partial derivatives. This solves the Poisson equation with cosmological constant term ($c = 1$),

$$\nabla^2\Phi + \Lambda = \tfrac{1}{2}\kappa\mu, \tag{16}$$

if

$$2\omega^2 \equiv \tfrac{1}{2}\kappa\mu - \Lambda. \tag{17}$$

We see that this Newtonian result coincides with the relation (3) for the Gödel solution in the $p = 0$ case and, in contrast to general relativity, can be satisfied even for $\Lambda = 0$ and some $\Lambda > 0$ values. In general relativity the additional geometrical constraint (11) leads to the stronger relation,

$$\omega^2 \equiv \tfrac{1}{2}\kappa\mu, \tag{18}$$

for Gödel's solution, which therefore requires $\Lambda < 0$.

2.7. The closed timelike curves

The most intriguing property of the Gödel spacetime is the existence of closed timelike curves, which makes time travel a theoretical possibility and has ensured an enduring interest in the model [44–47]. To demonstrate the presence of such closed timelike worldlines, consider the line element of the original Gödel solution (i.e. with $p = 0$ and metric signature -2) written in cylindrical coordinates (r, z, ϕ)

$$\mathrm{d}s^2 = 4a^2[\mathrm{d}t^2 - \mathrm{d}r^2 - \mathrm{d}z^2 + (\sinh^4 r - \sinh^2 r)\mathrm{d}\phi^2 + 2\sqrt{2}\sinh^2 r\,\mathrm{d}\phi\mathrm{d}t], \tag{19}$$

with $1/a^2 = \kappa\mu = -2\Lambda$ [1]. Then, introducing the coordinate transformation $t \to \tilde{t} = 2at$, $r \to \tilde{r} = 2ar$, $z \to \tilde{z} = 2az$ and $\phi \to \tilde{\phi} = \phi$, transforms the above line element into

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - \mathrm{d}z^2 + 4a^2\left[\sinh^4\left(\frac{r}{2a}\right) - \sinh^2\left(\frac{r}{2a}\right)\right]\mathrm{d}\phi^2 + 4\sqrt{2}a\sinh^2\left(\frac{r}{2a}\right)\mathrm{d}\phi\mathrm{d}t, \tag{20}$$

where the tildas have been suppressed. Clearly, if for some $r = r_0$ the quantity $\sinh^4(r/2a) - \sinh^2(r/2a)$ is positive, the circle defined by $r = r_0, t = 0 = z$ will be timelike everywhere [1]. Thus, on using (10) with $p = 0$, the condition $r > 2\ln(1+\sqrt{2})/\sqrt{\kappa\mu}$ for the existence of such closed timelike curves reads

$$r > R_G \equiv \frac{\sqrt{2}\ln(1+\sqrt{2})}{\omega}, \tag{21}$$

where R_G denotes the radius of the observer's 'causal region'. Hence, $R_G \to \infty$ as $\omega \to 0$, which means that the weaker the rotation of the model the more 'remote' the closed timelike curves become. Alternatively, one might say that the faster the Gödel universe rotates the smaller its causal region becomes. Note that one can include the fluid pressure in equation (21) by simply using the transformation $\mu \to \tilde{\mu} = \mu + p$.

3. The perturbed Gödel universe

3.1. Conservation laws

For a perfect fluid, the energy density and the momentum density conservation laws are given by the nonlinear expressions

$$\dot{\mu} = -(\mu + p)\Theta, \tag{22}$$

and

$$(\mu + p)\dot{u}_a = -\mathrm{D}_a p, \tag{23}$$

respectively. These formulae also hold after we have linearized about the Gödel background. In that case, however, we can substitute the zero-order relation $\mu + p = 2\omega^2$ on the right-hand side of (22). Also, when dealing with a barotropic perfect fluid (i.e. for $p = p(\mu)$), we have $\mathrm{D}_a p = c_s^2 \mathrm{D}_a \mu$, where $c_s^2 = \mathrm{d}p/\mathrm{d}\mu$ is the square of the adiabatic sound speed.

3.2. Density perturbations

Consider a general spacetime filled with a single barotropic perfect fluid. The nonlinear evolution of density inhomogeneities is monitored by the system [38]

$$\dot{\mathcal{D}}_{\langle a\rangle} = w\Theta\mathcal{D}_a - (\sigma_{ab} - \omega_{ab})\mathcal{D}^b - (1+w)\mathcal{Z}_a, \tag{24}$$

$$\dot{\mathcal{Z}}_{\langle a\rangle} = -\tfrac{2}{3}\Theta\mathcal{Z}_a - (\sigma_{ab} - \omega_{ab})\mathcal{Z}^b - \tfrac{1}{2}\kappa\mu\mathcal{D}_a + a\,\Re\dot{u}_a + aA_a - 2a\mathrm{D}_a(\sigma^2 - \omega^2), \tag{25}$$

where $\mathcal{D}_a = (a/\mu)\mathrm{D}_a\mu$, $\mathcal{Z}_a = a\mathrm{D}_a\Theta$ and $w \equiv p/\mu$ by definition[5]. Also

$$\Re = \kappa\mu - \tfrac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) + A + \Lambda \tag{26}$$

defines $\Re$ and $A_a = \mathrm{D}_a A$ with $A = \nabla^a \dot{u}_a$. The 4-acceleration satisfies the momentum-density conservation law, which for adiabatic perturbations is given by equation (23). The projected vectors $\mathcal{D}_a$ and $\mathcal{Z}_a$, respectively, describe local inhomogeneities in the matter energy density and in the volume expansion [38]. Note that in an exact Gödel universe $\mathrm{D}_a\mu = 0 = \mathrm{D}_a\Theta$. Hence, $\mathcal{D}_a$ and $\mathcal{Z}_a$ vanish identically to zero order and therefore both satisfy the gauge-invariant requirements [48].

The scalar $\Re$ plays an important and subtle role via its coupling with the 4-acceleration in equation (25). In particular, the sign of $\Re$ determines the effect of the pressure gradients associated with $\dot{u}_a$. When $\Re$ is positive these gradients will add to the gravitational pull of the

[5] The background scale a, which has been introduced to make $\mathcal{D}$ and $\mathcal{Z}$ dimensionless and appears in equations (24), (25), is a characteristic dimension of the Gödel universe corresponding to the radius of the smallest closed timelike curves [1, 37].

density inhomogeneities. Otherwise, they will contribute to the pressure support. The subtlety in $\Re$ is that the vorticity adds to the effect of the ordinary matter, whereas the shear opposes it. This is a purely relativistic effect, with no known Newtonian analogue. Note that in exact FRW models $\Re = 0$, which explains why the aforementioned behaviour has passed relatively unnoticed [38].

Linearizing equations (24), (25) about the Gödel universe and keeping only the zero-order component of (26) we obtain

$$\dot{\mathcal{D}}_a = \omega_{ab}\mathcal{D}^b - (1+w)\mathcal{Z}_a, \tag{27}$$

$$\dot{\mathcal{Z}}_a = \omega_{ab}\mathcal{Z}^b - \tfrac{1}{2}\kappa\mu\mathcal{D}_a + a\Re\dot{u}_a + aA_a + 2a\mathrm{D}_a\omega^2, \tag{28}$$

with

$$\Re = \kappa\mu + 2\omega^2 + \Lambda = 3\omega^2 > 0, \tag{29}$$

since $\kappa\mu - \omega^2 + \Lambda = 0$ in the exact Gödel spacetime. Also, $A = \mathrm{D}^a\dot{u}_a$ to first order and $\dot{u}_a$ is still given by equation (23). Then, the system (27), (28) reads

$$\dot{\mathcal{D}}_a = \omega_{ab}\mathcal{D}^b - (1+w)\mathcal{Z}_a, \tag{30}$$

$$\dot{\mathcal{Z}}_a = \omega_{ab}\mathcal{Z}^b - \frac{1}{2}\kappa\mu\mathcal{D}_a - \frac{3a\omega^2}{\mu(1+w)}\mathrm{D}_a p - \frac{a}{\mu(1+w)}\mathrm{D}_a(\mathrm{D}^2 p) + 2a\mathrm{D}_a\omega^2 \tag{31}$$

where $\omega^2 = \kappa\mu(1+w)/2$ to zero order (see equation (10)). Compared to the perturbed FRW case, the nonzero rotation of the Gödel background has added four extra terms to the system (30), (31). They are the first term on the right-hand side of (30), and the first, third and the last term on the right-hand side of (31). Note the second of the three vorticity terms in equation (31), which is proportional to ω^2 and adds to the overall gravitational pull. This relativistic effect seems to suggest that rotational energy also has 'weight'. Indirectly, it also seems to favour the de Felice and the Barrabès *et al* interpretation [49, 50] of the Abramowicz–Lasota 'centrifugal force reversal' effect [51, 52]. The third vorticity term in (31), on the other hand, is triggered by inhomogeneities in ω^2 and will also appear in a Newtonian treatment. It can resist the collapse, thus mimicking the effects of the ordinary pressure gradients. As we shall see below, the stability of the Gödel universe depends crucially on which of these two terms dominates over the other.

Density perturbations are related to gradients in the vorticity and the curvature by means of the Gauss–Codacci equation, according to which the linearized projected Ricci scalar is given by $\mathcal{R} = 2(\kappa\mu - \omega^2 + \Lambda)$. The above immediately implies the linear relation

$$\kappa\mu\mathcal{D}_a = a\mathrm{D}_a\omega^2 + \tfrac{1}{2}a\mathrm{D}_a\mathcal{R}, \tag{32}$$

between $\mathcal{D}_a$, $\mathrm{D}_a\omega^2$ and $\mathrm{D}_a\mathcal{R}$.

3.3. Two alternative types of perturbations

Following (32) we may consider two types of density perturbation, first by setting $\mathrm{D}_a\mathcal{R} = 0$ and then by assuming that $\mathrm{D}_a\omega^2 = 0$. For simplicity, we will label the former *isocurvature* and the latter *perturbations under rigid rotation*. In general one does expect these two pertrurbation types to propagate independently. However, this approximation scheme simplifies the system (30), (31) and we can investigate the effects of global rotation on the evolution of density perturbations analytically.

3.3.1. Isocurvature perturbations. Setting $\mathrm{D}_a\mathcal{R}=0$ in equation (32) means that $a\mathrm{D}_a\omega^2=\kappa\mu\mathcal{D}_a$ and the linear system (30), (31) takes the form

$$\dot{\mathcal{D}}_a=\omega_{ab}\mathcal{D}^b-(1+w)\mathcal{Z}_a, \tag{33}$$

$$\dot{\mathcal{Z}}_a=\omega_{ab}\mathcal{Z}^b+\frac{3}{2}\kappa\mu\big(1-c_{\mathrm{s}}^2\big)\mathcal{D}_a-\frac{c_{\mathrm{s}}^2}{1+w}\mathrm{D}_a(\mathrm{D}^b\mathcal{D}_b), \tag{34}$$

where we have used the background relation $\omega^2=\kappa\mu(1+w)/2$ and the barotropic expression $\mathrm{D}_a p=\big(\mu c_{\mathrm{s}}^2/a\big)\mathcal{D}_a$. The projected divergence of these equations, multiplied by the background scale a, gives the set of equations governing the linear evolution of isocurvature density perturbations in a perturbed Gödel universe.

$$\dot{\Delta}=-(1+w)\mathcal{Z}, \tag{35}$$

$$\dot{\mathcal{Z}}=\frac{3}{2}\kappa\mu\big(1-c_{\mathrm{s}}^2\big)\Delta-\frac{c_{\mathrm{s}}^2}{1+w}\mathrm{D}^2\Delta, \tag{36}$$

where $\Delta=a\mathrm{D}^a\mathcal{D}_a$ and $\mathcal{Z}=a\mathrm{D}^a\mathcal{Z}_a$ by definition[6]. Note that the scalar Δ describes local matter aggregations and it is the covariant analogue of the standard density contrast $\delta\rho/\rho$. The system (35), (36) leads to the following wavelike equation for the evolution of matter aggregations:

$$\ddot{\Delta}=-\tfrac{3}{2}\kappa\mu(1+w)\big(1-c_{\mathrm{s}}^2\big)\Delta+c_{\mathrm{s}}^2\mathrm{D}^2\Delta, \tag{37}$$

and subsequently leads to

$$\ddot{\Delta}_{(k)}=-\left[\frac{3}{2}\kappa\mu(1+w)\big(1-c_{\mathrm{s}}^2\big)+\frac{k^2c_{\mathrm{s}}^2}{a^2}\right]\Delta_{(k)}, \tag{38}$$

assuming the decomposition $\Delta=\sum_k\Delta_{(k)}Q^{(k)}$ with $\mathrm{D}_a\Delta_{(k)}=0$. Note that $Q^{(k)}$ are locally defined scalar harmonics with $\dot{Q}^{(k)}=0$ and $\mathrm{D}^2Q^{(k)}=-(k^2/a^2)Q^{(k)}$. The last equation has an oscillatory (i.e. neutrally stable) solution as long as $-\mu\leqslant p\leqslant\mu$. Note that the stability of the isocurvature modes is guaranteed even when the fluid pressure vanishes.

Incorporating $\mathrm{D}_a\omega^2$ in equation (34) has increased the overall resistance of the model against the gravitational pull of the matter, since gradients in the rotational energy act like pressure gradients. Their presence is responsible for the stability of the linear isocurvature perturbations demonstrated in (38).

3.3.2. Perturbations under rigid rotation. Setting $\mathrm{D}_a\omega^2=0$ in equation (31) and using the barotropic fluid relation $\mathrm{D}_a p=\big(\mu c_{\mathrm{s}}^2/a\big)\mathcal{D}_a$, transforms the system (30), (31) into

$$\dot{\mathcal{D}}_a=\omega_{ab}\mathcal{D}^b-(1+w)\mathcal{Z}_a, \tag{39}$$

$$\dot{\mathcal{Z}}_a=\omega_{ab}\mathcal{Z}^b-\frac{1}{2}\kappa\mu\big(1+3c_{\mathrm{s}}^2\big)\mathcal{D}_a-\frac{c_{\mathrm{s}}^2}{1+w}\mathrm{D}_a(\mathrm{D}^b\mathcal{D}_b). \tag{40}$$

Multiplying the above with the characteristic background scalar a and then taking their scalar parts, as before, we obtain

$$\dot{\Delta}=-(1+w)\mathcal{Z}, \tag{41}$$

$$\dot{\mathcal{Z}}=-\frac{1}{2}\kappa\mu\big(1+3c_{\mathrm{s}}^2\big)\Delta-\frac{c_{\mathrm{s}}^2}{1+w}\mathrm{D}^2\Delta, \tag{42}$$

[6] In deriving equation (35) we have used the linear result $a\mathrm{D}^a\dot{\mathcal{D}}_a=\dot{\Delta}+a\omega_{ab}\mathrm{D}^a\mathcal{D}^b$. An exactly analogous relation for the expansion gradients has been used to obtain (36).

which leads to the following wavelike equation for $\Delta_{(k)}$:

$$\ddot{\Delta}_{(k)} = \left[\frac{1}{2}\kappa\mu(1+w)\left(1+3c_{\rm s}^2\right) - \frac{k^2c_{\rm s}^2}{a^2}\right]\Delta_{(k)}, \tag{43}$$

on using the harmonic decomposition $\Delta = \sum_k \Delta_{(k)} Q^{(k)}$ of the previous section. For dust (i.e. $w = 0 = c_{\rm s}^2$) there is no pressure support and Δ grows unimpeded. In the presence of pressure, however, there is an effective Jeans length given by

$$\lambda_J = \frac{c_{\rm s}}{\omega\sqrt{1+3c_{\rm s}^2}}, \tag{44}$$

since $\kappa\mu(1+w) = 2\omega^2$ to zero order. Inhomogeneities on scales exceeding λ_J collapse under the gravitational pull of the matter, but short wavelength perturbations oscillate like sound waves. Interestingly, the Jeans scale given above is comparable to R_G, namely to the radius of the smallest closed timelike curves (see equation (21)). This means that the causal regions of a Gödel universe containing a fluid with nonzero pressure are stable with respect to linear matter aggregations.

By setting $\mathrm{D}_a\omega^2 = 0$ we have removed the supporting effect of the centrifugal energy gradients from equation (31). This is the reason for the instability pattern associated with (43). The only effect of the Gödel background is via the rotational energy contribution to the overall gravitational pull (see also section 3.2). As a result, the above given Jeans scale is smaller than that of the almost-FRW models [53]. Note that the Jeans criterion found here is analogous to that obtained in [37].

3.4. Directional effects on density perturbations

The rotation of the Gödel universe introduces a preferred direction because of the background vorticity vector. The analysis of density perturbations given in the previous sections examines the evolution of the scalar Δ, which describes the average matter aggregation and therefore it does not pick out any directional effects. These, however, are incorporated in $\mathcal{D}_a$, the projected vector that describes variations in the density distribution as seen between two neighbouring observers. To reveal the directional effects of the background rotation it helps to implement an additional splitting of the Gödel space, along and orthogonal to the vorticity vector. As a first step, we introduce the unit vector $n_a = \omega_a/\omega$ (where $\omega = \sqrt{\omega_a\omega^a}$) parallel to ω_a and use it to define the two-dimensional projection tensor

$$f_{ab} = h_{ab} - n_a n_b, \tag{45}$$

with $f_{ab} = f_{(ab)}$, $f_{ab}n^b = 0$, $f_a{}^b f_b{}^c = f_a{}^c$ and $f_a{}^a = 2$. This projects into the 2D space orthogonal to ω_a. Employing (45) we define the gradient $\tilde{\mathrm{D}}_a = f_a{}^b\mathrm{D}_b$, with $n^a\tilde{\mathrm{D}}_a = 0$, and introduce the decomposition $\mathrm{D}_a = \tilde{\mathrm{D}}_a + n_a n^b \mathrm{D}_b$ orthogonal to and along the rotation axis. Then, the density gradients split as

$$\mathcal{D}_a = \tilde{\mathcal{D}}_a + \tilde{\mathcal{D}}n_a, \tag{46}$$

where $\tilde{\mathcal{D}}_a = f_a{}^b\mathcal{D}_b = (a/\mu)\tilde{\mathrm{D}}_a\mu$ is the density perturbation orthogonal to n_a and $\tilde{\mathcal{D}} = n^a\mathcal{D}_a = (a/\mu)n^a\mathrm{D}_a\mu$ is the density perturbation parallel to the rotation axis. Similarly we write $\mathcal{Z}_a = \tilde{\mathcal{Z}}_a + \tilde{\mathcal{Z}}n_a$, with $\tilde{\mathcal{Z}}_a = a\tilde{\mathrm{D}}_a\Theta$ and $\tilde{\mathcal{Z}} = an^a\mathrm{D}_a\Theta$, for the expansion gradients.

Starting from equations (30) and (31), using the barotropic fluid relation $\mathrm{D}_a p = c_{\rm s}^2\mathrm{D}_a\mu$, expressing $\mathrm{D}_a(\mathrm{D}^2 p)$ with respect to $\mathrm{D}^2(\mathrm{D}_a p)$ (by changing the order of the covariant derivatives), and then applying the decomposition introduced above, we obtain

$$\ddot{\tilde{\mathcal{D}}}_a = 2\left(1-2c_{\rm s}^2\right)\epsilon_{abc}\omega^b\dot{\tilde{\mathcal{D}}}^c + \tfrac{1}{2}\kappa\mu(1+w)\left(2+c_{\rm s}^2\right)\tilde{\mathcal{D}}_a + c_{\rm s}^2\mathrm{D}^2\tilde{\mathcal{D}}_a - 2(1+w)a\tilde{\mathrm{D}}_a\omega^2, \tag{47}$$

for density perturbations orthogonal to n_a and

$$\ddot{\tilde{\mathcal{D}}} = \tfrac{1}{2}\kappa\mu(1+w)\left(1+3c_s^2\right)\tilde{\mathcal{D}} + c_s^2\mathrm{D}^2\tilde{\mathcal{D}} - 2(1+w)an^a\mathrm{D}_a\omega^2, \tag{48}$$

for those along the rotation axis. This reveals a qualitatively different evolution for perturbations orthogonal and parallel to ω_a. The gravitational pull of the matter gets stronger along the rotation axis as the fluid sound speed increases above the $c_s^2 = 1/2$ threshold. The role of the background vorticity, on the other hand, is more pronounced orthogonal to the rotation axis and depends on the nature of the medium and on the vector product $\epsilon_{abc}\omega^b\dot{\tilde{\mathcal{D}}}^c$. Note that the effects of pressure gradients and of gradients in the centrifugal energy density, orthogonal and parallel to ω_a, are effectively identical.

One can obtain more quantitative results by looking at certain particular cases. For example, consider inhomogeneities parallel to the rotation axis and assume that the vorticity gradients do not change along this direction (i.e. set $n^a\mathrm{D}_a\omega^2 = 0$). Then, it is easy to show that $\tilde{\mathcal{D}}$ is unstable for dust and that there is an associated Jeans length when the fluid has nonzero pressure, as predicted in [37]. Alternatively, we may allow for $n^a\mathrm{D}_a\omega^2 \neq 0$ and assume isocurvature perturbations (i.e. set $a\mathrm{D}_a\omega^2 = \kappa\mu\mathcal{D}_a$). In that case, inhomogeneities parallel to the rotation axis are neutrally stable, they oscillate, for all types of matter with $w, c_s^2 \geqslant 0$. Thus, the evolution of density perturbations in the direction of the rotation axis is identical to that of average matter aggregations.

The supporting role of the gradient $\mathrm{D}_a\omega^2$ against inhomogeneities orthogonal to the rotation axis is also clear, at least for perturbations of the isocurvature type. In this direction the background vorticity has an additional effect conveyed by the first term in the right-hand side of equation (47). This is harder to quantify, however, as it depends in an intricate way on the nature of the fluid and on the orientation of the vector $\epsilon_{abc}\omega^b\dot{\tilde{\mathcal{D}}}^c$. Interestingly, the effect of this term is reversed as the sound speed of the medium crosses the $c_s^2 = 1/2$ threshold[7].

3.5. Vorticity perturbations

Rotation is controlled by a pair of nonlinear propagation and constraint equations given respectively by

$$\dot{\omega}_{\langle a\rangle} = -\tfrac{2}{3}\Theta\omega_a - \tfrac{1}{2}\mathrm{curl}\,\dot{u}_a + \sigma_{ab}\omega^b, \tag{49}$$

and

$$\mathrm{D}_a\omega^a = \dot{u}_a\omega^a. \tag{50}$$

The same expressions also hold when we linearise about the Gödel background, although then only the zero-order vorticity vector contributes to the right-hand side of (49) and (50). For a barotropic fluid the vorticity propagation formula takes the linearized form

$$\dot{\omega}_{\langle a\rangle} = -\tfrac{2}{3}\left(1-\tfrac{3}{2}c_s^2\right)\Theta\omega_a + \sigma_{ab}\omega^b. \tag{51}$$

When deriving the above we have used the barotropic expression $\dot{u}_a = -\left[c_s^2\big/(\mu+p)\right]\mathrm{D}_a\mu$ for the 4-acceleration, the conservation law (22), and taken into account that on a rotating background the projected gradients of scalars do not commute. All these guarantee that $\mathrm{curl}\,\dot{u}_a = -2c_s^2\Theta\omega_a$.

Part of the rotational behaviour seen in equation (51) is already familiar from studies of the perturbed FRW universes. In particular, we see that expansion leads to less rotation

[7] The projected vector $\tilde{\mathrm{D}}_a$ contains information about scalar-matter aggregations as well as for turbulence and distortions in the density distribution of the fluid. One should be able to isolate and extract this information by developing further the decomposition introduced via (45) and (46). This, however, goes beyond the scope of this paper.

when $c_s^2 < 2/3$, but increases it as $\omega_a \propto a^{3c_s^2-2}$ in models with a stiffer equation of state for the matter. In the case of contraction the situation is, obviously, reversed—at least until the nonlinear terms cease to be negligible. The background rotation of the Gödel universe, however, has introduced additional effects which propagate through the shear term in the right-hand side of (51). To get some idea of impact of the shear, we will assume that ω_a is a shear eigenvector (i.e. set $\sigma_{ab}\omega^b = \sigma\omega_a$, where σ is the associated eigenvalue). Then, equation (51) simplifies to

$$\dot{\omega}_{\langle a\rangle} = -\left[\tfrac{2}{3}\left(1 - \tfrac{3}{2}c_s^2\right)\Theta - \sigma\right]\omega_a. \tag{52}$$

Accordingly, when $\sigma > 0$ the extra shear coupling will increase the overall rotation of the model, but it will lead to a reduction otherwise.

To explain the shear effect, recall that positive σ means that there is an extra expansion along the rotation axis due to the shear effects. Given, the trace-free nature σ_{ab}, this means that there is an overall contraction orthogonal to ω_a, which explains why rotation increases when $\sigma > 0$. Clearly, the opposite happens when σ is negative.

3.6. Gravitational-wave perturbations

3.6.1. Evolution of the shear. In addition to the vorticity, the shear also affects the linear evolution of gravitational-wave perturbations. Here we will only present the relevant propagation and constraint equations, which will be used to analyse linear gravity waves within a perturbed Gödel universe in the following sections.

For perfect-fluid matter, the nonlinear shear evolution is governed by a set of one propagation equation and one constraint

$$\dot{\sigma}_{\langle ab\rangle} = -\tfrac{2}{3}\Theta\sigma_{ab} - E_{ab} + \mathrm{D}_{\langle a}\dot{u}_{b\rangle} - \sigma_{c\langle a}\sigma_{b\rangle}{}^{c} - \omega_{\langle a}\omega_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle}, \tag{53}$$

$$\mathrm{D}^b\sigma_{ab} = \tfrac{2}{3}\mathrm{D}_a\Theta + \mathrm{curl}\,\omega_a - 2\epsilon_{abc}\omega^b\dot{u}^c. \tag{54}$$

When linearized about the Gödel background, the constraint equation (54) remains the same, whereas the propagation equation (53) reduces to

$$\dot{\sigma}_{ab} = -E_{ab} + \mathrm{D}_{\langle a}\dot{u}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle}. \tag{55}$$

3.6.2. Evolution of the Weyl components. A covariant description of the gravitational waves is provided by the electric (E_{ab}) and the magnetic (H_{ab}) parts of the Weyl tensor. The two Weyl components support the different polarization states of propagating gravitational radiation and obey evolution and constraint equations that are remarkably similar to Maxwell's equations. For a perfect fluid, the nonlinear evolution of the two Weyl components is determined by a set of two propagation equations [54]

$$\dot{E}_{\langle ab\rangle} = -\Theta E_{ab} + \mathrm{curl}\,H_{ab} - \tfrac{1}{2}\kappa(\mu + p)\sigma_{ab} + 2\dot{u}^c\epsilon_{cd\langle a}H_{b\rangle}{}^{d} + 3\sigma_{c\langle a}E_{b\rangle}{}^{c} - \omega^c\epsilon_{cd\langle a}E_{b\rangle}{}^{d}, \tag{56}$$

$$\dot{H}_{\langle ab\rangle} = -\Theta H_{ab} - \mathrm{curl}\,E_{ab} + 3\sigma_{c\langle a}H_{b\rangle}{}^{c} - \omega^c\epsilon_{cd\langle a}H_{b\rangle}{}^{d} - 2\dot{u}^c\epsilon_{cd\langle a}E_{b\rangle}{}^{d}, \tag{57}$$

supplemented by the constraints

$$\mathrm{D}^b E_{ab} = \tfrac{1}{3}\kappa\mathrm{D}_a\mu + \epsilon_{abc}\sigma^b{}_d H^{cd} - 3H_{ab}\omega^b, \tag{58}$$

$$\mathrm{D}^b H_{ab} = \kappa(\mu + p)\omega_a - \epsilon_{abc}\sigma^b{}_d E^{cd} + 3E_{ab}\omega^b. \tag{59}$$

In addition, the magnetic Weyl tensor is related to the kinematic variables through the nonlinear constraint

$$H_{ab} = \mathrm{curl}\,\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle}, \tag{60}$$

where $\mathrm{curl}\,T_{ab} = \epsilon_{cd\langle a}\mathrm{D}^c T^d{}_{b\rangle}$ for any symmetric, orthogonally projected tensor T_{ab}.

Linearized about the unperturbed Gödel background, the propagation equations (56) and (57) become

$$\dot{E}_{\langle ab\rangle} = -\Theta E_{ab} + \mathrm{curl}\,H_{ab} - 3\omega^c\sigma_{c\langle a}\omega_{b\rangle} - \omega^c\epsilon_{cd\langle a}E_{b\rangle}{}^d, \tag{61}$$

$$\dot{H}_{ab} = -\mathrm{curl}\,E_{ab} - \omega^c\epsilon_{cd\langle a}H_{b\rangle}{}^d - 2\dot{u}^c\omega_{c\langle a}\omega_{b\rangle}, \tag{62}$$

on using the zero-order relation (4) to express the background E_{ab} tensor with respect to the vorticity vector. Similarly, the two constraints (58) and (59) reduce to

$$\mathrm{D}^b E_{ab} = \tfrac{1}{3}\kappa\mathrm{D}_a\mu - 3H_{ab}\omega^b, \tag{63}$$

$$\mathrm{D}^b H_{ab} = \kappa(\mu + p)\omega_a + \epsilon_{abc}\sigma^b{}_d\omega^c\omega^d + 3E_{ab}\omega^b, \tag{64}$$

while (60) does not change.

3.6.3. Isolating the gravitational-wave perturbations. Given the symmetric and trace-free nature of E_{ab} and H_{ab}, we can isolate the pure tensor modes, namely the gravitational waves, by demanding that E_{ab} and H_{ab} are also divergence free at the linear level. In practice, this means setting the right-hand side of (63) and (64) to zero. When there is no background vorticity, as in perturbed FRW models, we can isolate the gravity waves by assuming a barotropic fluid (i.e. setting $p = p(\mu)$) and by switching off both scalar and vector perturbations. This is done by introducing the linear constraints $\mathrm{D}_a\mu = 0 = \mathrm{D}_a\Theta = \omega_a$, which are consistent to first order [55–57]. When the background is rotating, however, the aforementioned set of constraints is not enough. Moreover, switching off the model's rotation is no longer an option. Here, we are dealing with the rotating Gödel background, and we will isolate the pure tensor modes by imposing the following self-consistent linear constraints:

$$\mathrm{D}_a\mu = 0 = \mathrm{D}_a\Theta, \qquad \mathrm{D}_a\omega^2 = 0 = \mathrm{curl}\,\omega_a, \tag{65}$$

$$H_{ab}\omega^b = 0 = \sigma_{ab}\omega^b, \qquad E_{ab}\omega^b = -\tfrac{1}{3}\kappa(\mu + p)\omega_a, \tag{66}$$

in addition to the barotropic fluid assumption. Clearly these constraints guarantee the transversality of E_{ab} and H_{ab}, as well as that of σ_{ab}, while they maintain a non-zero background rotation. Note that constraint (65a) is the standard restriction imposed on perturbed FRW cosmologies, while the rest are new. Of the three extra constraints, (65b) guarantees that any perturbations in the centrifugal energy that happen to be present are switched off, and that the vorticity vector is curl free. Constraints (66), on the other hand, imply that ω_a is an eigenvector of H_{ab}, σ_{ab} and E_{ab} to linear order. In the first two cases the corresponding eigenvalues are zero, whereas in the third the eigenvalue is $-\kappa(\mu + p)/3$. Finally, we point out that the first of the constraints (65a), together with the barotropic fluid assumption, guarantees that $\mathrm{D}_a p = 0$. This in turn ensures that $\dot{u}_a = 0$ through the momentum-density conservation. As a result, $\mathrm{D}_a\omega^a = 0$ (see equation (50)), which means that the vorticity vector is solenoidal.

We check the consistency of any set of constraints by propagating them in time. If every constraint is still satisfied, without the need of any extra restrictions, we say that the set is self-consistent. Here we can show that the set (65), (66) is self-consistent only when the matter source is pressure-free dust (i.e. for $p = 0$).[8] In particular, the consistency of (65a) follows directly from the linear propagation equations (30), (31) of the density and the expansion gradients, respectively. The consistency of the two constraints in (65b) is shown by propagating them in time and then by using the linear commutation laws between time derivatives and projected covariant derivatives (see [58] for a comprehensive list). Finally, taking the time derivatives of $H_{ab}\omega^b$, $\sigma_{ab}\omega^b$ and $E_{ab}\omega^b$ we can show that constraints (66) are also consistent to linear order. In doing so, one needs to employ the zero-order relations (4), (10) and use the covariant identities given in [58].

3.6.4. Linear evolution of gravitational waves. Once the constraints (65), (66) have been imposed and the pure tensor modes have been isolated, equations (61) and (62) reduce to

$$\dot{E}_{ab} = -\Theta E_{ab} + \operatorname{curl} H_{ab} - \omega^c \epsilon_{cd\langle a} E_{b\rangle}{}^d, \tag{67}$$

$$\dot{H}_{ab} = -\operatorname{curl} E_{ab} - \omega^c \epsilon_{cd\langle a} H_{b\rangle}{}^d, \tag{68}$$

respectively. In the same situation, the constraint (60) reads

$$H_{ab} = \operatorname{curl} \sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle}, \tag{69}$$

implying that $H_{ab} \neq \operatorname{curl} \sigma_{ab}$ in contrast to the perturbed FRW case. In the Gödel spacetime the magnetic component of the Weyl tensor vanishes identically (this is a feature that permits the existence of the close Newtonian analogue discussed above). Its electric counterpart, however, has a nonzero value that is expressed in terms of the model's vorticity (see equation (4)). This implies, according to [48], that only the gauge invariance of H_{ab} is guaranteed when linearizing about a Gödel background. Therefore, to avoid any gauge-related ambiguities, we will monitor the linear gravity-wave perturbations by looking exclusively into the evolution of H_{ab}.

To obtain the wave equation for H_{ab} we take the time derivative of equation (68) and then employ a lengthy, but relatively straightforward, calculation. In the process we apply the constraints (65), (66), use the zero-order relations (4) and (10), to express E_{ab} and $\kappa(\mu + p)$ with respect to the background vorticity vector, and employ the commutation laws and the covariant identities of [58]. At the end we arrive at

$$\ddot{H}_{ab} - \mathrm{D}^2 H_{ab} = 0, \tag{70}$$

which means that the linear-order wave equation of H_{ab} has no vorticity-related source terms, despite the fact that the background rotation has not been switched off. Following the above, we claim that the Gödel universe is neutrally stable against these linearized gravity-wave perturbations.

4. The case of homogeneous perturbations

We can also consider the stability of the Gödel universe against homogeneous perturbations, by ignoring spatial gradients such as $\mathrm{D}_a\mu$, $\mathrm{D}_a\Theta$, $\mathrm{D}_a\omega^2$, etc. We will do this by focusing on the Raychaudhuri equation, which takes the linearized form

$$\dot{\Theta} = -\tfrac{1}{2}(\mu + 3p) + \omega^2 + \Lambda. \tag{71}$$

[8] In principle, there might be an alternative set of linear constraints, which is less restrictive than (65), (66) and still isolates the pure tensor modes of a perturbed Gödel universe. It is also possible that a modified set of constraints could isolate the linear gravity-wave perturbations, even for matter with nonzero pressure. As yet, however, we have not been able to identify any such sets.

We can determine the effects of small homogeneous perturbations of the metric that alter the matter density and rotation by comparing the relative growth of the vorticity and matter energy density terms in (71). The conservation of angular momentum for rotational perturbations of a perfect fluid with equation of state

$$p = (\gamma - 1)\mu \equiv c_s^2 \mu$$

requires that the angular momentum of an eddy of mass M and comoving scale a be constant; thus

$$Ma^2\omega = \text{const.}$$

Since $\mu \propto a^{-3\gamma}$ and $M \propto \mu a^3$ we have $\omega \propto a^{3\gamma-5}$ (in agreement with equation (52) when σ is negligible) and so

$$\frac{\omega^2}{\mu} \propto a^{9\gamma-10}.$$

Thus, for changes that cause expansion ($a \to \infty$) the rotation dominates the effects of self-gravity whenever the fluid state satisfies $\gamma > 10/9$. This includes the case of radiation ($\gamma = 4/3$) but excludes that of dust ($\gamma = 1$). For perturbations that produce gravitational collapse ($a \to 0$) the opposite conclusion holds: rotation becomes negligible with respect to the self-gravity of the fluid whenever $\gamma > 10/9$ but is dominant whenever $\gamma < 10/9$. Superficially, this implies that rotation would halt gravitational collapse whenever $\gamma < 10/9$, but the self-gravitating effect of the rotation actually contributes to the collapse and a singularity will result so long as the matter stresses obey the strong energy condition, in accord with the singularity theorems [4]. The detailed behaviour of the rotational collapse depends on the nonlinear behaviour of the perturbations as $a \to 0$.

5. Scalar fields in the Gödel universe

Scalar field dominated models are key to contemporary cosmology, given their prominent role in inflationary scenarios for the very early universe and as candidates for the dark energy content of the universe today. It is interesting to examine whether the high symmetry of homogeneous spacetimes might make them more likely as initial states. If so, they may constraint the existence of scalar fields in some way. The symmetries of the Einstein static universe, for example, mean that any scalar field that happened to be present will have constant kinetic energy and potential [35]. Here we will look into the implications of the Gödel symmetries for such a scalar field

Consider a general spacetime filled with a minimally coupled scalar filed ϕ and assume that $\nabla_a\phi\nabla^a\phi < 0$ [59]. Then, ϕ has a perfect-fluid description with respect to a 4-velocity field u_a that is normal to the surfaces $\{\phi = \text{constant}\}$. More specifically, the scalar field behaves as a perfect fluid with

$$\mu_\phi = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) \qquad \text{and} \qquad p_\phi = \tfrac{1}{2}\dot{\phi}^2 - V(\phi), \tag{72}$$

relative to $u_a = -\nabla_a\phi/\dot{\phi}$ [59]. Note that $\dot{\phi} = -(\nabla_a\phi\nabla^a\phi)^{1/2}$ determines the kinetic energy of the scalar field and $V(\phi)$ is the associated potential.

The perfect-fluid description of ϕ is necessary if one wants to allow scalar fields in the Gödel universe. However, since the covariant derivatives of scalars commute, it becomes clear that the 4-velocity field u_a defined above is irrotational. In other words, the vorticity vector associated with u_a vanishes [59], which is in direct contradiction with the generic rotation of the Gödel spacetime. Putting it in a different way, one can say that the presence of the above u_a implies the existence of global spacelike hypersurfaces, something strictly forbidden in the

Gödel spacetime. On these grounds, one could argue that, at least globally, minimally coupled scalar fields are not compatible with the symmetries of the Gödel universe. More generally, these considerations may offer some insight into the absence of observational evidence for rotation in the universe today, to very high precision [60]. If the initial state of the universe is dominated by scalar fields, which appear to exist in profusion in string theories [61], then a zero-vorticity initial state will be enforced and provides a simple explanation for Mach's 'principle' even without inflation. Of course, any subsequent bout of inflation driven by scalar fields will reduce the effects of pre-existing vorticity to levels far below the threshold of detectability today and would be unable to generate new vorticity from the scalar fluctuations. As a result, the observation of any large-scale vorticity in the universe would be a decisive piece of evidence against an early inflationary state [62] and reveal specific information about the nature of matter at very high energies.

It should be emphasized that in this section we have considered an exact Gödel spacetime containing a single scalar field with a timelike gradient (i.e. $\nabla_a\phi\nabla^a\phi < 0$) and a corresponding Segrè type [1, (111)]. However, scalar fields of different nature are not *a priori* excluded. For example, a scalar field with zero gradient (i.e. $\nabla_a\phi = 0$), which corresponds to an effective cosmological constant and has a Segrè type [(1, 111)] is clearly compatible with the symmetries of the Gödel metric. For more details on the classification of scalar fields with respect to the nature of their gradients the reader is referred to [63]. We should also point out that scalar fields are allowed in rotating Gödel-type spacetimes as it has been shown in [44].

6. Discussion

Gödel's universe is one of the most intriguing solutions of the Einstein field equations. The common factor behind its unique and exotic features is the rigid rotation of the model, which is why the Gödel solution has been widely used to illustrate the possible general-relativistic effects of global vorticity and time travel. In this paper we have looked into the implications of the model's generic rotation for the stability of the Gödel universe under a variety of perturbations.

We first considered scalar-matter aggregations and looked into two different types of inhomogeneities, namely isocurvature and perturbations under rigid rotation. We found different patterns of stability determined by the presence or absence of gradients in the centrifugal energy. These act as effective pressure gradients balancing the gravitational pull of the matter fields. Interestingly, the latter is found to have a contribution from the rotational energy as well. This is a purely relativistic effect, as opposed to the supporting effect of the vortical energy gradients which is Newtonian in nature. When the gradients in the rotational energy are included, as occurs for the isocurvature perturbations, the model is found to be stable against density inhomogeneities even for pressure-free fluids. In their absence, however, stability is possible only if there are pressure gradients and then only on scales below an effective Jeans length. The latter is of the order of the largest causal region and smaller than its counterpart in almost-FRW cosmologies. This is what happens when dealing with perturbations under rigid rotation.

The equation of state of the cosmic medium is decisive for the evolution of homogenous perturbations. These were studied by comparing the relative growth between the vorticity and the matter terms in the Raychaudhuri equation. For perturbations causing expansion, we found that rotation dominates over self-gravity when the equation of state obeys $p > \mu/9$, which includes the cases of radiation and stiff matter. In the case of contraction the situation is reversed.

The linear rotational behaviour of the perturbed Gödel universe was found to follow the general pattern familiar from the almost-FRW studies, with some additional effects due to shape distortions. We examined these effects by aligning the vorticity vector along the shear eigenvectors. We found that, when the shape distortions lead to an extra contraction orthogonal to the rotation axis, the vorticity of the model increased, but it decreased otherwise.

The covariant analysis of linear gravitational waves on rotating backgrounds is complicated by the need to select and impose a set of constraints that isolates the pure tensor modes without switching off the vorticity. Here, we have introduced for the first time, a set of seven linear constraints that does this. We then proceeded to study the gravity-wave evolution by looking at the behaviour of the magnetic Weyl tensor. Our choice was based on the fact that this tensor has no Newtonian analogue and vanishes in the exact Gödel spacetime. The latter property guarantees that our analysis is free of any gauge-related ambiguities. The result was a simple plane-wave equation for H_{ab} without any vorticity-related source terms, which argues for the neutral stability of linear gravity-wave perturbations.

Finally, we have considered the implications of the Gödel symmetries for the presence of scalar field. We argue that the absence of any global spacelike hypersurfaces in the Gödel spacetime, namely the absence of any global irrotational vector field, forbids the introduction of scalar fields with timelike gradient vectors.

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