Quantizing Gravity: Insights from Quasinormal Modes of Black Holes

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Introduction

One of the most radical predictions of General Relativity is the existence of black holes. Black holes have been studied extensively, especially since the 1960's. A great deal has been learnt during the last decades, from the stability of black holes to black-hole thermodynamics and linear perturbation theory. Recently the study of the latter has excited great interest because it seems to reveal certain quantum properties of the black hole.

A perturbation of a black hole can be either electromagnetic or gravitational (or scalar in general). For example the field that accompanies a particle of a certain mass falling along a geodesic of the Schwarzschild geometry, produced by the black hole, can be considered as a perturbation on the background Schwarzschild geometry.

The mathematical treatment of the black hole perturbation theory was first originated by Regge and Wheeler and later was continued by Zerilli. In a first order of approximation, these perturbations are analyzed into normal modes of harmonic oscillations. It is surprising the fact that these oscillations are not carried only by the horizon of the black hole, but also by the spacetime outside the horizon. This is something to be expected: the horizon is just a noetic structure, black holes are actually intriguing shapes of spacetime.

The frequencies of those oscillations are called quasinormal, instead of normal, since they are complex numbers. The real part of such a mode represents the frequency of the oscillation, while the imaginary part represents the damping. They are of great importance because they are associated directly with specific properties of the black hole, such as the mass. That is, a black hole can oscillate at frequencies that are characteristic for the black hole itself.

A remarkable property of those modes, for the case of the Schwarzschild black hole, is that in the large damping limit (large n) the real part of their frequencies ω_R becomes a non-zero constant, with value that approximately equals the following quantity

$$T_H \ln 3.$$

Here $T_H = 1/8\pi M$ is the Hawking temperature of the black hole. This conjecture was made by Hod and proved analytically by Motl. Hod's conjecture was the reason for a new approach for calculating by Dreyer the Barbero-Immirzi parameter γ that appears as a proportionality constant in the black hole entropy calculations in the contexts of Loop Quantum Gravity.

Loop Quantum Gravity is an attempt to quantize the gravitational field itself starting from the classical field equations and in the past years it has become a strong candidate for a non-perturbative quantum theory of gravity. The Barbero-Immirzi parameter is an unknown constant and it parameterizes an ambiguity in the choice of canonically conjugate variables that are to be quantized. One way to fix this parameter is to use the result from the LQG for the black hole entropy and adjust it to the Bekenstein-Hawking result

$$S = A/4$$

where S is the entropy and A the area of the horizon.

The new approach, however, seems to be very attractive since it uses very little information about LQG. Hod considered a black hole as a quantum system and based on the limiting behavior of the quasinormal frequencies he assumed that the quantum of energy that a black hole can emit or absorb is $\Delta M = \hbar\omega_{asy}$. Using Bekenstein-Hawking formula for the entropy he found an equally spaced black hole area spectrum. This pioneer conjecture was later also used by Dreyer to fix the Barbero-Immirzi parameter.

In my thesis I present a qualitative and quantitative analysis of all the above stated topics. The analysis is presented as follows: The first chapter is devoted to the perturbations of the Schwarzschild black hole. There are two ways of approach to the study of these perturbations. One can study, directly, the perturbations in the metric coefficients via the Einstein equations linearized about the unperturbed space-time; or, one can study the perturbations in the Weyl scalars via the equations of the Newman-Penrose formalism. Although the latter way appears more suitable to the study of perturbations of the space-time around Schwarzschild black holes, in my thesis I follow the first avenue, since I find it easier to understand. The second chapter is dedicated in the determination of the quasinormal mode frequencies I present various techniques from the bibliography for calculating numerically the frequencies of these modes. For two of those techniques the determination of the quasinormal frequencies reduces to a simple eigenvalue problem with certain boundary conditions, such as in the case of the WKB semianalytic method or in the evaluation of a continue fraction such as in the case of Nollert's analytical approach. The asymptotic spectrum is also presented both for the large-n limit and for the large-l limit.

In the third, and last, chapter I present the application of quasinormal modes in quantum gravity. Before any reference to that topic, I find plausible to outline the basic concepts of modern black hole thermodynamics. Black hole thermodynamics constitutes a first substantial step to quantum gravity, by emerging the purely quantum properties of black holes. Notions from black hole thermodynamics, Loop Gravity attempts to assimilate and incorporate in its mathematical and physical formulation through Hod's conjecture and Dreyer's method for calculating the Barbero-Immirzi parameter. This is the last part of the analysis. Finally, I complete with some general remarks and conclusion, regarding the role of the asymptotic spectrum of other solutions for static spacetimes (Reissner-Nordstöm and Kerr black holes) in Hod's conjecture and the significance of the whole endeavor in the foundations of the final theory for quantum gravity.

Contents

Chapter 1

Perturbation theory of Schwarzschild Black Holes

1.1 Linear perturbations of Schwarzschild spacetime

The starting point of our analysis of black hole perturbations will be the unperturbed Schwarzschild line element

$$ds^2 = \overset{\circ}{g}_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (1.1)$$

where the $\overset{\circ}{g}$ denotes the unperturbed metric, $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$, r is the radius and θ, ϕ are the colatitude and azimuthal angle.This line element describes the spacetime outside a static and spherical symmetric black hole. We now introduce small perturbations $h_{\mu\nu}$ and the new perturbed metric will be regarded as a sum of the unperturbed background metric[1] $\overset{\circ}{g}_{\mu\nu}$ and the perturbation $h_{\mu\nu}$, which is called the *metric perturbation*,

$$\widetilde{g}_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1 \qquad (1.2)$$

[1]The background metric can be any solution of the field equations; it does not have to be the Minkowski metric.

The linearized Einstein equations are obtained by inserting (1.2) into the field equations is vacuum, $\delta\mathfrak{R}_{\mu\nu} = 0$ and expanding it to first order terms in $h_{\mu\nu}$. In our analysis for the Schwarzschild solution we consider the case where the black hole is subjected to a small initial perturbation, e.g. a test particle thrown towards the black hole, which is investigated up to terms of the first order in the departure from sphericity. In order to do this we calculate first the perturbed Cristoffel symbols $\widetilde{\Gamma}^{\kappa}_{\mu\nu}$. We use the background metric to rise and lower indices, i.e.,

$$h^{\mu\nu} = \mathring{g}^{\mu\alpha}\,\mathring{g}^{\kappa\nu} h_{\alpha\kappa} \tag{1.3}$$

The perturbation of $\mathring{g}_{\mu\nu}$ with rased indices is written as follows:

$$\widetilde{g}^{\mu\nu} = \mathring{g}^{\mu\nu} - h^{\mu\nu} + O(h^2). \tag{1.4}$$

The perturbed Cristoffel symbols are given by

$$\begin{aligned}\widetilde{\Gamma}^{\kappa}_{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\kappa\alpha}(\widetilde{g}_{\alpha\nu,\mu} + \widetilde{g}_{\alpha\mu,\nu} - \widetilde{g}_{\nu\mu,\alpha}) \\ &= \frac{1}{2}(\mathring{g}^{\kappa\alpha} - h^{\kappa\alpha})(\widetilde{g}_{\alpha\nu,\mu} + \widetilde{g}_{\alpha\mu,\nu} - \widetilde{g}_{\nu\mu,\alpha}) \\ &= \frac{1}{2}\,\mathring{g}^{\kappa\alpha}(\widetilde{g}_{\alpha\nu,\mu} + \widetilde{g}_{\alpha\mu,\nu} - \widetilde{g}_{\nu\mu,\alpha}) - \frac{1}{2}h^{\kappa\alpha}(\widetilde{g}_{\alpha\nu,\mu} + \widetilde{g}_{\alpha\mu,\nu} - \widetilde{g}_{\nu\mu,\alpha})\end{aligned} \tag{1.5}$$

By replacing the expressions for the first-order derivatives of $g_{\mu\nu}$ and keeping terms linear in h, we find

$$\begin{aligned}\widetilde{\Gamma}^{\kappa}_{\mu\nu} = &\frac{1}{2}\,\mathring{g}^{\kappa\alpha}(\mathring{g}_{\alpha\nu,\mu} + \mathring{g}_{\alpha\mu,\nu} - \mathring{g}_{\nu\mu,\alpha}) + \frac{1}{2}\,\mathring{g}^{\kappa\alpha}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\nu\mu,\alpha}) \\ &- \frac{1}{2}h^{\kappa\alpha}(\mathring{g}_{\alpha\nu,\mu} + \mathring{g}_{\alpha\mu,\nu} - \mathring{g}_{\nu\mu,\alpha}) - \frac{1}{2}h^{\kappa\alpha}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\nu\mu,\alpha})\end{aligned} \tag{1.6}$$

The last term of (1.6) is consisted of second order terms of h, so we can omit it. We also observe that the term $h^{\kappa\alpha}$ can be rewritten as $h^{\kappa\alpha} = \mathring{g}^{\alpha\beta}\,\mathring{g}^{\kappa\lambda} h_{\beta\lambda}$. That is,

$$\frac{1}{2}\,\mathring{g}^{\alpha\beta}\,\mathring{g}^{\kappa\lambda} h_{\beta\lambda}(\mathring{g}_{\alpha\nu,\mu} + \mathring{g}_{\alpha\mu,\nu} - \mathring{g}_{\nu\mu,\alpha}) = \mathring{g}^{\kappa\lambda} h_{\beta\lambda}\,\mathring{\Gamma}^{\beta}_{\mu\nu} = \mathring{g}^{\kappa\alpha} h_{\beta\alpha}\,\mathring{\Gamma}^{\beta}_{\mu\nu}$$

We finally obtain the following compact form of the perturbed Cristoffel symbols,

$$\widetilde{\Gamma}^{\kappa}_{\mu\nu} = \mathring{\Gamma}^{\kappa}_{\mu\nu} + \delta\Gamma^{\kappa}_{\mu\nu} \tag{1.7}$$

where the term $\delta\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}\,\mathring{g}^{\kappa\alpha}(h_{\alpha\nu,\mu} + h_{\alpha\mu,\nu} - h_{\nu\mu,\alpha}) - \mathring{g}^{\kappa\alpha} h_{\beta\alpha}\,\mathring{\Gamma}^{\beta}_{\mu\nu}$ represents the variation of the Cristoffel symbols. It can be shown that the variation of the Christoffel symbols equals with the following quantity

$$\delta\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}\,\mathring{g}^{\kappa\alpha}(h_{\alpha\nu;\mu} + h_{\alpha\mu;\nu} - h_{\nu\mu;\alpha}) \tag{1.8}$$

which forms a tensor, although the unperturbed Christoffel symbols do not.

It is not difficult now to derive the expression for the Ricci tensor that corresponds to the perturbed metric $\overset{\circ}{g}_{\mu\nu} + h_{\mu\nu}$. By simply substituting (1.7) into the definition of $\mathfrak{R}_{\mu\nu}$, we get

$$\begin{aligned}
\widetilde{\mathfrak{R}}_{\mu\nu} &= \widetilde{\Gamma}^{\alpha}_{\mu\nu,\alpha} - \widetilde{\Gamma}^{\alpha}_{\mu\alpha,\nu} + \widetilde{\Gamma}^{\alpha}_{\beta\alpha}\widetilde{\Gamma}^{\beta}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\beta\nu}\widetilde{\Gamma}^{\beta}_{\mu\alpha} \\
&= \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu,\alpha} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\alpha,\nu} + \overset{\circ}{\Gamma}{}^{\alpha}_{\beta\alpha}\overset{\circ}{\Gamma}{}^{\beta}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\beta\nu}\overset{\circ}{\Gamma}{}^{\beta}_{\mu\alpha} + \delta\Gamma^{\alpha}_{\mu\alpha,\nu} - \delta\Gamma^{\alpha}_{\mu\nu,\alpha} + \delta\Gamma^{\beta}_{\mu\alpha}\overset{\circ}{\Gamma}{}^{\alpha}_{\nu\beta} \\
&+ \overset{\circ}{\Gamma}{}^{\beta}_{\mu\alpha}\delta\Gamma^{\alpha}_{\nu\beta} - \delta\Gamma^{\beta}_{\mu\nu}\overset{\circ}{\Gamma}{}^{\alpha}_{\alpha\beta} - \overset{\circ}{\Gamma}{}^{\beta}_{\mu\nu}\delta\Gamma^{\alpha}_{\alpha\beta} \\
&= \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\nu,\alpha} - \overset{\circ}{\Gamma}{}^{\alpha}_{\mu\alpha,\nu} + \overset{\circ}{\Gamma}{}^{\alpha}_{\beta\alpha}\overset{\circ}{\Gamma}{}^{\beta}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}_{\beta\nu}\overset{\circ}{\Gamma}{}^{\beta}_{\mu\alpha} + \delta\Gamma^{\alpha}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}_{\mu\nu;\alpha} \Rightarrow \\
\widetilde{\mathfrak{R}}_{\mu\nu} &= \overset{\circ}{\mathfrak{R}}_{\mu\nu} + \delta\mathfrak{R}_{\mu\nu}.
\end{aligned} \tag{1.9}$$

where $\delta\mathfrak{R}_{\mu\nu} = \delta\Gamma^{\alpha}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}_{\mu\nu;\alpha}$. Regard now our background to be the simply vacuum, and the equations (1.9) take the form

$$\delta\mathfrak{R}_{\mu\nu} = 0, \tag{1.10}$$

or as a function of the covariant derivatives of the Christoffel symbols

$$\delta\Gamma^{\alpha}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}_{\mu\nu;\alpha} = 0. \tag{1.11}$$

The physical meaning of the equations $\delta\mathfrak{R}_{\mu\nu} = 0$ is that the perturbed space is also empty of matter or energy. Indeed these equations form a system of ten uncoupled differential equations linear in the perturbation h. As we shall see later this system can be decoupled into a system of equations that describe separately the dependence on the angular coordinates θ and ϕ and the dependence on the radial coordinate r and the time t, a mathematical treatment that is usual in systems with spherical symmetry. The system of equations must decouple in the sense that they can be considered independently of each other. We shall find that this is indeed the case.

1.2 Analysis into spherical harmonics

An important constraint is possed by Birkoff's theorem: *Let the geometry of a given region of spacetime be spherically symmetric, and be a solution to the Einstein field equations in vacuum. Then that geometry is necessarily a piece of the Schwarzschild geometry.* The external field of any electrically neutral, spherical star satisfies the conditions of Birkhoff's theorem, whether the star is static, oscillating, or collapsing. We consider an equilibrium configuration that is unstable under gravitation collapse, such as a collapsing star, and has the Schwarzschild geometry as its external gravitational field. We perturb this configuration in a spherical symmetric way, so that it begins to collapse radially. The perturbation and subsequent collapse cannot effect the external gravitational field so long as spherical symmetry is maintained. There is no possible way for any gravitational influence of the radial collapse to propagate outward. Thus, spherically symmetric black holes can only be perturbed by nonradial perturbations and this forces us to study separately the angular part of the perturbation equations after decoupling them into a product of four separate parts each being a function of one coordinate (t, r, θ and ϕ) only. This approach is very helpful since it simplifies also the mathematical treatment a lot.

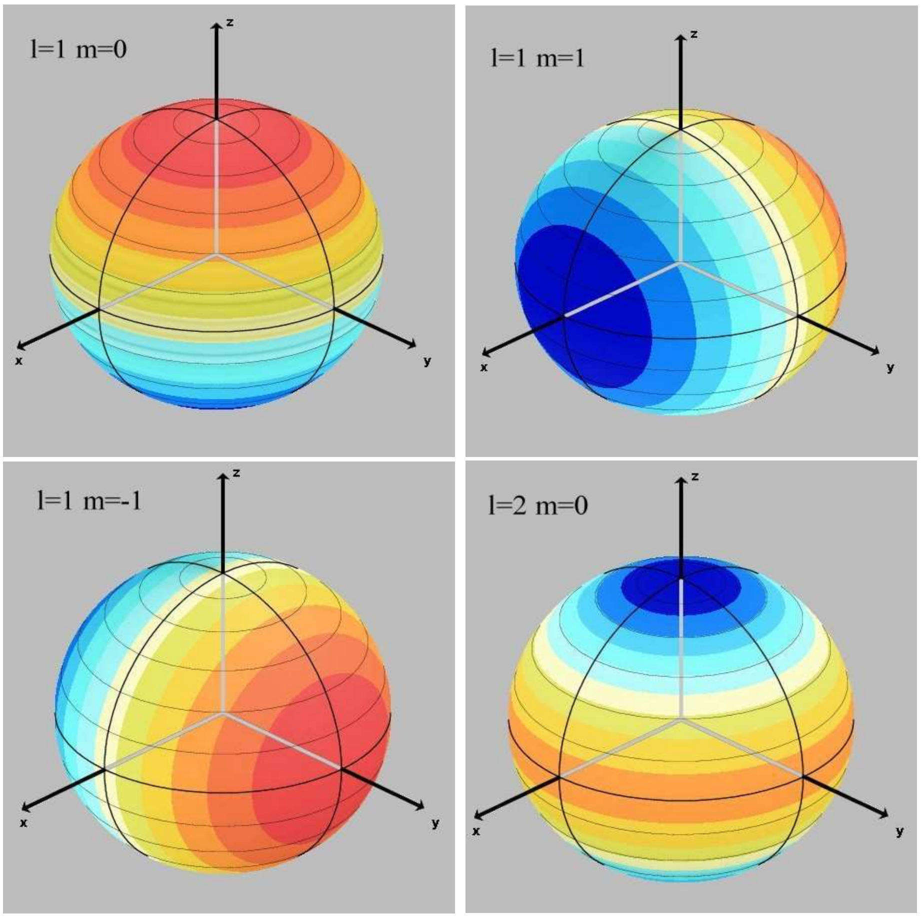
The best way to achieve this decomposition is to expand the angular part of the perturbing tensor $h_{\mu\nu}(t, r, \theta, \phi)$ into tensor spherical harmonics, having always in mind that the problem is unchanged under rotations around the origin. The functions describing the radial dependence will, of course, be unaffected by this decomposition in the same way that the radial wavefunction of an electron in a central field is independent of the quantum number m and orbital angular momentum l. The general expression that corresponds to the expansion of a second rank symmetric tensor $h_{\mu\nu}$ is the following

$$h_{\mu\nu}(t, r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{10} c^n(t, r)(Y^n_{lm})_{\mu\nu}(\theta, \phi)$$

In our case the ten independent components of $h_{\mu\nu}$ do not transform all in the

Nonradial oscillations

The simplest oscillation a star can undergo is a *radial* one. In that case, the star expands and contracts radially and spherically symmetry is preserved during the oscillation cycle. From a mathematical point of view,the differential equation describing the radial displacement is of the Sturm-Liouville type and thus allows eigensolutions that correspont to an infinitely countable amount of eigenfrequencies. The smallest frequency corresponds to the fundamental radial radial oscillation mode. The period of this mode is inversely proportional to the square root of the mean density of the star. Radial oscillations are characterized by the radial wavenumber n: the number of nodes of the eigenfunctions between the center and the surface of the star. Well known radial oscillators are the Cepheids, RR Lyrae stars and Red Giants.
If transverse motions occur in addition to radial motions, one uses the term *non-radial oscillation*. The oscillation modes are then not only characterized by a radial wavenumber n, but also by non-radial wavenumbers l and m. The latter numbers correspond to the degree and the azimuthal number of the spherical harmonic $Y_{lm}(\theta, \phi)$ that represents the dependance of the modes on the angular variables θ and ϕ for a star with a spherically symmetric equilibrium configuration. The degree l represents the number of surface nodal lines, while the azimuthal number m denotes the number of such lines that pass through the rotation axis of the star. The surface pattern of some nonradial oscillations is graphically depicted in Figure (1.1).


l=1 m=0
l=1 m=1
l=1 m=-1
l=2 m=0
x
y
z

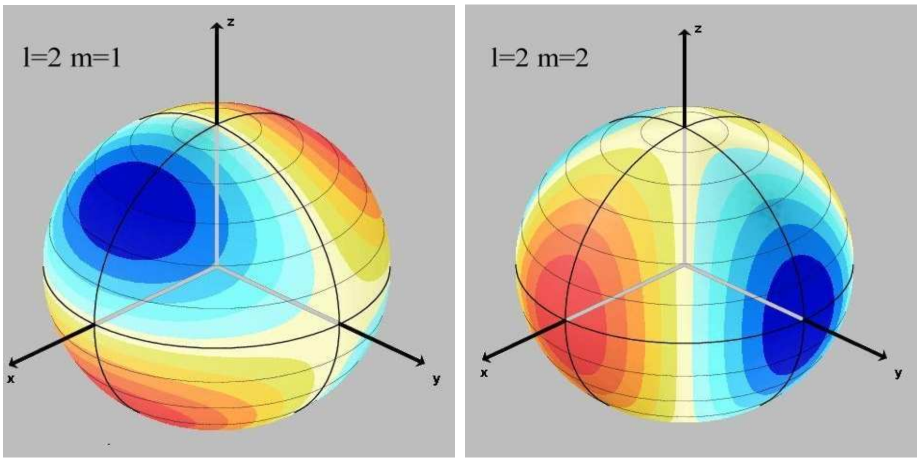


Figure 1.1: *Different examples of non-radial oscillations. The velocity field of a non-radial oscillator is represented by a spherical harmonic* Y_{lm}. *The meaning of the spherical wavenumbers* (l, m) *is visualized. The z axis indicates the symmetry axis of the oscillation, which corresponds to the rotation axis of the star. The coloring denotes the Doppler shift in an observed spectrum due to the oscillation, i.e., at this particular instance, the red parts are moving towards the stellar center (thus away from the observer) and therefore shift the spectrum to longer wavelengths (redshift) while the blue parts are moving outwards (towards the observer) and result in a shift to shorter wavelengths (blueshift).*

same way, when we study rotations on the 2-sphere[2]. That is, the components h_{00}, h_{01}, h_{11} transform like scalars, the components h_{02}, h_{03}, h_{12}, h_{13} transform like 2 vectors and finally the components h_{33}, h_{34}, h_{44} like the components of a 2×2 tensor,

$$h_{\mu\nu} = \begin{pmatrix} \boxed{S} & \boxed{S} & \boxed{V} \\ \boxed{S} & \boxed{S} & \boxed{V} \\ \boxed{V} & \boxed{V} & \boxed{T} \end{pmatrix}$$

Since not all the components of $h_{\mu\nu}$ transform like scalars, we need to introduce the generalized spherical harmonics for scalars, vectors and tensors (see Figure 1.1 and 1.2 for a schematic representation of the scalar and vector harmonics). The tensorial nature of the spherical harmonics refers to the unit 2-sphere. The analysis procedure is well treated in [1] and we shall follow from now own the same path.

[2]The angular momentum that is defined here by the index l is investigated by studying rotations on a 2-dimensional manifold ($t = constant, r = constant$).

The procedure of constructing vectors and tensors from scalar quantities is trivial. According to the expansion theorem any scalar function of t, r, θ and ϕ may be expanded in the form

$$S(t, r, \theta, \phi) = \sum_{l}^{\infty} \sum_{m=-l}^{l} a_{lm}(t, r) Y_l^m(\theta, \phi) \tag{1.12}$$

The spherical harmonics are defined as

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} (-1)^m e^{im\phi} P_{lm}(cos\theta)$$

where $P_{lm}(\cos\theta)$ are the associated Legendre functions of degree l and order m. Based on that form we can construct the two types of vector components of $h_{\mu\nu}$ as follows:

$$(V_{lm}^1)_\mu = (S_{lm})_{;\mu} = const. \frac{\partial}{\partial x^\mu} Y_{lm} \tag{1.13}$$

$$(V_{lm}^2)_\mu = \epsilon_\mu^\nu (S_l^m)_{;\mu} = const. \gamma^{\nu\alpha} \epsilon_{\mu\alpha} \frac{\partial}{\partial x^\nu} Y_{lm} \tag{1.14}$$

while the three types of tensor are the following:

$$(T_{lm}^1)_{\mu\nu} = const. Y_l^m{}_{;\mu\nu} \tag{1.15}$$

$$(T_{lm}^2)_{\mu\nu} = const. \gamma_{\mu\nu} Y_{lm} \tag{1.16}$$

$$(T_{lm}^3)_{\mu\nu} = const. \frac{1}{2} [\epsilon_\mu^\lambda (T_{lm}^1)_{\lambda\nu} + \epsilon_\nu^\lambda (T_{lm}^1)_{\lambda\mu}] \tag{1.17}$$

where the indices α, μ and ν run from 2 to 3. We use the metric tensor on the unit 2-sphere

$$\gamma_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & sin^2\theta \end{pmatrix}$$

and the totally antisymmetric tensor ϵ

$$\epsilon_{\mu\nu} \begin{pmatrix} 0 & -sin\theta \\ sin\theta & 0 \end{pmatrix}$$

to raise and lower indices. All tensorial operations (including covariant differentiation) are referred to this metric. Furthermore all the above quantities carry an angular momentum l with a projection on the z axis equal to m. However, they are not characterized by the same parity. When we study inversion in space, in order to formalize a mathematical description of them, we introduce the *parity operator* $\mathbf{P}$ whose rule of operator is to reflect $x \rightarrow -x$. Thus, for a function $\psi(x)$, the action of that operator gives

$$\mathbf{P}\psi(x) = \psi(-x)$$

The effect of that operation depends on the type of the function. For an even function we have

$$\mathbf{P}\psi(x) = \psi(x)$$

and for an odd function

$$\mathbf{P}\psi(x) = -\psi(x)$$

An attempt to generalize the act of the parity operator in relativistic perturbation theory leads to the conclusion that vector and tensor spherical harmonics are classified into two different types as well. For example, in the case of a 2×2 tensor spherical harmonic we can consider an operator that produces the following parity transformation

$$\mathbf{P} \ \ [Y_{lm}(\theta,\phi)]_{\mu\nu} \rightarrow [\widetilde{Y}_{lm}(\pi-\theta,\pi+\phi)]_{\mu\nu}$$

Under this action, i.e., inversion on a 2-sphere, the behavior of a tensor spherical harmonic comes in two distinct types: the even parity harmonic for which $\widetilde{Y}_{\mu\nu} = (-1)^l Y_{\mu\nu}$ and the odd parity harmonic for which $\widetilde{Y}_{\mu\nu} = (-1)^{l+1} Y_{\mu\nu}$[3]. The classification is exactly the same for the vector harmonics and it is reflected also on the metric perturbations (even and odd parity perturbations).

In practice, from the group of ten independent harmonics $\mathbf{S}_l^m$, $(\mathbf{V}_l^m)^1$, $(\mathbf{T}_l^m)^1$, and $(\mathbf{T}_l^m)^2$ are multiplied by the factor $(-1)^l$ under parity transformations, while $(\mathbf{V}_l^m)^2$ and $(\mathbf{T}_l^m)^3$ are multiplied by the factor $(-1)^{l+1}$. This result is based on the fact that the gradient of a scalar preserves parity, while the multiplication with the totally antisymmetric tensor ϵ inverts it. Thus, we have finally the following set of scalar, vector and tensor spherical harmonics:

- **Scalars**

$$S_{lm}^{1,2,3} = Y_{lm}, \qquad \textit{parity} \ \ (-1)^l \tag{1.18}$$

- **Vectors**

$$V_{lm}^1 = [\partial_\theta Y_{lm}, \ \ \partial_\phi Y_{lm}], \qquad \textit{parity} \quad (-1)^l \tag{1.19}$$

$$V_{lm}^2 = \left[\frac{1}{\sin\theta}\partial_\theta Y_{lm}, \ \ \sin\theta\partial_\phi Y_{lm}\right], \qquad \textit{parity} \quad (-1)^{l+1} \tag{1.20}$$

[3] According to the quantum mechanical formalism, the spherical harmonics are normalized eigenfunctions of angular momentum. Since angular momentum commutes with parity, the spherical harmonics are eigenfunctions of parity as well. Furthermore, evenness and oddness are properties that are time independent, which means that they can be considered as constants of the motion.

Terminology	First three	Last seven
Parity	$(-1)^{l+1}$	$(-1)^l$
Number of such tensorial harmonics	3	7
Regge and Wheeler [1]	Odd parity	Even parity
Thorne and Campolattaro	Odd or Magnetic	Even or Electric
Zerilli [2]	Electric	Magnetic
Chandrasekhar [3]	Axial	Polar

Table 1.1: Correlation between the terminology used in various papers (borrowed from Zerilli, [2]).

- **Tensors**

$$T^1_{lm} = \begin{bmatrix} \partial^2_{\theta\theta} & \frac{1}{2}X^m_l \\ * & (\partial^2_{\phi\phi} - \cos\theta\sin\theta\partial_\theta) \end{bmatrix} Y_{lm}, \quad parity \quad (-1)^l \quad (1.21)$$

$$T^2_{lm} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} Y_{lm}, \quad parity \quad (-1)^l \quad (1.22)$$

$$T^3_{lm} = \begin{bmatrix} -\frac{1}{2}\frac{1}{\sin\theta}X_{lm} & \frac{1}{2}\sin\theta W_{lm} \\ * & (\sin\theta\partial^2_{\phi\phi} - \cos\theta) \end{bmatrix} Y_{lm}, \quad parity \quad (-1)^{l+1} (1.23)$$

where $W_{lm} = (\partial^2_{\theta\theta} - \cot\theta\partial_\theta - 1/\sin^2\theta\partial^2_{\phi\phi}) = [l(l+1) + 2\partial^2_{\theta\theta}]$ and $X_{lm} = 2(\partial^2_{\theta\phi} - \cot\theta\partial_\phi)$.

The set of ten spherical harmonics that we have defined so far are not normalized; the normalization is not necessary since the constants in equations (1.12)-(1.17) disappear during the mathematical analysis. However, we list the complete set of normalized tensor spherical harmonics in Appendix A. Finally, it is worth mentioning the terminology that is used to describe these two families of harmonics (see Table 1.1). Zerilli [2] uses the notation electric and magnetic perturbations type, while Chandrasekhar [3] uses the terms axial and polar. Chandrasekhar's terminology is justified by the fact that the former induce a dragging of the inertial frame and induce a rotation to the black hole while the latter impart no such rotation. In this essay, from now own we shall use Chandrasekhar's terminology to describe the metric perturbations.

So far we have constructed the generalized form of spherical harmonics for scalars, vectors and 2×2 tensors and studied the behavior of those under parity transformation, namely the division into axial and polar harmonics. This division into two subsets is, as we earlier mentioned, reflected also on the metric perturbations. Due to the spherical symmetry of the background the equations for polar and axial perturbations are totally decoupled from each other. Hence, the metric perturbation can be thereby written as follows

$$h_{\mu\nu} = h^{axial}_{\mu\nu} + h^{polar}_{\mu\nu}$$

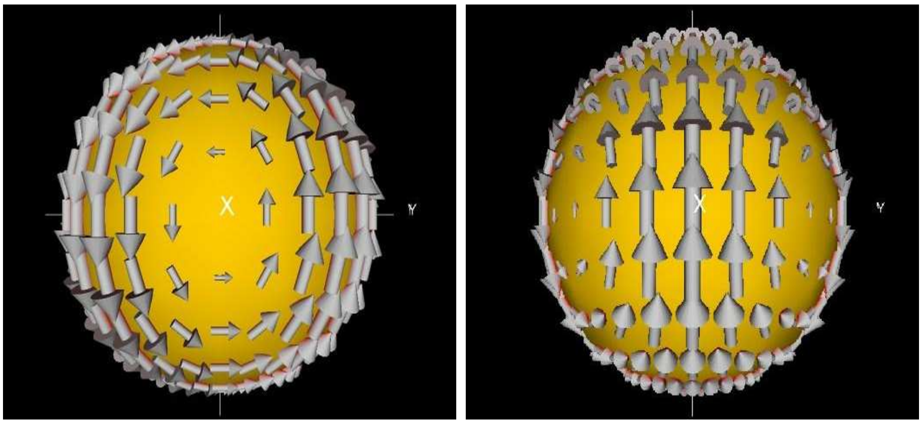


Figure 1.2: *The Vector Spherical Harmonics are angular vectors with Real and Imaginary part. Here we can see a schematic representation for the* $l = 2$, $m = 2$ *vectorial harmonic field.*

We now list below separately the straightforward expressions for the components of those parts analytically (in each case summation over l and m is implied):

- **Axial perturbations**

$$\mathrm{h}_{\mu\nu}^{\mathrm{axial}} : \; h_{tt} = h_{tr} = h_{rr} = 0 \tag{1.24}$$

$$h_{t\theta} = -h_{0a}(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{lm} \tag{1.25}$$

$$h_{t\phi} = h_{0a}(t,r)\sin\theta\partial_\theta Y_{lm} \tag{1.26}$$

$$h_{r\theta} = -h_{1a}(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{lm} \tag{1.27}$$

$$h_{r\phi} = h_{1a}(t,r)(\sin\theta\partial_\theta Y_{lm} \tag{1.28}$$

$$h_{\theta\theta} = h_{2a}(t,r)\frac{1}{\sin\theta}(\partial^2_{\phi\theta} - \frac{\cos\theta}{sin^2\theta}\partial_\phi)Y_{lm} \tag{1.29}$$

$$h_{\theta\phi} = \frac{1}{2}h_{2a}(t,r)\frac{1}{\sin\theta}(\partial^2_{\phi\phi} + \cos\theta\partial_\theta - \sin\theta\partial^2_{\theta\theta})Y_{lm} \tag{1.30}$$

$$h_{\theta\theta} = -h_{2a}(t,r)(\partial^2_{\theta\phi} - \cos\theta\partial_\phi)Y_{lm} \tag{1.31}$$

where the missing components are fixed by the symmetry $h_{\mu\nu} = h_{\nu\mu}$, i.e.,

$$h_{t\theta} = h_{\theta t}, \;\; h_{\theta r} = h_{r\theta}, \;\; h_{t\phi} = h_{\phi t}, \;\; h_{\phi r} = h_{r\phi}, \;\; h_{\phi\theta} = h_{\theta\phi}$$

- **Polar perturbations:**

$$\mathrm{h}^{\mathrm{polar}}_{\mu\nu} : h_{tt} = \left(1 - \frac{2M}{r}\right) H_0(t,r) Y_{lm} \tag{1.32}$$

$$h_{tr} = H_1(t,r) Y_{lm} \tag{1.33}$$

$$h_{t\theta} = h_{\theta t} = h_{0p}(t,r) \partial_\theta Y_{lm} \tag{1.34}$$

$$h_{t\phi} = h_{0p}(t,r) \partial_\phi Y_{lm} \tag{1.35}$$

$$h_{rt} = H_1(t,r) Y_{lm} \tag{1.36}$$

$$h_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} H_2(t,r) Y_{lm} \tag{1.37}$$

$$h_{r\theta} = h_{\theta r} = h_{1p}(t,r) \partial_\theta Y_{lm} \tag{1.38}$$

$$h_{r\phi} = h_{1p}(t,r) \partial_\phi Y_{lm} \tag{1.39}$$

$$h_{\theta\theta} = r^2 [K(t,r) + G(t,r) \partial^2_{\theta\theta}] Y_{lm} \tag{1.40}$$

$$h_{\theta\phi} = h_{\phi\theta} = r^2 G(t,r) (\partial^2_{\theta\phi} - \tan\theta \partial_\phi) Y_{lm} \tag{1.41}$$

$$h_{\phi\phi} = r^2 [K(t,r) \sin^2\theta + G(t,r)(\partial^2_{\phi\phi} + \sin\theta\cos\theta\partial_\theta)] Y_{lm} \tag{1.42}$$

The unknown coefficient functions h_{0a}, h_{1a}, h_{2a}, H_0, H_1, H_2, h_{0p}, h_{1p}, G and K that we have introduced in the above listed components are not independent between each other and are different for different values of l and m (we have omitted the indices l and m). Some of them we shall have the freedom to annul after the choice of a suitable gauge transformation, something that it will help us to simplify the rather lengthy expressions of the components.

1.3 Proper gauge transformation

In the preceding sections no restrictions were made concerning the choice of coordinate frame. The choice of coordinates in our mathematical analysis and interpretation of physical phenomena is arbitrary. Every choice is equivalent to any other choice (provided the mapping is one-to-one and differentiable), since complete invariance is ensured. By careful choice of coordinates the solutions of the perturbation equations are simplified sufficiently. The ten equations $\delta\mathfrak{R}_{\mu\nu} = 0$ cannot determine $h_{\mu\nu}$ uniquely because their values could be changed by changing the coordinate system, without changing the geometry. In (1.2) we assumed a coordinate system in which the metric takes the form of Schwarzschild spacetime (or any solution of the field equations) and $h_{\mu\nu}$ is an unknown perturbation of it. However, that assumption does not uniquely fix the coordinates.

Let us consider an infinitesimal coordinate transformation of the form:

$$x'^{\alpha} = x^{\alpha} - \xi^{\alpha}(x)$$

where the displacements ξ^{α} transform like a vector, whose components are functions of position. We have demanded ξ^{α} to be infinitesimal, meaning that $\xi^{\alpha} \ll x^{\alpha}$ (or if we operate with a gradient, $|\xi^{\alpha}_{;\beta}| \ll 1$). Such a transformation, as we will thereupon see, preserves the form of (1.2) but change the functional form of the $h_{\mu\nu}$.

In the new coordinate system we shall have

$$\widetilde{g}'_{\mu\nu}(x') = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\widetilde{g}_{\kappa\lambda}(x - \xi(x)) \tag{1.43}$$

According to the transformation that we introduced and considering the fact that $\xi^{\alpha} \ll x^{\alpha}$, we take for the partial derivative,

$$\frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \frac{\partial x'^{\alpha}}{\partial x'^{\beta}} - \frac{\partial \xi^{\beta}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \frac{\partial \xi^{\alpha}}{\partial x'^{\beta}} \simeq \delta^{\alpha}_{\beta} - \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} \tag{1.44}$$

By substituting the form of the partial derivative in (1.43) we take

$$\begin{aligned}
\widetilde{g}'_{\mu\nu} &= \left(\delta^{\kappa}_{\mu} - \frac{\partial \xi^{\kappa}}{\partial x^{\mu}}\right)\left(\delta^{\lambda}_{\nu} - \frac{\partial \xi^{\lambda}}{\partial x^{\nu}}\right)\widetilde{g}_{\kappa\lambda}(x - \xi(x)) \\
&= \left(\delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu} - \delta^{\kappa}_{\mu}\frac{\partial \xi^{\lambda}}{\partial x^{\nu}} - \delta^{\lambda}_{\nu}\frac{\partial \xi^{\kappa}}{\partial x^{\mu}} + \frac{\partial \xi^{\kappa}}{\partial x^{\mu}}\frac{\partial \xi^{\lambda}}{\partial x^{\nu}}\right)\underbrace{\left(\widetilde{g}(x)_{\kappa\lambda} - \xi^{\alpha}\frac{\partial \widetilde{g}_{\kappa\lambda}}{\partial x^{\alpha}} + \dots\right)}_{taylor\ expansion} \\
&= \delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu}\widetilde{g}_{\kappa\lambda} - \delta^{\kappa}_{\mu}\widetilde{g}_{\kappa\lambda}\frac{\partial \xi^{\lambda}}{\partial x^{\nu}} - \delta^{\lambda}_{\nu}\widetilde{g}_{\kappa\lambda}\frac{\partial \xi^{\kappa}}{\partial x^{\mu}} - \delta^{\kappa}_{\mu}\delta^{\lambda}_{\nu}\xi^{\alpha}\frac{\partial \widetilde{g}_{\kappa\lambda}}{\partial x^{\alpha}} + \mathcal{O}(|\xi|^2) \\
&\simeq \widetilde{g}_{\mu\nu} - \underbrace{\widetilde{g}_{\mu\lambda}\frac{\partial \xi^{\lambda}}{\partial x^{\nu}}}_{\lambda\to\kappa} - \widetilde{g}_{\kappa\nu}\frac{\partial \xi^{\kappa}}{\partial x^{\mu}} - \underbrace{\xi^{\alpha}\frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^{\alpha}}}_{\alpha\to\kappa}
\end{aligned}$$

$$\begin{aligned}
&= \mathring{g}_{\mu\nu} + h_{\mu\nu} - \widetilde{g}_{\mu\kappa}\frac{\partial \xi^{\kappa}}{\partial x^{\nu}} - \widetilde{g}_{\kappa\nu}\frac{\partial \xi^{\kappa}}{\partial x^{\mu}} - \xi^{\kappa}\frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^{\kappa}} \\
&= \mathring{g}_{\mu\nu} + h_{\mu\nu} - \frac{\partial}{\partial x^{\nu}}(\widetilde{g}_{\mu\kappa}\xi^{\kappa}) + \xi^{\kappa}\frac{\partial \widetilde{g}_{\mu\kappa}}{\partial x^{\nu}} - \frac{\partial}{\partial x^{\mu}}(\widetilde{g}_{\kappa\nu}\xi^{\kappa}) + \xi^{\kappa}\frac{\partial \widetilde{g}_{\kappa\nu}}{\partial x^{\mu}} - \xi^{\kappa}\frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^{\kappa}}
\end{aligned}$$

	Linearized Perturbation theory	Electromagnetism
Basic potentials	Linearized metric perturbation $h_{\mu\nu}(x)$	Vector and scalar potentials $\Phi(t,x), \mathbf{A}(t,x)$
Field quantities	Linearized Ricci tensor $\delta R_{\mu\nu}$	Electric and Magnetic fields $\mathbf{E}(t,x), \mathbf{B}(t,x)$
Gauge transformation	$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$	$\mathbf{A} \to \mathbf{A} + \nabla\Lambda$ $\Phi \to \Phi - \partial_t \Lambda$
Examples of gauge condition	$h_2(t,r) = 0$	$\nabla \cdot \mathbf{A} + \partial_t \Phi = 0$

Table 1.2: Gauge transformations in Linearized Perturbation theory of Schwarzschild black holes and the theory of Electromagnetism

$$= \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} - \frac{\partial}{\partial x^\nu}(\widetilde{g}_{\mu\kappa}\xi^\kappa) - \frac{\partial}{\partial x^\mu}(\widetilde{g}_{\kappa\nu}\xi^\kappa) + \xi^\kappa \left(\frac{\partial \widetilde{g}_{\mu\kappa}}{\partial x^\nu} + \frac{\partial \widetilde{g}_{\kappa\nu}}{\partial x^\mu} - \frac{\partial \widetilde{g}_{\mu\nu}}{\partial x^\kappa} \right) \quad (1.45)$$

Now the last term of the last equation reminds readily the definition of the Cristoffel symbols, while the gradient of $\widetilde{g}_{\mu\nu}\xi^\mu$, keeping first orders in ξ, simply gives $\overset{\circ}{g}_{\mu\nu}\,\xi^\mu$. Thus, we have

$$\begin{aligned} \widetilde{g}'_{\mu\nu} &= \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} - \frac{\partial}{\partial x^\nu}(\overset{\circ}{g}_{\mu\kappa}\,\xi^\kappa) - \frac{\partial}{\partial x^\mu}(\overset{\circ}{g}_{\kappa\nu}\,\xi^\kappa) + 2\xi^\kappa \Gamma_{\kappa\mu\nu} \\ &= \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} - \frac{\partial \xi_\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x^\mu} + 2\xi^\kappa\,\overset{\circ}{g}_{\kappa\alpha}\,\Gamma^\alpha_{\mu\nu} \\ &= \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} - \frac{\partial \xi_\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x^\mu} + 2\xi^\alpha \Gamma^\alpha_{\mu\nu} \end{aligned} \quad (1.46)$$

from which we obtain the simple form

$$\widetilde{g}'_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} = \overset{\circ}{g}_{\mu\nu} + h'_{\mu\nu} \quad (1.47)$$

meaning that the metric of the form (1.2) transforms into a metric of the same form but with a perturbation that has to be defined again as,

$$h^{new}_{\mu\nu} \to h^{old}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad (1.48)$$

which is called *gauge field*. If all $|\xi^\alpha_{,\beta}|$ are small then the new defined $h_{\mu\nu}$ is small as well, that is, we remain in an acceptable coordinate system. The above coordinate transformation is called *gauge transformation* and the vector ξ^α, *gauge vector* and it is strong related to the gauge transformations in Electromagnetism.

The freedom of choosing coordinates in our equations means that we can choose arbitrarily that vector ξ^α, that simplifies our equations the most. However, we have to be careful since the gauge transformation must respect the decomposition of the perturbation tensor into tensor spherical harmonics, as well as the dissociation into axial and polar perturbations. This requirement imposes directly a first constraint concerning the form of ξ^α: it's components have to be functions of vector spherical harmonics of the same l and m as the part of the perturbation $h_{\mu\nu}$ that we concern and it has to be different for axial and polar perturbations. We will define separately those vectors during our analysis.

1.4 Exploring the Axial Perturbations

In the case of the axial perturbations we use the following gauge vector:

$$\xi^\mu = [0,\ \ 0,\ \ \Lambda(t,r)\frac{1}{\sin\theta}\partial_\phi Y_{lm},\ \ \Lambda(t,r)\sin\theta\partial_\theta Y_{lm}] \tag{1.49}$$

where we have introduced an arbitrary function $\Lambda(t,r)$ that will allow us later to annul the radial factor $h_2(t,r)$. This gauge vector, as we will directly see, simplifies substantially the form of the perturbation tensor $h_{\mu\nu}$.

As we saw earlier the perturbation tensor in the gauge that we used is written as

$$h^{new}_{\mu\nu} = h^{old}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

Its components can be explicitly derived; by simply substituting the covariant derivatives[4] of quantity ξ in the above formula we obtain the following expressions for the components of the axial perturbation $h^{new}_{\mu\nu}$:

$$h^{new}_{tt} = h^{new}_{tr} = h^{new}_{rr} = 0 \tag{1.50}$$

$$h^{new}_{r\theta} = [h_1 + r^2\partial_r\Lambda(t,r)]\frac{1}{\sin\theta}\partial_\phi Y_{lm} \tag{1.51}$$

$$h^{new}_{r\phi} = [h_1 + r^2\sin^2\theta\partial_r\Lambda(t,r)]\sin\theta\partial_\theta Y_{lm} \tag{1.52}$$

$$\begin{aligned} h^{new}_{\theta\theta} = & \left[-h_2 + 2r^2\frac{1}{\sin^2\theta}\Lambda(t,r)\right]\frac{1}{\sin\theta}\partial^2_{\theta\phi}Y_{lm} + \left[h_2\frac{\cot\theta}{\sin^2\theta} - 2r^2\Lambda(t,r)\cot\theta\right]\times \\ & \times\frac{1}{\sin\theta}\partial_\phi Y_{lm} \end{aligned} \tag{1.53}$$

$$\begin{aligned} h^{new}_{\theta\phi} = & -\frac{1}{2}\left[h_2 + 2r^2\Lambda(r,t)\sin^2\theta\right]\sin\theta\partial^2_{\theta\theta}Y_{lm} + \frac{1}{2}\left[h_2 - 2r^2\Lambda(r,t)\right]\partial^2_{\phi\phi}Y_{lm} + \\ & +\frac{1}{2}[h_2\cot\theta\partial_\theta - 2r^2\Lambda(r,t)\sin^2\theta\cos\theta\partial_\phi]Y_{lm} \end{aligned} \tag{1.54}$$

$$h^{new}_{\theta t} = -[h_0 Y_{lm} + r^2\partial_t\Lambda(t,r)]\frac{1}{\sin\theta}\partial_\phi Y_{lm} \tag{1.55}$$

$$\begin{aligned} h^{new}_{\phi\phi} = & -\left[\frac{1}{2}h_2 - 2r^2\sin^2\theta\Lambda(t,r)\right]\sin\theta\partial^2_{\theta\phi}Y_{lm} + \left[\frac{1}{2}h_2 - 2r^2\Lambda(t,r)\right]\times \\ & \times\cot\theta\sin\theta\partial_\phi Y_{lm} \end{aligned} \tag{1.56}$$

$$h^{new}_{\phi t} = [h_0 + r^2\sin^2\theta\partial_t\Lambda(t,r)]\sin\theta\partial_\theta Y_{lm} \tag{1.57}$$

[4]In order to calculate the covariant derivatives of ξ we use the unperturbed Christoffel symbols, as they have been defined in (1.7).

which they have, according to equations (1.24)-(1.31), the correct general form of an axial perturbation.

We will now proceed to the simplification of the components of $h_{\mu\nu}$ by using the freedom to choose a proper gauge. We introduce the *Regge-Wheeler* gauge which was first introduced by Regge and Wheeler [1] in the context of a stability analysis of the Schwarzschild black hole. The Regge-Wheeler gauge is complete in the sense that it does not allow for additional gauge transformations. Furthermore, in this gauge all the highest derivatives in the angles (θ, ϕ) have been eliminated. We achieve this by setting the function $\Lambda(t, r)$ as

$$\Lambda(t, r) = -\frac{1}{2} h_2(t, r)$$

In this gauge we have the freedom to annul the coefficient $h_2(t, r)$ and consequently all the contributions of the highest derivatives in the angles (θ, ϕ). In this gauge the axial metric perturbation takes the simplified form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_0 \frac{1}{\sin\theta}\partial_\phi & h_0 \sin\theta\partial_\theta \\ * & 0 & -h_1 \frac{1}{\sin\theta}\partial_\phi & h_1 \sin\theta\partial_\theta \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} Y_{lm} \tag{1.58}$$

Since we have simplified the form of our perturbation significantly, we can now substitute the expression (1.83) into the variation of the Einstein field equations

$$\delta\mathfrak{R}_{\mu\nu} = \delta\Gamma^{\alpha}_{\mu\alpha;\nu} - \delta\Gamma^{\alpha}_{\mu\nu;\alpha} = 0 \tag{1.59}$$

The expressions for the equations $\delta\mathfrak{R}_{\mu\nu} = 0$ although rather lengthy, are listed below:

$$\begin{aligned} \delta\mathfrak{R}_{tt} &= \delta\mathfrak{R}_{tr} = \delta\mathfrak{R}_{rr} = 0 \\ \delta\mathfrak{R}_{r\theta} &= \frac{1}{2r^3 \sin\theta(1-\cos^2\theta)(1-2M/r)} \times \\ &\times [-h_1 r\partial^3_{\phi\phi\phi} Y_{lm} - 2h_1 r\partial_\phi Y_{lm} + 2h_1 r\cos^2\theta\partial_\phi Y_{lm} + 2h_1 M\partial^3_{\phi\phi\phi} Y_{lm} \\ &- h_1 r\partial^3_{\phi\theta\theta} Y_{lm} + h_1 r\cos^2\theta\partial^3_{\phi\theta\theta} Y_{lm} + 2r^2\partial_t h_0 \partial_\phi Y_{lm} - 2r^2\cos^2\theta\partial_t h_0\partial_\phi Y_{lm} \\ &- r^3\partial^2_{tr} h_0\partial_\phi Y_{lm} - r^3\cos^2\theta\partial^2_{tr} h_0\partial_\phi Y_{lm} + r^3\partial^2_{tt} h_1\partial_\phi Y_{lm} - r^3\cos^2\theta\partial^2_{tt} h_1\partial_\phi Y_{lm} \\ &+ 4h_1 M\partial_\phi Y_{lm} - 4h_1 M\cos^2\theta\partial_\phi Y_{lm} - h_1 r\cos\theta\sin\theta\partial^2_{\phi\theta} Y_{lm} \\ &+ 2h_1 M\cos\theta\sin\theta\partial^2_{\phi\theta} Y_{lm} + 2h_1 M\partial^2_{\theta\theta} Y_{lm} - 2h_1\partial^2_{\theta\theta} Y_{lm}] \end{aligned} \tag{1.60}$$

$$
\begin{aligned}
\delta\mathfrak{R}_{r\phi} \;=\; & \frac{1}{2r^3(1-\cos^2\theta)(1-2M/r)}\times \\
\times\; & [h_1\sin\theta r\partial_\theta Y_{lm} - 2h_1\sin\theta r\cos^2\theta\partial_\theta Y_{lm} \\
-\; & 2h_1M\sin\theta\partial_\theta Y_{lm} + 4h_1M\sin\theta\cos^2\theta\partial_\theta Y_{lm} + h_1\sin\theta r\partial^3_{\phi\phi\theta}Y_{lm} \\
+\; & h_1\sin\theta r\partial^3_{\theta\theta\theta}Y_{lm} - h_1\sin\theta r\cos^2\theta\partial^3_{\theta\theta\theta}Y_{lm} - 2h_1r\cos\theta\partial^2_{\phi\phi}Y_{lm} \\
-\; & 2r^2\sin\theta\partial_t h_0\partial_\theta Y_{lm} + 2r^2\cos^2\theta\sin\theta\partial_t h_0\partial_\theta Y_{lm} + r^3\sin\theta\partial_\theta Y_{lm}\partial^2_{rt}h_0 \\
-\; & r^3\cos^2\theta\sin\theta\partial_\theta Y_{lm}\partial^2_{tr}h_0 - r^3\sin\theta\partial^2_{tt}h_1\partial_\theta Y_{lm} + r^3\cos^2\theta\sin\theta\partial^2_{tt}h_1\partial_\theta Y_{lm} \\
+\; & h_1r\cos\theta\partial^2_{\theta\theta}Y_{lm} - h_1r\cos^3\theta\partial^2_{\theta\theta}Y_{lm} - 2h_1M\cos\theta\partial^2_{\theta\theta}Y_{lm} \\
+\; & 2h_1M\cos^3\theta\partial^2_{\theta\theta}Y_{lm} - 2h_1M\sin\theta\partial^3_{\theta\theta\theta}Y_{lm} \\
+\; & 4h_1M\cos\theta\partial^2_{\phi\phi}Y_{lm}]
\end{aligned}
\tag{1.61}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\theta\theta} \;=\; & -\frac{1}{2r^3(1-\cos^2\theta)(1-2M/r)}\times \\
\times\; & [(-2h_1Mr + 4h_1M^2 - \partial_r h_1 r^3 + 4r^3M\partial_r h_1 - 4rM^2\partial_r h_1 + r^3\partial_t h_0) \\
\times\; & (\sin\theta\partial^2_{\phi\theta}Y_{lm} - \cos\theta\partial_\phi Y_{lm})]
\end{aligned}
\tag{1.62}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\theta\phi} \;=\; & \frac{1}{2r^3\sin\theta(1-2M/r)}\times \\
\times\; & [(-2h_1rM + 4h_1M^2 - r^3\partial_r h_1 + 4r^2M\partial_r h_1 - 4rM^2\partial_r h_1 + r^3\partial_t h_0) \\
\times\; & (-\partial^2_{\theta\theta}Y_{lm} + \cos^2\theta\partial^2_{\theta\theta}Y_{lm} + \cos\theta\sin\theta\partial_\theta Y_{lm} + \partial^2_{\phi\phi}Y_{lm})]
\end{aligned}
\tag{1.63}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\theta t} \;=\; & \frac{1}{2r^3(\cos^2\theta-1)\sin\theta}\times \\
\times\; & [-2\partial_\phi Y_{lm}\partial_t h_1 r^2 + 2\partial_\phi Y_{lm}\partial_t h_1 r^2\cos^2\theta + 4\partial_\phi Y_{lm}\partial_t h_1 Mr \\
-\; & 4\partial_\phi Y_{lm}\partial_t h_1 Mr\cos^2\theta + 4\partial_\phi Y_{lm}h_0M - 4\partial_\phi Y_{lm}h_0M\cos^2\theta \\
+\; & 2\partial_\phi Y_{lm}\partial^2_{rt}h_1Mr^2 - 2\partial_\phi Y_{lm}\partial^2_{rt}h_1Mr^2\cos^2\theta - 2\partial_\phi Y_{lm}\partial^2_{rr}h_0Mr^2 \\
+\; & 2\partial_\phi Y_{lm}\partial^2_{rr}h_0Mr^2\cos^2\theta - \partial_\phi Y_{lm}\partial^2_{rt}h_1r^3 + \partial_\phi Y_{lm}\partial^2_{rt}h_1r^3\cos^2\theta \\
+\; & \partial_\phi Y_{lm}\partial^2_{rr}h_0r^3 - \partial_\phi Y_{lm}\partial^2_{rr}h_0r^3\cos^2\theta + h_0r\partial^3_{\phi\phi\phi}Y_{lm} \\
+\; & h_0\cos\theta r\partial^2_{\phi\theta}Y_{lm}\sin\theta + h_0\partial^2_{\phi\theta\theta}Y_{lm}r - h_0\partial^2_{\phi\theta\theta}Y_{lm}r\cos^2\theta]
\end{aligned}
\tag{1.64}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\phi\phi} \;=\; & \frac{1}{r^3(1-2M/r)}\times \\
\times\; & [(-2h_1Mr + 4h_1M^2 - \partial_r h_1r^3 + 4\partial_r h_1r^2M - 4r\partial_r h_1M^2 + \partial_t h_0r^3) \\
\times\; & (-\partial_\phi Y_{lm}\cos\theta + \partial^2_{\phi\theta}Y_{lm}\sin\theta)]
\end{aligned}
\tag{1.65}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\phi t} &= -\frac{1}{2r^3(1-\cos^2\theta)} \times \\
&\times\ [2\sin\theta\partial_\theta Y_{lm}\partial_t h_1 r^2 - 2\sin\theta\partial_\theta Y_{lm}\partial_t h_1 r^2\cos^2\theta - 4\sin\theta\partial_\theta Y_{lm}\partial_t h_1 rM \\
&+\ 4\sin\theta\partial_\theta Y_{lm}\partial_t h_1 rM\cos^2\theta - 4h_0\partial_\theta Y_{lm}\sin\theta M + 4h_0\partial_\theta Y_{lm}\sin\theta M\cos^2\theta \\
&-\ 2\sin\theta r^2\partial_\theta Y_{lm}\partial^2_{rt}h_1 M + 2\sin\theta r^2\partial_\theta Y_{lm}\partial^2_{rt}h_1 M\cos^2\theta + 2\sin\theta r^2\partial_\theta Y_{lm}\partial^2_{rr}h_0 M \\
&-\ 2\sin\theta r^2\partial_\theta Y_{lm}\partial^2_{rr}h_0 M\cos^2\theta + \sin\theta r^3\partial_\theta Y_{lm}\partial^2_{rt}h_1 - \sin\theta r^3\partial_\theta Y_{lm}\partial^2_{rt}h_1\cos^2\theta \\
&-\ \sin\theta r^3\partial_\theta Y_{lm}\partial^2_{rr}h_0 + \sin\theta r^3\partial_\theta Y_{lm}\partial^2_{rr}h_0\cos^2\theta - h_0 r\sin\theta\partial^3_{\phi\phi\theta}Y_{lm} \\
&+\ h_0 r\sin\theta\partial_\theta Y_{lm} - h_0 r\partial^2_{\theta\theta}Y_{lm}\cos\theta + h_0 r\partial^2_{\theta\theta}Y_{lm}\cos^3\theta \\
&-\ h_0 r\sin\theta\partial^3_{\theta\theta\theta}Y_{lm} + h_0 r\sin\theta\partial^3_{\theta\theta\theta}Y_{lm}\cos^2\theta \\
&+\ 2h_0\cos\theta r\partial^2_{\phi\phi}Y_{lm}]
\end{aligned}
\tag{1.66}
$$

By setting the above listed equations equal to zero we obtain the equations governing the perturbed spherically symmetric vacuum spacetime. Out of the ten Einstein equations three are satisfied trivially ($\delta\mathfrak{R}_{tt} = \delta\mathfrak{R}_{tr} = \delta\mathfrak{R}_{rr} = 0$), while the rest seven are satisfied only under specific conditions, namely for $m = 0$. The reductions can be further facilitated by noting that, besides being simpler, Einstein's equations in the Regge-Wheeler gauge are independent of m, which can therefore set to be zero. This is something that is allowable due to the spherical symmetry of the background. Thus, after replacing the definition for the spherical harmonics functions

$$
Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m e^{im\phi}P_{lm}(cos\theta),
$$

we obtain the following three equations

$$
\begin{aligned}
\delta\mathfrak{R}_{r\phi} &= -\left(\frac{2l+1}{\pi}\right)^{1/2}\frac{1}{4r^3(\cos^2\theta-1)(1-2M/r)} \times \\
&\times\ [-h_1\sin\theta r\partial_\theta P_l + 2h_1\sin\theta r\cos^2\theta\partial_\theta P_l + 2\sin\theta\partial_\theta P_l h_1 M \\
&-\ 4h_1\sin\theta\cos^2\theta\partial_\theta P_l M - h_1\sin\theta r\partial^3_{\theta\theta\theta}P_l + h_1\sin\theta r\partial^3_{\theta\theta\theta}P_l\cos^2\theta \\
&+\ 2\sin\theta\partial_\theta P_l\partial_t h_0 r^2 - 2\sin\theta\partial_\theta P_l\partial_t h_0 r^2\cos^2\theta - \sin\theta\partial_\theta P_l\partial^2_{rt}r^3 h_0 \\
&+\ \sin\theta\partial_\theta P_l\partial^2_{rt}r^3 h_0\cos^2\theta + \sin\theta\partial_\theta P_l r^3\partial^2_{tt}h_1 - \sin\theta\partial_\theta P_l r^3\partial^2_{tt}h_1\cos^2\theta \\
&-\ h_1 r\cos\theta\partial^2_{\theta\theta}P_l + h_1 r\cos^3\theta\partial^2_{\theta\theta}P_l + 2h_1\cos\theta\partial^2_{\theta\theta}P_l M \\
&-\ 2h_1\cos^3\theta\partial^2_{\theta\theta}P_l M + 2\sin\theta h_1\partial^3_{\theta\theta\theta}P_l M - 2\sin\theta h_1\partial^3_{\theta\theta\theta}P_l M\cos^2\theta]
\end{aligned}
\tag{1.67}
$$

$$
\begin{aligned}
\delta\mathfrak{R}_{\theta\phi} &= -\left(\frac{2l+1}{\pi}\right)^{1/2}\frac{1}{4r^3\sin\theta(1-2M/r)} \times \\
&\times\ [(-2h_1 Mr + 4h_1 M^2 - \partial_r h_1 r^3 + 4\partial_r h_1 r^2 M - 4r\partial_r h_1 M^2 + \partial_t h_0 r^3) \\
&\times\ (\cos\theta\partial_\theta P_l\sin\theta - \partial^2_{\theta\theta}P_l + \partial^2_{\theta\theta}P_l\cos^2\theta)]
\end{aligned}
\tag{1.68}
$$

$$\begin{aligned}
\delta\mathfrak{R}_{t\phi} &= \left(\frac{2l+1}{\pi}\right)^{1/2} \frac{1}{4r^3(\cos^2\theta-1)(1-2M/r)} \times \\
&\times\ [2\sin\theta\partial_\theta P_l\partial_t h_1 r^2 - 2\sin\theta\partial_\theta P_l\partial_t h_1 r^2\cos^2\theta - 4\sin\theta\partial_\theta P_l\partial_t h_1 rM \\
&+\ 4\sin\theta\partial_\theta P_l\partial_t h_1 rM\cos\theta^2 - 4h_0\partial_\theta P_l\sin\theta M + 4h_0 M\partial_\theta P_l\sin\theta\cos^2\theta \\
&-\ 2\sin\theta r^2\partial_\theta P_l\partial^2_{rt}h_1 M + 2\sin\theta r^2\partial_\theta P_l\partial^2_{rt}h_1 M\cos^2\theta + 2\sin\theta r^2\partial_\theta P_l\partial^2_{rr}h_0 M \\
&-\ 2\sin\theta r^2\partial_\theta P_l\partial^2_{rr}h_0 M\cos^2\theta + \sin\theta r^3\partial_\theta P_l\partial^2_{rt}h_1 - \sin\theta r^3\partial_\theta P_l\partial^2_{rt}h_1\cos^2\theta \\
&-\ \sin\theta r^3\partial_\theta P_l\partial^2_{rr}h_0 + \sin\theta r^3\partial_\theta P_l\partial^2_{rr}h_0\cos^2\theta + h_0 r\sin\theta\partial_\theta P_l - h_0 r\cos\theta\partial^2_{\theta\theta}P_l \\
&+\ h_0 r\cos^3\theta\partial^2_{\theta\theta}P_l - h_0 r\sin\theta\partial^3_{\theta\theta\theta}P_l + h_0 r\sin\theta\partial^3_{\theta\theta\theta}P_l\cos^2\theta] \qquad (1.69)
\end{aligned}$$

The required reductions can be simplified by noting that the eigenvalue problem for the angular momentum in Quantum mechanics can be formulated explicitly as follows:

$$L^2 Y_{lm} = -\left[\frac{1}{\sin^2\theta}\partial^2_{\phi\phi} + \frac{1}{\sin\theta}\left(\partial_\theta\sin\theta\partial_\theta\right)\right] Y_{lm} = l(l+1)Y_{lm} \qquad (1.70)$$

In each case we can analyze the three differential equations into an angular and a non-vanishing angle part. Such an analyzation, for the Schwarzschild metric, yields to the following set of nontrivial differential equations:

$$\delta\mathfrak{R}_{\theta\phi} = 0 \Rightarrow \left(\frac{1}{1-\frac{2M}{r}}\right)\partial_t h_0 - \partial_r\left(1-\frac{2M}{r}\right)h_1 = 0 \qquad (1.71)$$

$$\delta\mathfrak{R}_{r\phi} = 0 \Rightarrow$$

$$\left(\frac{1}{1-\frac{2M}{r}}\right)^{-1}\left(\partial^2_{tt}h_1 - \partial^2_{tr}h_0 + \frac{2}{r}\partial_t h_0\right) + \frac{1}{r^2}[l(l+1)-2]h_1 = 0 \qquad (1.72)$$

$$\delta\mathfrak{R}_{t\phi} = 0 \Rightarrow$$

$$\frac{1}{r^2}\left[r\partial_r\left(\frac{1}{1-\frac{2M}{r}}\right) - \frac{1}{2}l(l+1)\right]h_0 + \left(\frac{1}{1-\frac{2M}{r}}\right)\left(\partial^2_{rr}h_0 - \partial^2_{tr}h_1 - \frac{2}{r}\partial_t h_1\right) = 0 \qquad (1.73)$$

where the last equation is a consequence of the other two. The only unknown functions here are $h_0(t,r)$ and $h_1(t,r)$.

It is useful now to introduce a new function of t and r defined as follows

$$Q(t,r) = \frac{1}{r}\left(1-\frac{2M}{r}\right)h_1(t,r) \qquad (1.74)$$

From the above definition we take for the equation (1.71)

$$\partial_t h_0 = \left(1-\frac{2M}{r}\right)\partial_r(rQ) \qquad (1.75)$$

which, if we substitute in (1.72) we take the following differential equation

$$\left(1-\frac{2M}{r}\right)^{-1/2}\partial^2_{tt}(rQ)-\partial_r\left[rQ\left(1-\frac{2M}{r}\right)\right]+\frac{2}{r}\left(1-\frac{2M}{r}\right)\partial_r(rQ)$$
$$+\frac{1}{r}[l(l+1)-2]Q=0 \quad (1.76)$$

It is now time to introduce another auxiliary coordinate transformation, or better a new radial coordinate, the so called *tortoise* coordinate x which is expressible in the form:

$$x\equiv r+2Mln\left(\frac{r}{2M}-1\right)$$

It is quite easy to show that it stands

$$\frac{dx}{dr}=\left(1-\frac{2M}{r}\right)^{-1}$$

From that expression we derive for the partial derivative ∂_r:

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial x}\frac{\partial x}{\partial r}=\left(1-\frac{2M}{r}\right)^{-1}\frac{\partial}{\partial x} \quad (1.77)$$

Under this coordinate transformation our differential equation takes the final form:

$$\partial^2_{tt}Q(t,x)-\partial^2_{xx}Q(t,x)+\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right]Q(t,x)=0 \quad (1.78)$$

where r is thought of as function of x. This equation is known as the *Regge-Wheeler* equation and it can be considered as a one-dimension wave equation in a scattering potential barrier $V^{(-)}(r)$

$$V^{(-)}(r)\equiv\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right] \quad (1.79)$$

which is called *Regge-Wheeler* potential, namely

$$\partial^2_{tt}Q(t,x)-\partial^2_{xx}Q(t,x)+V^{(-)}(r)Q(t,x)=0 \quad (1.80)$$

The presented analysis was carried out having assumed only gravitational perturbations. However, the followed analysis can also be applied to electromagnetic perturbations or perturbations of any scalar field. The form of the equations remains the same for every kind of perturbations. The only difference is the appearance of parameter τ that distinguishes the type of perturbation into the potential. That is,

$$V^{(-)}(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M\sigma}{r^3}\right] \quad (1.81)$$

Tortoise coordinates

The problem with our current coordinates is that $dt/dr \rightarrow \infty$ along radial null geodesics which approach $r = 2M$. The equation that governs the radial null geodesics is

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

and it comes from the demand that

$$ds^2 = 0 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2$$

We see that progress in the r direction becomes slower and slower with respect to the coordinate time t. We can try to fix this problem by replacing t with a coordinate which moves more slowly along null geodesics. First we have to notice that we can explicitly solve the condition characterizing radial null curves to obtain

$$t = \pm x + constant$$

where the tortoise coordinate is

$$x = r + 2Mln\left(\frac{r}{2M} - 1\right)$$

The tortoise coordinate is only sensibly related to r when $r \geq 2M$. The price we pay, however, is that the surface at $r = 2M$ has been pushed to ∞:

$$x \rightarrow -\infty \text{ as } r \rightarrow 2M^+ \text{ and } x \rightarrow r \text{ as } r \rightarrow \infty$$

$$\tau = \begin{cases} +1 & \text{scalar field} \\ 0 & \text{electromagnetic field} \\ -3 & \text{gravitational field} \end{cases}$$

The value of the parameter τ derives from the expression $\tau = 1 - s^2$, where $s = 0, 1, 2$ is the spin of the perturbed field.

1.5 Exploring the Polar Perturbations

The mathematical analysis is exactly the same for the polar perturbations. However, the components of the various tensors over exceeds the purpose of our mathematical interpretation. We will limit our outline only to the basic equations and constraints governing the polar perturbations.

We will start with the choice of the polar gauge vector. We choose the following form

$$\xi^\mu = \left[M_0(t,r)Y_{lm},\ M_1(t,r)Y_{lm},\ M(t,r)\partial_\theta Y_{lm},\ M(t,r)\frac{1}{\sin^2\theta}Y_{lm}\right] \quad (1.82)$$

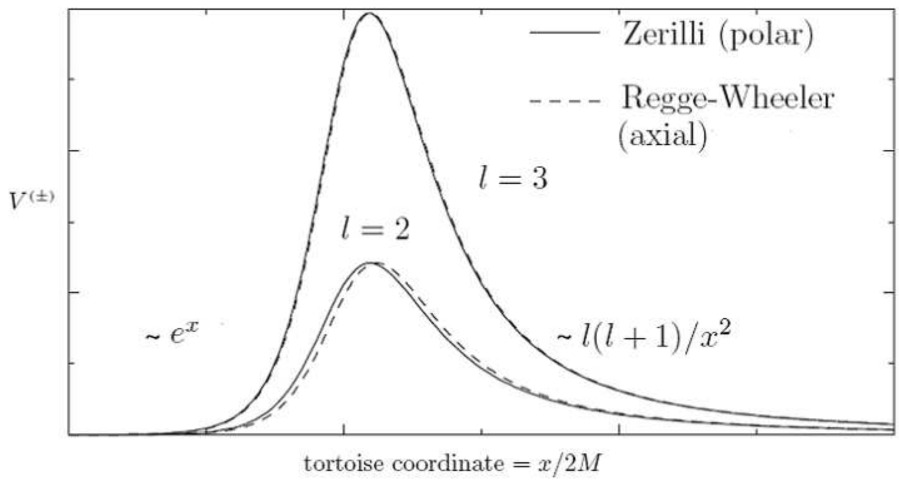


Figure 1.3: *The Regge-Wheeler and Zerilli potentials for $l = 2$ and $l = 3$.*

where the unknown radial functions M_0, M_1 and M are chosen in such a way so they annul the functions G, h_0 and h_1. Under the introduction of such a gauge vector, we obtain for the polar perturbation tensor the following form

$$h_{\mu\nu} = \begin{pmatrix} \left(1-\frac{2M}{r}\right) H_0 & H_1 & 0 & 0 \\ * & \left(1-\frac{2M}{r}\right) H_2 0 & 0 & 0 \\ * & * & r^2 K & 0 \\ * & * & * & r^2 \sin^2\theta K \end{pmatrix} Y_{lm} \tag{1.83}$$

Following the same procedure, we linearize the field equations for polar perturbations and we obtain again a one-dimension wave equation of the form

$$\partial^2_{tt} Z(t,x) - \partial^2_{xx} Z(t,x) + V^{(+)}(r) Z(t,x) = 0 \tag{1.84}$$

where the corresponding scattering potential, called *Zerilli* potential, is

$$V^{(-)}(x) \equiv \left(1 - \frac{2M}{r}\right) \left[\frac{72M^3}{r^5\lambda^2} - \frac{12M}{r^3\lambda^2}(l-1)(l+2)\left(1-\frac{3M}{r}\right) + \right. \\ \left. + \frac{l(l-1)(l+2)(l+1)}{r^2\lambda} \right] \tag{1.85}$$

Here $\lambda = l(l+1) + 6M/r - 2$ and r is thought of as a function of the tortoise coordinate x. The Regge-Wheeler and Zerilli potentials $V^{(\mp)}$ governing the axial and polar perturbations, in spite of appearance, are complete equivalent. Indeed, as shown by Chandrasekhar, the two potential are given by the following expression

$$V^{(\pm)} = \pm 6M \frac{df}{dx} + (6M)^2 f^2 + \kappa f \tag{1.86}$$

where
$$f = \frac{r - 2M}{r^2(\mu^2 r + 6M)}$$
$\kappa = \mu^2(\mu^2 + 2)$ and $\mu^2 = (l-1)(l+2)$. The two potentials can be considered totally equivalent, in a sense that the frequencies corresponding to the solutions of the wave equations that they govern, are identical. Thus, when we later attempt to study the solutions of the perturbed equations, we shall concentrate only into the axial equations since they have simpler analytical expressions.

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Chapter 2

Quasinormal Modes of Schwarzschild Black Hole

2.1 Definition of Quasinormal Modes

In the preceding chapter we found that the radial component of an axial or polar perturbation of the Schwarzschild metric outside of the event horizon satisfies the following master differential equation:

$$\partial^2_{xx} Q(t,x) - [\partial^2_{tt} + V] Q(t,x) = 0 \qquad -\infty < x < +\infty \tag{2.1}$$

So far we have not made any assumption concerning the nature of the perturbation. A black hole can be perturbed in a variety of ways. By the incidence of gravitational waves, by an object falling into it, or by the accretion of matter surrounding it. However, the analysis that we followed was independent of the form of the perturbation; we demanded only to be infinitesimal. We now assume that our perturbation has an harmonic time dependence, namely we seek harmonic solutions of the form

$$Q_{\omega_n}(t,x) = e^{i\omega_n t}\Psi(x), \tag{2.2}$$

where ω_n denotes a discrete spectrum of, n in number, oscillation frequencies (normal modes) and is a complex number of the type

$$\omega_n = \omega_n^{(re)} + i\omega_n^{(im)}, \qquad n = 0, 1, 2 \ldots \tag{2.3}$$

the real part of the complex frequency determines the oscillation frequency ($f = \omega^{(re)}/2\pi$) and the imaginary parts determines the rate at which each mode is damped as a result of the emission of radiation ($\tau = 1/\omega^{(im)}$). For a given kind of physical perturbation that radiation can be gravitational, electromagnetic or in general radiation of any scalar field.

Inserting (2.2) into the one-dimensional wave equation (2.1) yields to

$$\Psi'' + [\omega^2 - V]\Psi = 0, \tag{2.4}$$

where the prime denotes differentiation with respect to x. The solutions of the above equation defines the *quasinormal modes* of the black hole while the associated frequencies to those modes, *quasinormal frequencies* [4]. We may describe the quasinormal modes as the pure tones of a black hole. A characteristic sound, namely a selected superposition of complex modes that represents the response to any external excitation. The term "normal" refers to the normal modes of compact classical linear oscillating systems, while the nomenclature "quasi" is justified mainly by the fact that the frequencies are complex, thus they show strong damping. Finally, we must say that the two potentials $V^{(\pm)}$ that correspond to the potential of (2.4) are, though quite different, as we have already stated, totally equivalent, meaning that for the Schwarzschild black hole the axial and polar perturbations are isospectral. Thus, for the analysis that follows we will concentrate only on axial perturbations, since the Regge-Wheeler potential $V^{(-)}$ has an analytical form which is simpler to handle.

2.2 Stability of the Schwarzschild Black Hole

If we take the existence of black hole for granted, then we can assume in no way that as astrophysical object are isolated and inactive. In ordinary astrophysical systems that host a black hole it is customary matter to be fitted into the black hole continually. Thus, the black hole are perturbed all the time by accreting mass. But what is the dynamical response to that physical perturbations? Is it stable of unstable? To unveil this, let us recall some properties of fundamental physical systems. Consider an harmonically oscillating system that is subject to a small initial perturbation. The equation of motion for such system usually reads

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + kx = 0$$

since we have allowed frictional forces to be present. The resultant motion is then proportional to the term $e^{-\gamma t}e^{i\omega t}$, that is, the system undergoes a damped oscillation and remains stable. It turns out that the picture is more or less the same for the black holes! Whichever is the analytic solution of (2.4), the complex form of the quasinormal frequencies assures that the oscillation of the black hole will be damped. Of course this request that the imaginary part of the frequency is negative: as we shall see, this is something that is ensured by the boundary conditions and the potential of the physical problem. The physical meaning of that result is that the black hole spacetime right after an initial

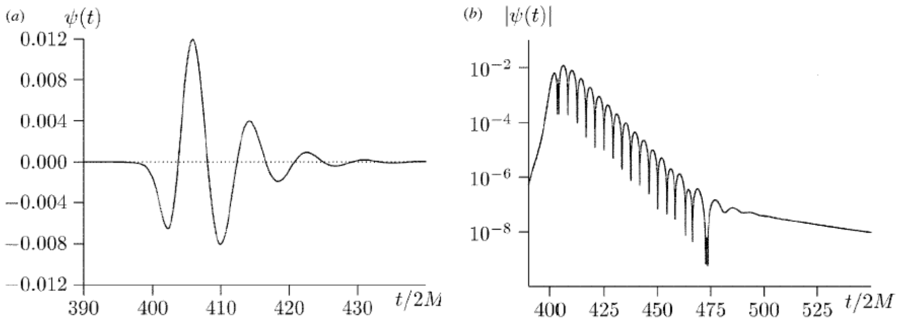

Figure 2.1: (a) *Evolution of a gravitational perurbation near a Schwarzschild black hole. One can see that the waveform is dominated by a characteristic ringing. That ringing down frequency and damping time is independent of the perturbation.* (b) *The same in logaritmic scale. The dominance of quasinormal ringing is clear, and at very late times, a power-law falloff appears. Although the signal seems to contain only one frequency, it consists of a discrete set of quasinormal frequencies.*

perturbation loses energy through gravitational wave emission (or any massless field) and becomes stable. The certain response of spacetime is unavoidable: every time the spacetime oscillates gravitational waves are generated that carry energy away. Thus a black hole is stable under gravitational perturbations.

This was shown also numerically by integrating the Regge-Wheeler equation. As shown in Figure (2.2) both in linear and logarithmic scale, the signal is dominated, during a certain time, by damped single frequency oscillations. The frequency and damping of these oscillations depend only on the parameters characterizing the black hole, which in the Schwarzschild case is its mass. The initial perturbation here arise from close limit approximation of black hole - black hole collisions, though the numerical results always shows that the response is completely independent of the particular initial configuration that caused the excitation of such vibrations. Finally, in a logarithmic scale, the same calculations shows an appearance of quasinormal modes only over a limited time interval. At very late times quasinormal ringing gives way to a power-law falloff. This short appearance is due to the incompleteness of the quasinormal modes that prohibits a possible superposition for all times as in the case of the normal mode systems.

2.3 Boundary Conditions

The motion of compact linear oscillating systems such as finite strings with both ends fixed can always be described as a superposition of preferred harmonic states. The mathematical formulation of the problem is described by the following equation

$$\frac{\partial \Phi}{\partial t} = \mathbf{A}\Phi \tag{2.5}$$

where $\mathbf{A}$ is a differential operator acting on the spatial variables. The boundary conditions of such a system indicates that the problem is *self-adjoint*[1], thus the differential operator $\mathbf{A}$ has a discrete point spectrum on the Hilbert space. The eigenfunctions and the eigenvalues of the operator determines the solutions of the problem, i.e., the normal modes. Due to the completeness[2] of those eigenfunctions, the physical motion of the system corresponds to a superposition of the normal modes.

Completely different is, however, the behavior and the mathematical properties of systems that may lose energy at infinite such the one that we study here. Having in mind the forms of the potentials (1.81) and (1.85), we realize that the potential V is positive towards the horizon and spatial infinity (cf. Figure 1.3) and satisfies

$$V \to 0 \quad as \quad x \to \pm\infty$$

while it can be characterized of short range in the sense that the integral

$$\int_{-\infty}^{+\infty} V(x)dx \tag{2.6}$$

is finite. However, such potentials do not allow bound states, meaning that we cannot impose that the solutions vanish as $x \to \pm\infty$. Therefore, this precludes a possible normal mode analysis. Nevertheless, an idea that the general solution may be written as a superposition of QNMs seems to be correct, a task that we will later prove by using Laplace transformations.

The exact boundary conditions to be applied here are "radiation boundary conditions" and can be determined by studying the flux of radiation at infinity. As $|x| \to +\infty$ the spacetime becomes asymptotically flat and

$$(\omega^2 - V) \to \omega^2$$

The same holds when $|x| \to -\infty$, hence, the solutions $\Psi(x)$ at spatial infinity and near the horizon, resemble to spherical plane waves. At infinity, one can detect *ingoing* and *outgoing* waves, while at the horizon only waves propagating towards the black hole are present, namely *ingoing* waves that falls into the black hole. Ingoing and outgoing waves at spatial infinity correspond, respectively, to the radial solutions proportional to $e^{-i\omega x}$ and $e^{i\omega x}$. The physically correct boundary condition at the horizon correspons to the solution proportional to

[1] An operator is said to be *self-adjoint* if $\mathbf{A} = \mathbf{A}^\dagger$, where $\mathbf{A}^\dagger$ is the **adjoint** operator of $\mathbf{A}$ defined by the following relation

$$\langle u, \mathbf{A}v\rangle = \langle \mathbf{A}^\dagger u, v\rangle$$

[2] A set of eigenfunctions of any self-adjoint operator is a complete set in the sense that any well behaved function can be expanded as a linear combination of these functions

$e^{-i\omega x}$. Thus, the boundary conditions defining QNMs are those that toward the infinity are purely outgoing and toward the horizon, purely ingoing,

$$\Psi \sim e^{+i\omega x}, \qquad x \to +\infty \tag{2.7}$$

$$\Psi \sim e^{-i\omega x}, \qquad x \to -\infty \tag{2.8}$$

The boundary conditions at the edges of spacetime that we imposed here are fairly suitable for our analysis. They allow us to study the evolution of a perturbation without any external contributions, as for example gravitational waves coming from infinity. The fact that near the horizon we allow solutions only proportional to $e^{-i\omega x}$, was something that it was to be expected since no scalar field can travel trough the horizon to the outside region of the black hole. Before we close this section we should notice that the above boundary conditions lay the foundations of the physical content of problem; the reflection and transmission of incident waves from $\pm\infty$ by the one-dimensional potential barrier $V^{(\pm)}$.

2.4 Definition of Quasinormal Modes via Laplace Transformations

2.4.1 Existence of the Solutions

In order to explore the quasinormal modes of Schwarzschild black hole we could use Fourier transforms in time, so that we would be able to construct a discrete frequency spectrum. However, the quasinormal mode solutions do not form a complete set, as the normal modes in the case of the compact linear systems stated above. The solutions become unbounded at the edges of spacetime, and that causes a lot of problems, both mathematically and numerically. Only under special circumstances, namely certain initial conditions to be taken into account, the quasinormal modes can form a complete spectrum. Nevertheless, the Fourier transform does not allow initial conditions at some fixed time to be considered into our calculations. This problem can be solved explicitly if we use a Laplace transformation on (2.1) instead of a Fourier transformation, a mathematical confrontation first introduced by Nollert & Schmidt [6].

In general a Laplace transformation can be used to solve differential equations that involve compact initial value problems and for a function $f(t)$ defined for $0 \leq t \leq \infty$ is the ordinary calculus integration problem denoted as

$$\mathcal{L}f(t) \equiv \hat{f}(s,t) = \int_0^\infty e^{-st} f(t) dt \quad [f(t) = 0 \ \ for \ \ t < 0] \tag{2.9}$$

where s is the *transform variable* that may be complex, $s = \sigma + i\omega$. The Laplace transform can therefore be viewed as a generalization of the Fourier transform from the real line (a simple frequency axis) to the complex plane. It is necessary that $f(t)$ is exponentially bounded, or more precisely is of *exponential order*, that is $f(t)$ should not increase very fast. For this class of functions the relation

$$\lim_{t\to\infty} |f(t)e^{-\alpha t}| = 0$$

is required to hold for some real number α, or equivalently, for some constants M and α,

$$|f(t)| \leq Me^{\alpha t}$$

when $f(t)$ is piecewise continuous. In addition, the Laplace transform is analytic for $Re(s) = \sigma = 0$ (the so-called *right-half plane*).

The Laplace transform satisfies particular properties that allow us to solve explicitly differential equations usually with non-constant coefficients. Some of those are the following:

$\mathcal{L}[f(t) + g(t)] = \mathcal{L}f(t) + \mathcal{L}g(t)$	The integral of a sum is the sum of the integrals.
$\mathcal{L}[cf(t)] = c\mathcal{L}f(t)$	Constants c pass through the integral sign.
$\mathcal{L}f'' = s^2\mathcal{L}f(t) - sf\mid_{t=0} - f'\mid_{t=0}$	The t-derivative rule, or integration by parts (the initial values are called *Cauchy data*).

We now turn back to our case where we have to solve the following time-dependent differential equation

$$\partial^2_{xx}Q(t,x) - [\partial^2_{tt} + V(x)]Q(t,x) = 0 \qquad -\infty < x < +\infty, \;\; 0 \leq t \leq \infty \tag{2.10}$$

which can be rewritten as

$$\partial^2_{tt}Q(t,x) + \mathbf{A}Q(t,x) = 0 \tag{2.11}$$

where $\mathbf{A}$ is the following differential operator

$$\mathbf{A} = -\partial^2_{xx} + \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M\tau}{r^3}\right]$$

that in first site to do not defines a self-adjoint problem. Applying to both terms of (2.10) the Laplace transform, we obtain

$$\mathcal{L}\left[\partial^2_{xx}Q(t,x)\right] - \mathcal{L}[\partial^2_{tt}Q(t,x)] - \mathcal{L}[VQ(t,x)] = \mathcal{L}(0) \Rightarrow$$

$$\frac{d^2}{dx^2}\hat{f}(s,x) - s^2\hat{f}(s,x) + sQ(t,x)\mid_{t=0} + Q'(t,x)\mid_{t=0} - V(x)\hat{f}(t,x) = 0 \Rightarrow$$

$$\frac{d^2}{dx^2}\hat{f}(s,x) - [s^2 + V(x)]\hat{f}(s,x) = -sQ(t,x)\mid_{t=0} - Q'(t,x)\mid_{t=0}$$

$$\frac{d^2}{dx^2}\hat{f}(s,x) - [s^2 + V(x)]\hat{f}(s,x) = \mathcal{I}(s,x) \tag{2.12}$$

Here we have defined the Laplace transformation $\hat{f}(s,x)$ of the solution $Q(t,x)$ as the integral

$$\int_0^\infty e^{-st} Q(t,x) ds$$

It is remarkable the fact that the Laplace transformation allow us to include initial data, concerning the perturbations that we study, via the Cauchy data on the initial surface $t = 0$, namely the *source term* $\mathcal{I}(s,x)$. We may restrict now our treatment only to initial data of compact support. In this way we force the operator **A**, despite beig self-adjoint now, to have a purely continuous spectrum. The lack of discreteness leads to plane wave solutions that although they form a set of improper eigenfunctions, when treated properly they can build proper superpositions and thus, construct solution that can be considered as normal eigenfunctions. In this way the solutions can be viewed as bounded in a sense that there is a positive constant α such that,

$$\Psi(t,x,\theta,\phi) \leq \alpha, \ \ \forall \ \ t,x,\theta \text{ and } \phi \tag{2.13}$$

We then know from the standard theory of PDE's that the corresponding general solution $Q(t,x)$ must have a Laplace transform which is bounded as well and analytic in the right-half plane and can be obtained from the *Bromwich Inversion Formula*

$$Q(t,x) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\sigma - i\omega}^{\sigma + i\omega} e^{st} \hat{f}(s,x) ds \tag{2.14}$$

where $\hat{f}$ is the solution of the *inhomogeneous* equation (2.12).

2.4.2 Quasinormal Expansion

Up until now, that we have not specified what σ in $s = \sigma + i\omega$ is. For a given $\hat{f}(s,x)$, σ must be chosen so that the right hand side of the inversion integral is zero for $t < 0$ (to match the left side). To do this, we start with $\sigma > 0$ and close the contour of integration using a semicircle in $\text{Re}(s) > \sigma > 0$ to form a closed contour C. Now, for $t < 0$, the factor ensures that the contribution from the integral over the semicircle at infinity vanishes. Thus, for the integral around C to yield zero, $\hat{f}(s,x)$ must have no singularities inside C. Therefore, all singularities of $\hat{f}(s,x)$ must lie to the left of the line $\text{Re}(s) = \sigma$. This fixes σ. For $t > 0$, we close the contour in $\text{Re}(s) < \sigma$. This gives

$$Q(t,x) = \frac{1}{2\pi i} \oint_C e^{st} \hat{f}(s,x) ds \tag{2.15}$$

If the only singularities of $\hat{f}(s,x)$ are isolated poles, the inversion integral can be performed by the *Cauchy residue theorem*

$$\begin{aligned} Q(t,x) &= \sum \text{Res}[e^{st} \hat{f}(s,x)] \\ &= \sum_q e^{s_q t} \hat{f}(s_q,x) \end{aligned} \tag{2.16}$$

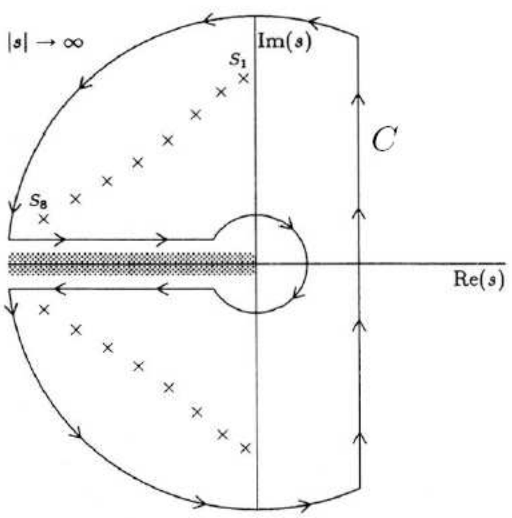

Figure 2.2: *The path of integration for the Laplace function $\hat{f}(s,x)$.*

for poles at $s = s_q$.

Now, the solution of the inhomogeneous differential equation (2.12) is unique up to a solution of the *homogeneous* equation. If $\text{Re}(s) > 0$, then any two linearly independent solutions, f_- and f_+, of the homogeneous differential equation

$$\frac{d^2}{dx^2}\hat{f}_\pm - [s^2 + V(x)]\hat{f}_\pm = 0 \tag{2.17}$$

define a particular Green's function $G(s,x,x')$ of the inhomogeneous equation such that

$$\hat{f}(s,x) = \int_{-\infty}^{+\infty} G(s,x,x')\mathcal{I}(s,x')dx' \tag{2.18}$$

The integrability boundary conditions that are to be satisfied from f_- and f_+ in order to be bounded are

$$\begin{aligned} f_-(s,x) &\sim e^{sx}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \quad as\ x \to -\infty \\ f_+(s,x) &\sim e^{-sx}\left(1 + \mathcal{O}\left(\frac{1}{x}\right)\right) \quad as\ x \to +\infty \end{aligned} \tag{2.19}$$

In the literature of the PDE's the Green's function is usually defined as the solution of the differential equation

$$\partial^2_{xx}G(s,x,x') + s^2G(s,x,x') = \delta(x - x') \tag{2.20}$$

The solution is then derived from the formula

$$G(s,x,x') = \frac{1}{W(s)} \begin{cases} f_-(s,x')f_+(s,x) & (x' < x) \\ f_-(s,x)f_+(s,x') & (x' > x) \end{cases}$$

where $W(s)$ is the Wronskian of f_- and f_+,

$$W(s) = f_-(s,x)\frac{d}{dx}f_+(s,x) - \frac{d}{dx}f_-(s,x)f_+(s,x)$$

The solutions $f_\pm(s,x)$ of the homogeneous differential equation, and consequently the Green's function, have been defined so far only into the right-half complex plane. In order to do this, we need to close the contour of integration to the left-plane of s. A prospective continuation of those solution also into the left-half plane requires $f_\pm(s,x)$ to be analytic. However, even if we assume initially that $\hat{f}_\pm$ are analytical, it turns out that the Wronskian of f_- and f_+ has isolated zeros which lead therefore to poles of the Green's function in the left-half plane. Since the poles are isolated, the Cauchy theorem of residua for $Q(x,t)$ yields to

$$\begin{aligned} Q(t,x) &= \frac{1}{2\pi i}\oint_C e^{st}\hat{f}(s,x)ds \\ &= \frac{1}{2\pi i}\oint_C e^{st}\int_{-\infty}^{+\infty} G(s_q,x,x')\Im(s_q,x')dx'ds \\ &= \sum \mathrm{Res}[e^{st}\int_{-\infty}^{+\infty} G(s_q,x,x')\Im(s_q,x')dx'] \\ &= \sum_q e^{s_q t}\mathrm{Res}\Big(\frac{1}{W(s)}\Big)\Big[f_+(x,s_q)\int_{-\infty}^{x} f_-(x',s_q)\Im(s_q,x')dx' \\ &\qquad + f_-(x,s_q)\int_{x}^{+\infty} f_+(x',s_q)\Im(s_q,x')dx'\Big] \end{aligned} \tag{2.21}$$

We need now to calculate the residue of the function $1/W(s)$. The function of interest can be written in the form

$$f(s) = \frac{g(s)}{W(s)}$$

where $g(s)=1$. Furthermore, $g(s)$ and $W(s)$ are analytical near s_q and $g(s_q) \neq 0$, $W(s_q)=0$, $dW/ds\mid_{s=s_q}$, thus the residue is given from

$$\mathrm{Res}\left(\frac{1}{W(s)}\right) = \frac{1}{dW/ds}$$

Since we have restrict our treatment only to initial data of compact support, we may assume an interval $[x_-, x_+]$ of compact Cauchy data. If x is located, say, at the left of the interval such that $x \equiv x_+$, then we may develop the solution $Q(t,x)$ into a series of quasinormal modes, namely a *quasinormal mode expansion*, of the form

$$Q(t,x) = \sum_q c_q u_q(t,x) \tag{2.22}$$

where

$$c_q = \frac{1}{dW(s_q)/ds} \int_{x_-}^{x_+} f_-(x', s_q)\mathfrak{I}(s_q, x')dx'$$

and

$$u_q(t,x) = e^{s_q t} f_+(s_q, x)$$

It is now time to follow a more intuitive approach for the definition of quasinormal modes. As we have already stated, the Laplace transform can be viewed as a generalization of the Fourier transform. Indeed, when evaluated along the $s = i\omega$ axis (i.e., $\sigma = 0$), the Laplace transform reduces to the Fourier transform. Such a replacement gives the following asymptotic behavior for $f_\pm$

$$f_\pm(s,x) \sim e^{\mp i\omega x} \; as \; x \to \pm\infty$$

thus, the obey the quasinormal mode boundary conditions as they have given out in (2.8). In addition, even though $f_\pm$ have been defined as linearly independent solutions of (2.17), at the roots s_q of the Wronskian they become identical, up to a constant.

$$\begin{aligned} W(s = s_q) &= f_-(s_q, x)\frac{d}{dx}f_+(s_q, x) - \frac{d}{dx}f_-(s_q, x)f_+(s_q, x) \equiv 0 \\ &\Rightarrow f_-(s_q, x)df_+(s_q, x) = df_-(s_q, x)f_+(s_q, x) \\ &\Rightarrow f_-(s_q, x) = A(s_q)f_+(s_q, x) \end{aligned} \tag{2.23}$$

The vanishing of the Wronskian implies that we have found values of s (and therefore of ω) where there exists a solution $f_-(s,x) = A(s_q)f_+(s,x)$ that satisfies the conditions at both boundary conditions simultaneously and it is called *quasi eigenfunction*. We therefore regard the roots of the Wronskian (or respectively the poles of the Green's function) as the quasinormal modes of the Regge-Wheeler potential! Thus, we have finally found that the corresponding spacetime solution defined for compact initial Cauchy data, is of exponential decay and has the form

$$Q(t,x) = \sum_q c_q e^{s_q t} f_+(s_q, x) \tag{2.24}$$

As we shall later see there is an infinite number of such roots for $\mathrm{Re}(s) < 0$. It has to be noticed, however, that these roots are not the only singularities of the Green function. Indeed, although the roots are confined in the left half-plane, Leaver in [8] found that there is an essential singularity at $s = 0$ that at late times leads to a polynomial decay in the solution, of the form $Q(t,x) \sim t^{-2l-3}$ (see also Figure (2.2)).

2.4.3 Construction of the Solutions

We will now seek for the solutions of the homogeneous part of equation (2.12). In the standard radial Schwarzschild coordinate r the homogeneous part reads

$$r(r-1)\frac{d^2 f(s,r)}{dr^2} + \frac{df(s,r)}{dr} - \left[\frac{s^2 r^3}{r-1} + l(l+1) - \frac{\tau}{r}\right] f(s,r) = 0 \qquad (2.25)$$

For simplicity the radial coordinate r has been scaled such that $2M = 1$. In the new form the homogeneous part is a second-order ordinary differential equation known as *generalized spheroidal wave equation*. In the region of interest equation (2.25) has a regular singular point at $r = 1$ (in tortoise coordinates, $x = -\infty$) and an irregular singular point at $r = \infty$ ($x = \infty$). For the above stated form of the wave equation obeyed by f, the linearly independent solutions f_- and f_+ have the following series expansions accordinh to ([8])

$$\begin{aligned} f_-(s,r) &= (r-1)^s \sum_{n=0}^{\infty} a_n (r-1)^n, \quad \text{with} \quad f_-(s,r) \overset{r \to 1}{\to} (r-1)^s \\ f_+(s,r) &= e^{-sr} r^{-s} \sum_{n=0}^{\infty} a_n r^{-n}, \qquad \text{with} \quad f_+(s,r) \overset{r \to \infty}{\to} e^{-sr} r^{-s} \end{aligned} \qquad (2.26)$$

For Re(s)>0, f_- is bounded and converges absolutely for $|r-1| < 1$. The analytic continuation and convergence of f_- in the left-half plane is also trivially. Thus, f_- is analytic in s, both in the right-half and left-half domain. The picture, however, is not the same for the solution f_+. The asymptotic expansion of f_+, for Re(s)>0, although it is bounded as $r \to \infty$, it does not converge for any value of r. What one should do mathematically in order to construct a well-defined solution is to integrate f_+ over large values of r, for various initial conditions, until he obtains a solution that stays bounded at infinity. Then, the well-behaved solution should be calculated for many s in the right-half plane until those that converges absolutely to be found and then to continued analytically into the left-half plane. Of course for the computation of the quasinormal modes, we do not have to construct $f_+(s,r)$ explicitly. It is sufficient to find the roots of $f_-(s,r)$. Then, according to (2.23), for those roots the Wronskian is zero and therefore the Green's function has a singularity that corresponds to a quasinormal frequency (for a detailed description see [6]).

2.5 Numerical Results

Nollert's method for calculating the quasinormal modes that we described previously is straightforward, accurate, it is not based on any approximations and its mathematical foundation is rigorous (cf. Figure 2.3 for numerical results). All expressions for the computation of the zeros of the Wronskian are given in terms of converged series. In the same way the Wronskian is expressed as a series of the form

$$W_n(r) \equiv K_n w(r) \qquad (2.27)$$

It turns out, however, that the series K_n may first grow very large before approaching its limiting value. This effect becomes stronger as the real part of s grows more negative. While this obstacle may be overcome, it precludes the evaluation of frequencies with Re(s)< -6.

There are many other numerical methods for computing the quasinormal modes in the left-half plane. The most popular are the *WKB method* originally introduced by Schutz and Will [8] and the *continued fractions* pioneered by Leaver, with the latter to be the most reliable. We will now proceed to the development of those techniques as well as to the representation of numerical results.

2.5.1 First-order WKB approximation

This semianalytic technique is based on the similarity between the wave equation describing the perturbations of the Schwarzschild black hole and the Schrödinger equation of a particle encountering a potential barrier. It yields a simple analytic formula that gives the real and imaginary part of the frequency in terms of the parameters of the black hole, namely he mass, and of the field whose perturbation is under study, and in terms of the quantity $(n+\frac{1}{2})$, where $n = 0, 1, 2, \ldots$ labels the fundamental mode, the first overtone mode, and so on.

Recalling (2.4), we rewrite the wave equation into the form

$$\Psi'' + S(\omega, r)\Psi = 0 \tag{2.28}$$

In the first order of approximation the WKB method leads to the following condition satisfied by a quasinormal mode frequency

$$2S_0 \left(\frac{d^2 S_0}{dx^2} \right)^{-1/2} = i\Big(n + \frac{1}{2}\Big) \tag{2.29}$$

Since S is frequency dependent, this condition will lead to discrete, complex values for the quasinormal mode frequencies, that is

$$(2M\omega)^2 = \left(1 - \frac{2}{r}\right) \left[\frac{l(l+1)}{r_0^2} + \frac{2\tau}{r_0^3} \right] + i \left[\frac{d^2 V_0}{dx^2} \right]^{1/2} \left(n + \frac{1}{2}\right) \tag{2.30}$$

where S and V are calculated in r_0, i.e., the peak of the Regge-Wheeler potential,

$$r_0 = \frac{3}{2l(l+1)} \{ l(l+1) - \tau + [l^2(l+1)^2 + l(l+1)(14/9)\tau + \tau^2] \}^{1/2}$$

and the second derivative of the potential $V(x)$ with respect to x is given by

$$\frac{d^2 V}{dx^2} = \left(1 - \frac{2}{r}\right) \left[- \frac{l(l+1)}{r^4} \right] \left(6 - \frac{40}{r} + \frac{60}{r^2} - \frac{2\tau}{r^5}\right) \left(12 - \frac{70}{r} + \frac{96}{r^2}\right)$$

Using the first-order WKB approximation we can compute the fundamental modes, $n = 0$, for different values of l with great accuracy, in comparison with the

	$l = 2$	$l = 3$	$l = 4$	$l = 5$
Re($2M\omega$)	0.7776	1.2332	1.6446	2.0458
Im($2M\omega$)	-0.1766	-0.1846	-0.1878	-0.1894

Table 2.1: *Representative Schwarzschild fundamental mode frequencies ($n = 0$) for gravitational perturbations ($\tau = -3$), calculated using the standard WKB method.*

numerical results of Chandrasekhar and Detweiler in [10]. The numerical results for gravitational perturbations for various values of the angular momentum index l are listed in Table 2.1. The WKB method gives the fundamental modes of Schwarzschild black hole with great accuracy and allows the calculation of modes with large l, providing an analytic formula for the large-l limit and it can be extended to higher-orders providing thereby better accuracy. However it fails at the calculation of modes with large imaginary parts.

2.5.2 Continued Fractions

This method derives from a technique, employed by Jaffè, for the determination of the electronic spectra of the H_2^+ ion in 1934. Leaver observed that the time-independent wave equation (2.4) is similar to Schrödinger equation for H_2^+ and adopted the technique for calculating the eigenvalues of the problem[3].

We have already mentioned that, the radial equation

$$r(r-1)\frac{d^2\Psi(s,r)}{dr^2} + \frac{d\Psi(s,r)}{dr} - \left[\frac{s^2r^3}{r-1} + l(l+1) - \frac{\tau}{r}\right]\Psi(s,r) = 0 \qquad (2.31)$$

belongs to a certain type of differential equations, the *generalized spheroidal wave equations.* That type of equations permits solutions in the form of series expansions, thus, it is wise to consider a solution of (2.31) as a product of the previously defined solutions $f_-(s,r)$ and $f_+(s,r)$, namely

$$\Psi(s,r) \equiv f_-(s,r)f_+(s,r) = \underbrace{(r-1)^s r^{-2s} e^{-s(r-1)}}_{\text{due to boundary conditions}} \sum_{n=0}^{\infty} \alpha_n \left(\frac{r-1}{r}\right)^n \qquad (2.32)$$

where the series converges for $\frac{1}{2} < r < \infty$. We now substitute the solution (2.32) into the wave equation (2.31) and we obtain a three term recursion relation for the coefficient α_n

$$\alpha_n\alpha_{n+1} + \beta_n\alpha_n + \gamma_n\alpha_{n-1} = 0, \quad n = 0, 1, 2, \ldots \qquad (2.33)$$

[3]There is, however, a major difference between those two problems. The necessary boundary conditions for the determination of the eigenvalues of H_2^+ defines a self-adjoint problem, a picture that is not similar for our problem, though, it turns out that the method is numerically stable for each case.

where the first two terms of α_n are determined from

$$\alpha_0\alpha_1 + \beta_0\alpha_0 = 0 \tag{2.34}$$

The three recurrence terms α_n, β_n and γ_n are given from

$$\begin{aligned} \alpha_n &= n^2 + (2s+2)n + 2s + 1 \\ \beta_n &= -\{2n^2 + (8s+2)n + 8s^2 + 4s + l(l+1) + \tau\} \\ \gamma_n &= n^2 + 4sn + 4s^2 + \tau - 1 \end{aligned} \tag{2.35}$$

Hence, only the parameters of the perturbation determines the expansion coefficients α_n (remember that $s = i\omega$). Leaver's substantial contribution was the determination of under what circumstances the series expansion converges. If there is a particular conditions to be satisfied by the recurrence coefficients then, because of the ω, l and τ dependence in equations (2.35), we could extract a relation for calculating the quasinormal mode frequencies. Indeed, it turns out that the recurrence relation convergence only if for the terms α_n and α_{n+1} holds

$$\frac{\alpha_{n+1}}{\alpha_n} \overset{n\to\infty}{\rightarrow} 1 - \frac{(2s)^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{2s - \frac{3}{4}}{n} + \ldots \tag{2.36}$$

a relation that is valid if ω in s is a quasinormal mode frequency. In that case the ratio of two continued terms α_n is given by a continued fraction representation

$$\frac{\alpha_{n+1}}{\alpha_n} = -\cfrac{\gamma_{n+1}}{\beta_{n+1} - \cfrac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \cfrac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}\ldots}}} \tag{2.37}$$

Such a representation, after some manipulations, is sometimes written in the form

$$\frac{\alpha_{n+1}}{\alpha_n} = -\frac{\gamma_{n+1}\alpha_{n+1}\gamma_{n+2}\alpha_{n+3}\gamma_{n+3}}{\beta_{n+1} - \beta_{n+2} - \beta_{n+3} -} \ldots \tag{2.38}$$

and it can be considered as a boundary condition on n at ∞. This relation evaluated at $n = 0$ gives

$$\frac{\alpha_1}{\alpha_0} = -\frac{\gamma_1\alpha_1\gamma_2\alpha_2\gamma_3}{\beta_1 - \beta_2 - \beta_3} \ldots$$

while, from (2.34) we take

$$\frac{\alpha_1}{\alpha_0} = -\frac{\beta_0}{\alpha_0} \tag{2.39}$$

Thus, by equating the right-hand sides, we obtain the following expression for calculating the quasinormal frequencies

$$0 = \beta_0 - \frac{\alpha_0\gamma_1\alpha_1\gamma_2\alpha_2\gamma_3}{\beta_1 - \beta_2 - \beta_3} \ldots$$

This relation, however, is not suggested for deriving a large number of frequencies, meaning the asymptotic spectrum of the black hole. The topology of the equation does not allow us to locate all the complex roots. What we could do is to invert (2.36) n times so that we could separate the infinite continued fraction into a finite and an infinite part, i.e.,

$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n\alpha_{n-2}\gamma_{n-1}}{\beta_{n-1} - \beta_{n-2}} \cdots \frac{\alpha_0\gamma_1}{\beta_0}\right] = \left[\frac{\alpha_n\gamma_{n+1}\alpha_{n+1}\gamma_{n+2}}{\beta_{n+1} - \beta_{n+2} -} \cdots\right] \quad (n = 1, 2, 3 \ldots)$$

We then choose a truncation index $N = n$, usually large, and we sum the infinite part backwards instead of normal left-to-right summation. We repeat the same procedure for various truncation indices N until we track all the frequencies up to $n = N = 1$. It must be noticed that the two equation written above for evaluating the quasinormal frequencies are for every $n > 0$ completely equivalent as far as the roots are concerned. By changing, however, the value of N, namely the number of inversions, we change the topology of the complex plane and that help us to track all the roots of the infinite continued fraction explicitly; the n-th mode is found to be the most stable root of the n-th inversion.

Leaver's method that we just described works well and with fine accuracy. Later Nollert [11] improved Leaver's method and he managed to evaluate modes with $n \ggg 100 \Rightarrow$, $\mathrm{Im}(\omega) \ggg 100$ (n up to 2,000). Nollert, after calculating almost 100,000 frequencies, found that for large values of n, the frequencies are well fitted by the following verified asymptotic expression

$$(2M\omega_N) = 0.0874247 + \hat{\Omega}N^{-1/2} + i\left(\frac{N - \frac{1}{2}}{2} - \hat{\Omega}N^{-1/2}\right) + O^{-1}(N^{-1/2}) \quad (2.40)$$

where

$$\hat{\Omega} = \begin{cases} 0.4850 & \text{for } l = 2 \\ 1.067 & \text{for } l = 3 \\ 3.97 & \text{for } l = 6 \end{cases}$$

and the original units have been reintroduced. Having a formula for evaluating the asymptotic behavior of the quasinormal frequencies it is now easy to construct and study the frequency spectrum for the gravitational perturbations of the Schwarzschild black hole. The schematic representation of the spectrum for $l = 2$ and $l = 3$ is depicted in Figure (2.4) and the first 41 overtones for $l = 2$ and $l = 3$ are depicted in Table (2.2).

We see that solution with negative real part are also included, since they constitute physical solutions of the above equations. Leaver bears up the fact that there is an infinite number of frequencies, but there is no formal proof about this. The sure is the existence of algebraically "special" modes, that is modes with zero real part, within the error of the computer. Those almost purely imaginary solutions for each l were analytically predicted by Chandrasekhar and they are given by

$$(2M\omega_l) = \pm i\frac{(l-1)(l+1)(l+2)}{6} \quad (2.41)$$

	$l = 2$	$l = 3$
n = 1	(0.747343369, -0.177924631)	(1.198887000, -0.185406000)
n = 2	(0.693421994, -0.547829751)	(1.165287606, -0.562596227)
n = 3	(0.602106909, -0.956553966)	(1.103369802, -0.958185502)
n = 4	(0.503009924, -1.410296405)	(1.023923822, -1.380674192)
n = 5	(0.415029160, -1.893689782)	(1.103369802, -0.958185502)
n = 6	(0.338598806, -2.391216108)	(0.940348012, -1.831298785)
n = 7	(0.266504681, -2.895821253)	(0.862772957, -2.304302724)
n = 8	(0.185645101, -3.407681809)	(0.795319048, -2.791824485)
n = 9	(0.000000000, -4.000000000)	(0.737984552, -3.287689057)
n = 10	(0.126620794, -4.605291864)	(0.689236637, -3.788065608)
n = 11	(0.153114211, -5.121615413)	(0.647366263, -4.290797900)
n = 12	(0.165216340, -5.630858722)	(0.610921804, -4.794709101)
n = 13	(0.171477284, -6.137418749)	(0.578768197, -5.299159211)
n = 14	(0.174757778, -6.642428268)	(0.550038792, -5.803799142)
n = 15	(0.176472347, -7.146697858)	(0.524072319, -6.308438956)
n = 16	(0.177251065, -7.650230047)	(0.500359431, -6.812977185)
n = 17	(0.177354514, -8.153298127)	(0.478502441, -7.317362326)
n = 18	(0.177070030, -8.656072139)	(0.458186005, -7.821571443)
n = 19	(0.176514490, -9.158666975)	(0.439155978, -8.325598043)
n = 20	(0.175590405, -9.661043449)	(0.421204073, -8.829445112)
n = 21	(0.174592249, -10.16280325)	(0.404156586, -9.333121030)
n = 22	(0.174145883, -10.66479799)	(0.387866111, -9.836637115)
n = 23	(0.172596136, -11.16677533)	(0.372205434, -10.34000692)
n = 24	(0.172304623, -11.66814457)	(0.357059818, -10.84324503)
n = 25	(0.170455493, -12.16969337)	(0.342330863, -11.34636289)
n = 26	(0.170164121, -12.67176553)	(0.327916109, -11.84938531)
n = 27	(0.169461568, -13.17222851)	(0.313742821, -12.35231382)
n = 28	(0.167573456, -13.67327646)	(0.299681309, -12.85517029)
n = 29	(0.166194668, -14.17531786)	(0.285679084, -13.35801775)
n = 30	(0.165702871, -14.67714853)	(0.271676672, -13.86081457)
n = 31	(0.165376445, -15.17841554)	(0.257498222, -14.36354344)
n = 32	(0.164861843, -15.67944371)	(0.242962822, -14.86627972)
n = 33	(0.164125255, -16.18050526)	(0.227924284, -15.36910824)
n = 34	(0.163125419, -16.68169171)	(0.212212592, -15.87207622)
n = 35	(0.161710641, -17.18287254)	(0.195591813, -16.37519443)
n = 36	(0.159803793, -17.68353519)	(0.177789459, -16.87844243)
n = 37	(0.158150890, -18.18290567)	(0.158641741, -17.38186933)
n = 38	(0.158569419, -18.68181517)	(0.138176412, -17.88616173)
n = 39	(0.160292868, -19.18311345)	(0.115716028, -18.39361153)
n = 40	(0.159786738, -19.68669637)	(0.075295600, -19.41554504)
n = 41	(0.155451285, -20.18805815)	(0.000258865, -20.01565305)

Table 2.2: *The first* 41 *quasinormal frequencies for* $l = 2$ *and* $l = 3$ *of Schwarzschild black hole.*

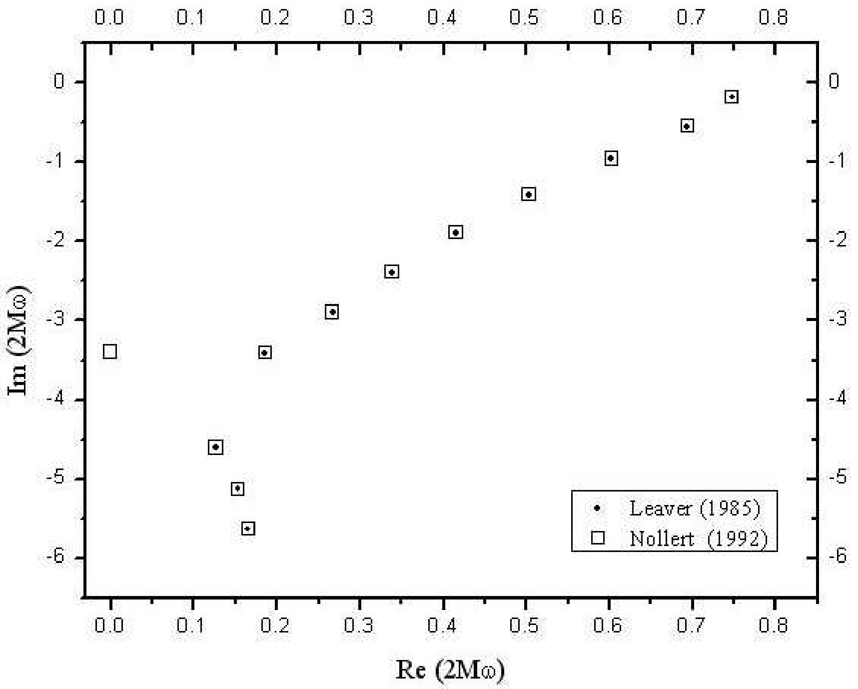

Figure 2.3: *Comparison between the first 12 quasinormal frequencies for $l = 2$ as calculated by Nollert (zeros of K_n) and Leaver (continued fractions). Although the divergences are neglected the later technique is always preferable since it allows the reliable derivation of frequencies with n up to 100.000, while it is fast as well. Note that for $n = 9$ Nollert's method does not give results.*

For $l = 2$ the special mode is located at (0.000, 3.998) and for $l = 3$ at (0.0002, 20.0157). As it is expected from the asymptotic expression (2.40), as the order of the mode increases the real part of the frequency tends to an l-independent constant value, something significant as we will see directly, and the imaginary part increases to high values proportionally to n. Does this finite limit for the real parts have any physical significance? That the behavior of all highly damped quasinormal frequencies is uniquely determined, expresses a suspicion that there must be a more fundamental relation between the response of a black hole to a general perturbation and the parameters of the black hole itself. Indeed, in 1998 Hod (see [13]) observationally and Motl (see [12]) after analytical manipulations of the infinite continued fraction derived by Leaver, related the high frequency external response of a black to a purely quantum property of the black hole, the *Hawking temperature* T_H. More precisely, the numerical results combined with the analytical calculations seems to suggest that

$$\omega_n = T_H \ln 3 + 2\pi T_H i(n + \frac{1}{2}) + \mathcal{O}(n^{-1/2}) \tag{2.42}$$

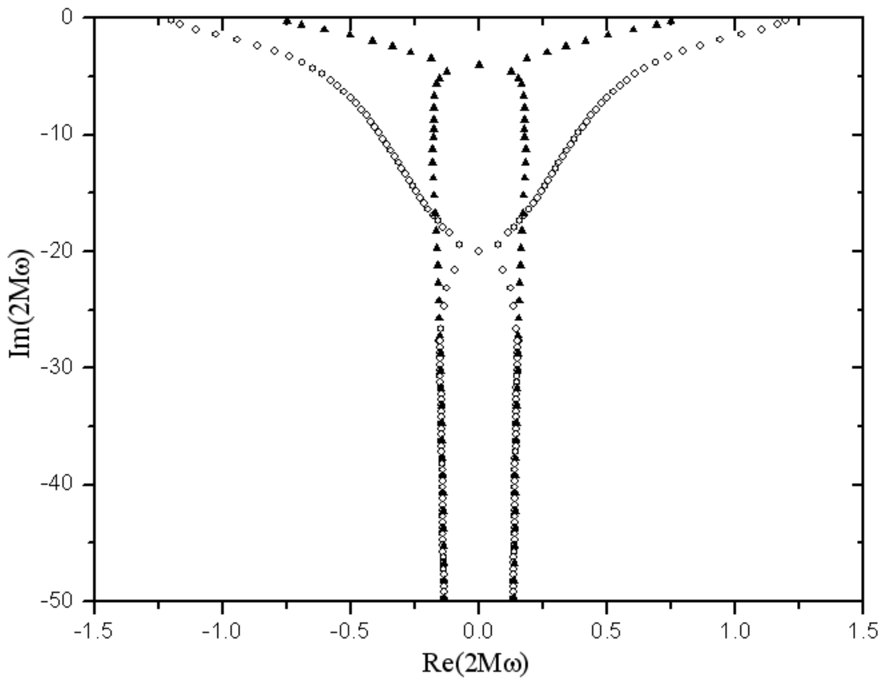

Figure 2.4: *Quasinormal modes of gravitational perturbations ($l = 2$ and $l = 3$) for large imaginary parts (up to* $\mathrm{Im}(2M\omega) = -50$*).*

where the Hawking temperature is given by

$$T_H = \frac{\hbar}{8\pi M} \tag{2.43}$$

or if we again chose our magnitudes to be expressed in physical units, i.e., $2M = 1$ and $\hbar = 1$, simply one reads

$$T_H = \frac{1}{4\pi} \tag{2.44}$$

The remarkable notations here are two. Firstly, in the large n-limit the imaginary parts of the ω_n are equidistantly spaced and thus this could be of relevance for a quantum description of black holes. Secondly, the real part of ω_n,

$$\omega_R \equiv T_H \ln 3 = \frac{\ln 3}{8\pi GM} \approx 8.85 \frac{M_\odot}{M} kHz \tag{2.45}$$

introduces a universality, since it is n and l-independent, and it can be considered from now own as a property of the black hole itself. It must be pointed out that these results and conclusions are the most significant ones of our so far analysis and they will constitute the key of the foundation of the following chapter.

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Chapter 3

Loop Quantum Gravity

3.1 Motivation

The aim of the last chapter of this report is to review the basis and the main results of a newly-born issue in the area of quantum gravity, the one that deals with the application of the quasinormal mode hints in the foundation of a possible quantum theory of gravity. To illustrate this let us recall first some basic ideas and principles of General Relativity.

From Newton to the beginning of this century, physics has had a solid foundation in a small number of key concepts such as space, time, causality and matter. In spite of substantial evolution, these concepts remained rather stable and self-consistent. In the first quarter of this century, Quantum theory and General Relativity have modified this foundation in depth. Before Albert Einstein postulate General Relativity, scientists thought of space as a fixed background, an independent entity, that has a geometrical structure - something like a "stage" where matter could travel independently. The picture changed slightly when Maxwell and Faraday introduced the concept of the *field*. The fields have the property that they can move through the space. They where first described as lines that filled up the space due to the presence of electric or magnetic charges. Later Maxwell and Faraday realized that the presence of those charges were not necessary; the fields constitute independent entities that can exist alone in space. In this way they reformulate the laws of electricity and magnetism by introducing the electric and magnetic field. The concept of the field later became even more appealing! Nowadays the fundamental description of nature simply reduces to the existence of various types of fields, with the characteristic property of undergoing quantum fluctuations, that is a temporary change in the amount of energy in a point in space, arising from Heisenberg's uncertainty principle. The usefulness of the fields is such that the fundamental forces are described by fields called *Yang-Mills*, the elementary particles by the *fermionic* fields, while massive particles, that rises from the *Higgs* particles, are described by the *scalar* fields. These fields are the elements of the *Quantum Field Theory*, a theory that attempts to unify the three (weak, strong and electromagnetic) fundamental forces of nature with the fourth fundamental force (i.e., gravity) and depends on particle fields embedded in the flat, Minkovskian spacetime of Special Relativity.

However, the scheme change dramatically when we study the gravitational field in General Relativity. General Relativity not only changed the way we understand gravity as a force but also reintroduced the concept of spacetime. Although Einstein originally tried only to describe gravity with terms of fields,

Why Quantum Field Theory?

Quantum Field Theory replaces the conventional Quantum mechanics in domains, smaller than atomic and molecular systems, where energy can manifest itself in a variate of ways, e.g., pair creation and annihilation or particles moving at relativistic speeds. Therefore, the Schrödinger equation has to be replaced by field equations such as the *Klein-Gordon* equation (for particle with no spin) or *Dirac* equation (for particle with spin $\frac{1}{2}$). These field equations are invariant (unchanged) under a change of the spacetime coordinate system (Lorentz transformation). It is referred to as relativistic invariance, which ensures the validity of the field equation at relativistic speed. To account for the creation/annihilation of particles in high energy interaction, the field is considered to be an operator. It is expanded into Fourier series in terms of harmonic functions and coefficients. These coefficients are then subjected to some quantization rules. Depending on whether the particle has integer or half integer spin, these operators satisfy the commutation or anti-commutation relations (for example: $ab + ba = 1$; the Pauli exclusion principle is guaranteed by quantization with the anti-commutation relations for spin $\frac{1}{2}$ particles). They are the creation and annihilation operators, which operate on state vectors describing the number of particles in different states. This is called the second quantization, which endows particle property to the field (field + second quantization = quantum field). Thus, in QFT the particles are just bundles of energy and momentum of the fields, which constitute the basic ingredient.

in order to be consistent with Special Relativity, he eventually found that the gravitational field that he had introduced was actually the background space that Newton had considered. Now spacetime or equivalently the gravitational field is a dynamical entity that changes as matter moves within it. While easy to embrace in principle, this is the hardest idea to understand about General Relativity, and its consequences are profound and not fully explored, even at the classical level. To a certain extent, General Relativity can be seen to be a relational theory, in which the only physically relevant information is the relationship between different events in spacetime.

The two theories, General Relativity and Quantum Field Theory, have obtained great success and vast experimental confirmation, and can be now considered as established knowledge. Each of the two theories modifies the conceptual foundation of classical physics in a (more or less) internally consistent manner, but we do not have conceptual foundation capable of supporting both theories. Such a foundation is called *quantum gravity*, and it is a theory that tries to unify Quantum mechanics and General Relativity in a physical regime in which it seems that both theories should be relevant, the regime of *Planck scale phenomena*, $10^{-33}cm$.

Any attempt to formulate a quantum gravity must incorporate an appropriate behavior of the gravitational field at the Plank scales. We know that in General Relativity space and time are considered as dynamical quantities, while spacetime location is relational only. This teach us that there is no fixed spacetime background, as found in Newtonian mechanics and Special Relativity. On the other hand, Quantum mechanics has taught us that any dynamical entity is subject to Heisenbergs uncertainty at small scales. Thus, even the gravitational field, in the Plank scale, should be subject to such a constraint. It seems that a quantum theory for gravitation should originate from the basic concepts of

Classical-Quantum inconsistencies

The fundamental principles of General Relativity and Quantum Field Theory seems to collide in the classical Einstein equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}}_{\text{Geometry (General Relativity)}} = \underbrace{8\pi T(g)_{\mu\nu}}_{\text{Matter (QFT, standard model)}}$$

These equations relate matter density in form of the energy momentum tensor $T_{\mu\nu}$ and geometry in form of the Ricci curvature tensor $R_{\mu\nu}$. Notice that the metric tensor $g_{\mu\nu}$ enters also the definition of the energy momentum tensor. However, while the left hand side is described until today only by a classical theory, the right hand side is governed by a Quantum Field Theory.

Quantum mechanics (or Quantum Field Theory) and General Relativity, and to conclude to a new synthesis, where space and time are deeply reshaped, in order to keep into account what we have learned from both our present “fundamental” theories and to unravel what take place in the Planck scale nature.

But what is the picture of quantum gravity nowadays? In modern physics, there have been established three different ways for approaching quantum gravity. Given that such a theory must be a unification of Quantum theory and General Relativity, the two of the three roads are not that unforseen. Hence, there is a path that starts from Quantum Field Theory and every idea and method that has been introduced is based exclusively on Quantum mechanics. There is also, as expected, another path that initiates from the basic principles of General Relativity and tries to modify them in order to include quantum phenomena. Each of those two roads has led to a well worked out and partially successful quantum theory of gravity; the first road gave birth to *String Theory*, while the second to the so-called *Loop Quantum Gravity* (or *Quantum General Relativity*). In the following sections we shall explore the basic remarks of those two elegant theories. Especially for the Loop Quantum Gravity we shall explore some intriguing features that arise from the classical black hole perturbation theory, which is actually the subject of this report.

Before that, it would be of great interest to explore which is the third road. Thus, apart from the first two roads, there is always another road for reaching quantum gravity. This path has been followed by scientists that threw down Relativity and Quantum mechanics as starting points. They decided that the new theory must incorporate and treat fundamental concepts such us the nature of time or the origins of the universe. While sometimes based on well established theories, scientist in this road do not hesitate to invent pioneering mathematical formalisms and abstract structures. By using these formalisms and accumulating of as many hints as possible from the known theories, they attempt to build a new theory from the ground up.

This approach has been already given prominent results concerning black holes, and so far the most important and encouraging are the *Area Theorem* and the Hawking temperature for black holes argued by Bekenstein and established by Hawking. Incredibly innovative, Bekenstein’s and Hawking’s results established the physical and mathematical foundations of the so far phenomenological area of *black hole thermodynamics*. Black hole thermodynamics, or quantum black hole thermodynamics after the contribution of Bekenstein and Hawking, leads to a series of significant conclusion about black holes when treated as physical systems. In this report we devote an individual section to that aspect, not only to catch on fundamental concepts, such as the Hawking temperature, but also to understand how black holes behave as dynamical physical objects that undergo the same processes that usual thermodynamical systems also undergo.

3.2 Black Hole Thermodynamics

Bekenstein, in the concepts of classical General Relativity, argued that a black hole should have a temperature T which is proportional to its *surface gravity* κ and an entropy S that is proportional to its area A. Using Quantum Field Theory on curved spacetime, Hawking discovered later a mechanism by which black holes have non-zero temperature. His formulation was based on the disquiet of a quantum mechanical vacuum state; that is, quantum fluctuations, namely, particle and anti-particle pair production and annihilation is continually occurring in empty space. By the Heisenberg uncertainty principle, the non-conservation of energy involved in creating the pair is balanced by the shortness of the time span, after which the pair annihilates. In modern theory, the vacuum more closely resembles a "quantum foam", as inspiringly described Wheeler, than it resembles empty, static space. Hawking assumed an initial vacuum field on the Schwarzschild metric, and calculated the particle mode populations at infinity at late times. He found a distribution that correspond to a perfect black body spectrum, with a temperature of

$$T_H = \frac{\kappa\hbar}{2\pi} \tag{3.1}$$

where κ in the case of the Schwarzschild black hole equals $1/4M$. That is, in the presence of the strong surface gravity just outside the event horizon, virtual pairs are separated, and one particle is absorbed while the other is radiated away from the black hole. The surface gravity, relates to the magnitude of the force exerted on a test particle at a particular position on the event horizon and simply expresses the local gravitational field strength at the surface of the black hole. It is defined only for a static or stationary spacetime, that is, whenever the event horizon of the black hole is a Killing horizon. Hence, the stronger the surface gravity is, the higher will be the observed temperature. The introduction of Hawking temperature resolved many flaws of classical black hole thermodynamics and constituted a transition to quantum black hole thermodynamics. Actually the classical black hole thermodynamics is just an analogy. It relies on the fact that energy can flow not just into black holes but also out of them. This is suggested by the famous *Penrose process*, a classical process where one can exploit the existence of the ergoregion in the rotating[1] black holes to extract rotational energy. Thus, a black hole can be considered as an ordinary thermodynamical system.

Rather than jumping now into the subject of quantum black hole thermodynamics, it would be worth discussing the aspects of the classical theory. These are important in their own right, but also it is intriguing to see what it can be inferred for our analysis without invoking to much the quantum theory.

At first sight it seems that we cannot correlate any classical concept of thermodynamics, such as the temperature or the entropy, to black holes. As a consequence, it seems to be even more peculiar how could a whole theory

[1] This is true also for non-rotating charged black holes: one can extract energy from them by neutralizing them.

of black hole thermodynamics to be established from classical definitions. As far as the temperature is concerned we saw that, as defined for black holes, it originates from a purely quantum effect; this has nothing to do with the classical definition of temperature, where it just reflects the internal energy of motion and vibration of the molecules of substance. However, the definition seems to work well. The proportionality relation between the surface gravity and the temperature leads to a thermal spectrum identical to that of a black body. But what are the physical implications of entropy? Is it possible to correlate the classical definition with physical (or statistical) properties of the black hole? The answer is yes, and it can be proved easily from simple quantum arguments concerning the properties of matter and energy[2].

Let us consider the simple case of a singularity in the center surrounded by a spherical event horizon. We know that when a black hole is created by a collapsing star that the ordinary matter is crushed out of existence. However the total mass-energy remains. Outside the event horizon all properties of that matter are gone except for the total mass-energy, rotation, and electric charge, where the total mass-energy is manifested as the curvature of spacetime around the singularity. This is sometimes called the *Black hole has no hair* theorem.

It is plausible, however, to consider representing all the mass-energy inside the black hole as waves, as Quantum mechanics predicts. Now, what kind of waves are possible to be generated inside the black hole? The answer is standing waves, waves that “fit” inside the black hole with a node at the event horizon. The possible wave states are very similar to the standing waves on a fixed string, a system that we presented in the preceding chapter. The energy represented by a particular wave state is related to the frequency and amplitude of its oscillation. The higher overtones have a higher frequency and thus these quantum mechanical waves contain more energy. If we assume that the total mass-energy inside the event horizon is fixed, then we shall have various standing waves, each with a certain amount of energy, and the sum of the energy of all these waves equals the total mass-energy of the black hole. There are a large number of ways that the total mass-energy can distribute itself among the standing waves. We could have it in only a few high energy waves or a larger number of low energy waves.

It turns out that all the possible standing wave states are equally probable. Thus, we can calculate the probability of a particular combination of waves containing the total mass-energy of the black hole the same way we calculated the probability of getting various combinations for dice. But we know that the entropy is just a measure of the probability. Thus we can calculate the entropy of a black hole. We also know that the entropy measures the heat divided by the absolute temperature. The “heat” here is just the total mass-energy of the black hole, and if we know that and we know the entropy, we can calculate a temperature for the black hole. So, as Hawking realized, we can apply all of thermodynamics to a black hole. Indeed, the introduction of the entropy of a black hole leads to the formulation of, as in classical Thermodynamics, the four

[2]The approach presented here is called the “Holographic Principle”

laws of black hole thermodynamics.

3.2.1 Zeroth Law

The Zeroth Law of black hole mechanics defines an equilibrium state for black holes. One expression of it states that any static or any stationary and axisymmetric black hole has a uniform surface gravity κ over the event horizon. By definition, κ is constant over each null geodesic on the event horizon; this law means that κ has the same value on every such null geodesic. Physically, the Zeroth Law defines a state of thermal equilibrium for a black hole. As we have seen, the Hawking temperature is proportional to κ, thus, κ is effectively the temperature of a black hole; hence, uniform surface gravity over the event horizon means a uniform temperature. This is readily comprehended when considering the symmetry of the relevant class of black holes, for which the event horizon is a Killing horizon.

3.2.2 First Law

The First Law of black hole thermodynamics relates perturbations in the mass to perturbations in area and angular momentum. In form, it very closely resembles the conservation of energy expressed in the First Law of thermodynamics (cf the following box) and, for a rotating charged black hole is stated as follows

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta Q \tag{3.2}$$

where J and Q are the angular momentum and the charge of the black hole and Ω and Φ the angular velocity and electric potential of the horizon. By simply using the law of entropy of a black hole one readily find another form of the first law as a function of the Hawking temperature and entropy

$$\delta M = T_H\delta S + \Omega\delta J + \Phi\delta Q \tag{3.3}$$

The surface gravity κ evidently plays here the role of temperature As mentioned, one can see similarities between (3.3) and the First Law of thermodynamics. For a stationary black hole κ, Ω and Φ, although locally defined on the horizon, are always constant over the horizon. This is an emanation of the Zeroth Law.

However, an even greater correlation is evident when the concept of entropy is introduced for a black hole. To first order, the entropy can be calculated to be directly proportional to the area; equation (3.3) then becomes

$$\delta M = \frac{\kappa}{2\pi}\delta S + \Omega\delta J \tag{3.4}$$

That is, the entropy of a black hole is calculated as

$$S = A/4$$

Equation (3.4) not only lends itself to comparison with the First Law of thermodynamics, it also highlights the correlation between the surface gravity and the temperature of a black hole, as the factor proportioning the entropy variation term.

3.2.3 Second Law (Area Theorem)

The Second Law is of course Hawking's area theorem, stating that the horizon area can never decrease. The Third Law also has an analog in black hole physics, namely, the surface gravity of the horizon cannot be reduced to zero in a finite number of steps.

Hawking proved that suggestion by assuming the null energy conditions in (3.3). Under these constraints, the Raychaudhuri equation implies that the expansion, θ, of the congruence of null geodesics (in part, those comprising the event horizon) will tend to infinity if it becomes negative. Hence, Hawking reasoned that θ is always non-negative, and thus the area of a black hole never decreases. In analogy to the Second Law of thermodynamics, the Area theorem

The Classical Laws of Thermodynamics

- **Zeroth Law**
  The Zeroth Law defines the thermodynamic equilibrium of systems. It states that if two systems are each in thermal equilibrium with a third system, then the two much be in thermal equilibrium with each other. Although seemingly obvious, this law lays the foundations on which the others are developed.
- **First Law**
  The First Law is an expression of the law of conservation of energy. It states that the increase in internal energy of a system is equal to the sum of the heat flow into the system and the work done on the system. For isolated systems, which have no interaction with their surroundings, this law implies that their internal energy is constant (such is the case for the universe). For interacting systems, the First Law can be expressed mathematically as
  $$dE = TdS - PdV$$
  In the above equation, T is the temperature of the system, S is the entropy of the system (see below), and the last term represents mechanical work in the form of forced compression or expansion done on the system.
- **Entropy and the Second Law**
  The Second Law applies to the entropy of a system. Entropy is a measure of disorder; and, disorder can take various forms, depending on the system. For instance, for particles in a box, the most ordered configuration has all the particles localized in a particular region.

As such, diffusion of a gas can be understood as a process that increases the entropy of the system. In statistical mechanics, entropy is defined as the natural logarithm of the density of states at a particular energy, as shown

$$S = \ln n$$

Entropy also relates to the reversibility of a process. A process that conserves the entropy of a system is reversible; and, dynamic equilibrium in chemical reactions is an example of an entropy-conserving process. The Second Law of thermodynamics describes the change of the entropy of a system. For an isolated system, the entropy is never decreasing. For a non-isolated system, the change in entropy is related to the heat flow across the system boundaries, as follows

$$dS = \frac{dQ}{T}$$

Since the universe is an isolated system, the Second Law implies that the entropy of the universe is never decreasing (moreover, because we have not reached a state of maximum disorder in the universe thermodynamically speaking, at least the entropy of the universe is continually increasing). This is the most famous expression of the Second Law, and its relation to black hole mechanics is very important. The equation below expresses the Second Law, as applied to the universe

$$dS_{universe} \geq 0$$

is expressed as

$$dA \geq 0 \tag{3.5}$$

3.2.4 The Third Law

The Third Law of thermodynamics, does not apply to black holes. There are, in fact, extremal black holes, for which the surface gravity is zero, but which have a non-zero area (i.e., entropy). It has been postulated, however, that the Third Law of thermodynamics is not, in fact, a fundamental law; rather, it is a consequence of the make up of many common systems. In fact, certain quantum mechanical systems have been shown to violate this law.

Generalized Second Law (GSL): One major past question surrounding the existence of black holes was what happens to the entropy of matter that falls in. When black holes were shown to possess a quantity serving as entropy, Bekenstein proposed an alteration to the Second Law of thermodynamics, as follows

$$\Delta S' \geq 0 \text{ where } S' = S + S_{bh} \tag{3.6}$$

That is, Bekenstein redefined the entropy of the universe to be the sum of the entropy of everything outside the black hole and that of the black hole. He proposed that this quantity is never decreasing.

Difficulties for the Area Theorem

Hawking radiation presents a problem for the Area Theorem, because when the black hole loses energy as radiation, it decreases in mass, which causes a decrease in area. A potential way around this dilemma is presented by the Generalized Second Law. Rather than the entropy of a black hole and the entropy of the matter outside it both individually constrained to be increasing, the sum of these is considered to increase. In fact, the loss in area due to Hawking radiation is balanced by a corresponding increase in the entropy outside, due to the particles being radiated. Hence, although the Area Theorem may be invalid, the thermodynamic implications of it are not necessarily so. The Generalized Second Law, itself, in part relies on Hawking radiation for validity. A possible refutation of the GSL involves lowering matter slowly into a black hole, such that no energy is transferred to it. In this ideal scenario, the area of the black hole would not increase, hence neither would its entropy, but the entropy of the matter would still be lost. A resolution to this problem involves the thermal atmosphere of a black hole. Trying to lower matter slowly through the temperature gradient surrounding the black hole would put pressure on it, and in doing so, energy would be transferred as work. This would cause a slight increase in the black hole area just enough to account for the lost entropy of the in-falling matter. One consequence of Hawking radiation is the eventual evaporation of a black hole, by radiating away all its energy. The lifetime of a black hole is theorized to be finite, and can be calculated for a number of common black hole geometries.

3.3 Loops, Networks and Spinfoam

After having explore the more issues of black hole thermodynamics let us return back to our main topic, the formulation of the quantum gravity. Up until now physicist have developed a large collection of theories that are treated as possible candidates for quantum gravity. Among of them, are theories with names, such as non-commutative geometry, supergravity and twistor theory. A more popular approach, however, is the String Theory.

According to the String Theory, the fundamental constituents of the material world are not point-like elementary particles, but tiny one-dimensional strings having a length of about $l_p = 10^{-33} cm$ (the Planck length) that lie on a elevendimensional spacetime. Like the string of a violin, they can vibrate in many different ways (different modes), which correspond to the different elementary particles observed in nature. It is a quantum theory that incorporates gravity naturally. In its larger framework of *M-theory*, the strengths of all the four fundamental forces merge together at very small distance ($10^{-33}cm$).

There are two classes of strings, those with ends (open strings), and those without (closed strings, i.e., a loop). The particles associated with the open strings are the spin-1 gauge bosons and fermions. Their movement is restricted on the surface of a membrane by some boundary conditions. Graviton with spin-2 is an example of closed string, which can travel freely in all spatial dimensions.

When a point particle moves through spacetime, it follows a geodesic and sweeps out a one-dimensional curve which is referred to as its world-line. However, when a string propagates through spacetime, it sweeps out a two-dimensional surface which, by analogy, is called its world-sheet, and moves along a surface of minimum area. When supersymmetry is incorporated into the original String Theory, it resolves the problem with tachyon (square of mass is negative), accommodates the ferminonic vibrational pattern, and merge General Relativity with Quantum mechanics. The Theory of Strings then becomes the *Theory of Superstrings*.

There is, however, and another more consistent and best-developed alternative approach, which is the subject of this report, that seems to be more promising and reliable than String Theory: the Loop Quantum Gravity. Loop Quantum Gravity is a mathematically well-defined and non-perturbative quantization of General Relativity in four-dimensions, that serves the minimality of structures. Minimality here means that one explores the logical consequences of bringing together the two fundamental theories of modern physics, without adding any experimentally unverified additional structures (as in String Theory) such as extra dimensions, extra symmetries or extra particle content beyond the standard model. Loop Quantum Gravity therefore is, by definition, not a unified theory of all interactions in the standard sense since such a theory would require a new symmetry principle. However, it unifies all presently known interactions in a new sense by quantum mechanically implementing their common symmetry group, the four-dimensional diffeomorphism group, which is almost completely broken in perturbative approaches.

Loop Quantum Gravity is an attempt to quantize gravity that does not make any assumptions beyond the experimentally well tested principles of General Relativity and Quantum theory. In particular it relies on two key principles of General Relativity and performs calculation with standard techniques of quantum mechanics. The first key principle is already known: the background independence. Although we have explained what is the background independency, it is worth mentioning the fact that the String Theory, as currently formulated, is not background independent; the equations describing the strings are set up in a predetermined classical (that is, non-quantum) flat spacetime.

The second principle, known as name diffeomorphism invariance, is closely related to background independence. This principle implies that the choice of

Loops vs. Strings

The main merits of String Theory are that it provides an elegant unification of known fundamental physics, and that it has a well defined perturbation expansion, finite order by order. Its main incompletenesses are that its non-perturbative regime is poorly understood, and that we do not have a background-independent formulation of the theory. In a sense, we do not really know what the theory we are talking about is. Because of this poor understanding of the non perturbative regime of the theory, Planck scale physics and quantum gravitational phenomena are not easily controlled: Except for a few computations, there has not been much Planck scale physics derived from String Theory so far. There are, however, two sets of remarkable physical results.
The first is given by some very high energy scattering amplitudes that have been computed. An intresting aspect of these results is that they indirectly suggest that geometry below the Planck scale cannot be probed and thus in a sense does not exist in String Theory. The second physical achievement of String Theory (which followed the *d-branes* revolution) is the recent derivation of the Bekenstein-Hawking black hole entropy formula for certain kinds of black holes.
he main merit of Loop Quantum Gravity, on the other hand, is that it provides a well-defined and mathematically rigorous formulation of a background-independent, non-perturbative generally covariant Quantum Field Theory. The theory provides a physical picture and quantitative predictions of the world at the Planck scale. The main incompleteness of the theory is regarding the dynamics, formulated in several variants. So far, the theory has lead to two main sets of physical results. The first is the derivation of the (Planck scale) eigenvalues of geometrical quantities such as areas and volumes. The second is the derivation of black hole entropy for normal black holes.
Finally, strings and loop gravity may not necessarily be competing theories. There might be a sort of complementarity between the two. This is due to the fact that the open problems of String Theory are with respect to its background-independent formulation, and Loop Quantum Gravity is precisely a set of techniques for dealing non-perturbatively with background independent theories. Perhaps the two approaches might even, to some extent, converge. Undoubtedly, there are similarities between the two theories: first of all the obvious fact that both theories start with the idea that the relevant excitations at the Planck scale are one dimensional objects call them loops or strings!

coordinates to map spacetime and express equations is totally free. A point in spacetime is defined only by what physically happens at it, not by its location according to some special set of coordinates (no coordinates are special). Diffeomorphism invariance is very powerful and is of fundamental importance in General Relativity.

The most revolutionary idea mediated by Loop Quantum Gravity is the change of our perception about space and time. The consideration of space and time as a continuous and smooth entity is now totally revised. Here space, as well as time, appears to be quantized. Although this seems to be an intuitive initial condition it is not. The discreteness of space and time reveals from calculation on the mathematical language of the theory. Indeed, when Smolin, Ashtekar, Rovelli and Jacobson, the founder fathers of Loop Quantum Gravity, started working on the quantization of gravity at the late 80's, they found that space is like atoms: it comes in distinct pieces or "quantum units of area and volume" (see Figure 3.1). We can call them as the quantum of area and volume, so long as we bear in mind that not all areas are integer multiples of this one – at least, not in the most popular version of the theory. This quanta determines the energy level spacing of the quantum states. The description of those quantum states is simple and arises from the definition of the Faraday's lines: a Faraday line can be viewed as a quantum excitation of the field in the presence of charges. In the absence of charges these lines must close and form a loop. Now in Loop Quantum Gravity the the quantum gravitational field is described in terms of these loops. That is, the loops are quantum excitations of the Faraday lines of the gravitational field.

The possible values of volume and area are measured in units of the Planck length[3]. The Planck length is also related to the strength of gravity, the size of quanta and the speed of light. It measures the scale at which the geometry of space is no longer continuous. Remember that the Planck length is very small: $10^{33} cm$. The smallest possible nonzero area is about a square Planck length, or $10^{66} cm^2$. The smallest nonzero volume is approximately a cubic Planck length, $10^{99} cm^3$. Thus, the theory predicts that there are about 10^{99} atoms of volume in every cubic centimeter of space. The quantum of volume is so tiny that there are more such quanta in a cubic centimeter than there are cubic centimeters in the visible universe ($\sim 10^{85}$).

However, space is more than just a collection of volume elements. Rather, the space is represented by diagrams, the so-called *spin networks* that represent the quantum states of area and volume (Figure 3.4). Each network carry some dots that are called "nodes", and represent the points where the loops intersect. A "link" of the net, i.e., the portion of loop between two nodes, indicates precisely the quanta of space that are adjacent to one another. Two adjacent elements of space are separated by a surface, and the area of this surface turns out to be quantized as well. In fact, it soon became clear that nodes carry quantum numbers of volume elements and links carry quantum numbers of area elements (Figure 3.2). But what does time represents in Loop Quantum Gravity? Space is replaced by a spin network and spacetime is therefore described by a history of spin networks. Time here is not thought continuous but rather as "ticks" of a clock, that last $10^{-43} sec$, i.e., the Plank time. The history of spin networks is called "spinfoam", and it has a simple geometrical structure. The history of

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[3]The value of that unit derives intuitionally from the following: The uncertainty principle states that in order to observe a small region of spacetime we need to concentrate a large amount of energy and momentum. However, General Relativity implies that if we concentrate too much energy and momentum in a small region, that region will collapse into a black hole and disappear. Putting in the numbers,we find that the minimum size of such a region is of the order of the Planck length about $1.6 \times 10^{33} cm$.

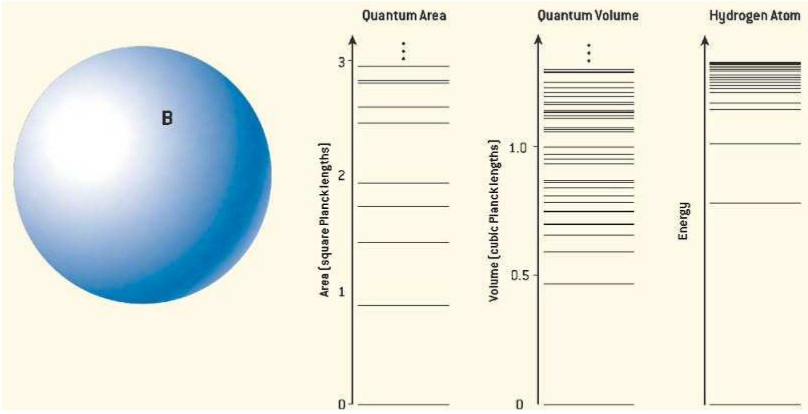

Figure 3.1: *Quantum states of volume and area: A central prediction of the Loop Quantum Gravity theory relates to volumes and areas. Consider a spherical shell that defines the boundary, B, of a region of space having some volume. According to classical (non-quantum) physics, the volume could be any positive real number. The loop quantum gravity theory says, however, that there is a nonzero absolute minimum volume (about one cubic Planck length, or* 10^{99} *cubic centimeter), and it restricts the set of larger volumes to a discrete series of numbers. Similarly, there is a nonzero minimum area (about one square Planck length, or* 10^{66} *square centimeter) and a discrete series of larger allowed areas. The discrete spectrum of allowed quantum areas (left) and volumes (center) is broadly similar to the discrete quantum energy levels of a hydrogen atom (right).*

a point is a line, and the history of a line is a surface. A spinfoam is therefore formed by surfaces called faces, which are the histories of the links of the spin network, and lines called edges, which are the histories of the nodes of the spin network (Figure 3.3). Now, the fundamental result of Relativity that time flows differently in different frames, is naturally incorporated in the physical result of the theory that time in the universe is represented by the ticking of innumerable clocks: every time a quantum of space moves in a spinfoam the clock at that location has ticked once!

3.4 Area and Entropy in Loop Gravity

By quantizing a theory, certain physical quantities take only discrete values, such as the energy levels in the hydrogen atom. Computing these quantized values involves solving the eigenvalue problem for the "operator" that represents a particular physical quantity. The volume of a region of space, or a certain number of loops, which in General Relativity is determined by the gravitational field. By solving the eigenvalue problem of the volume operator in Loop Gravity, it was found that the eigenvalues were discrete, that is, there are elementary

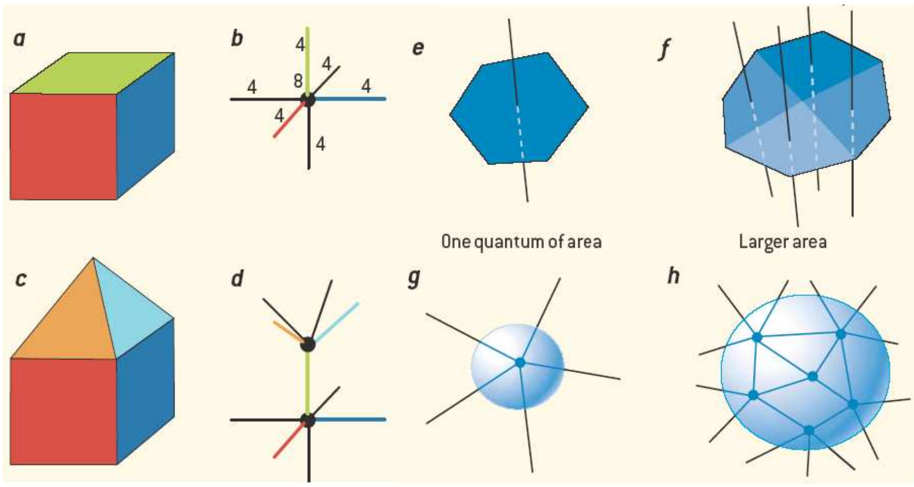

Figure 3.2: *Diagrams called spin networks are used by physicists who study Loop Quantum Gravity to represent quantum states of space at a minuscule scale. Some such diagrams correspond to polyhedra-shaped volumes. For example, a cube* (a) *consists of a volume enclosed within six square faces. The corresponding spin network* (b) *has a dot, or node, representing the volume and six lines that represent the six faces. The complete spin network has a number at the node to indicate the cubes volume and a number on each line to indicate the area of the corresponding face. Here the volume is eight cubic Planck lengths, and the faces are each four square Planck lengths. (The rules of loop quantum gravity restrict the allowed volumes and areas to specific quantities: only certain combinations of numbers are allowed on the lines and nodes.) If a pyramid sat on the cubes top face* (c), *the line representing that face in the spin network would connect the cubes node to the pyramids node* (d). *The lines corresponding to the four exposed faces of the pyramid and the five exposed faces of the cube would stick out from their respective nodes. (The numbers have been omitted for simplicity. In general, in a spin network, one quantum of area is represented by a single line* (e), *whereas an area composed of many quanta is represented by many lines* (f). *Similarly, a quantum of volume is represented by one node* (g), *whereas a larger volume takes many nodes* (h). *If we have a region of space defined by a spherical shell, the volume inside the shell is given by the sum of all the enclosed nodes and its surface area is given by the sum of all the lines that pierce it. The spin networks are more fundamental than the polyhedra: any arrangement of polyhedra can be represented by a spin network in this fashion, but some valid spin networks represent combinations of volumes and areas that cannot be drawn as polyhedra. Such spin networks would occur when space is curved by a strong gravitational field or in the course of quantum fluctuations of the geometry of space at the Planck scale.*

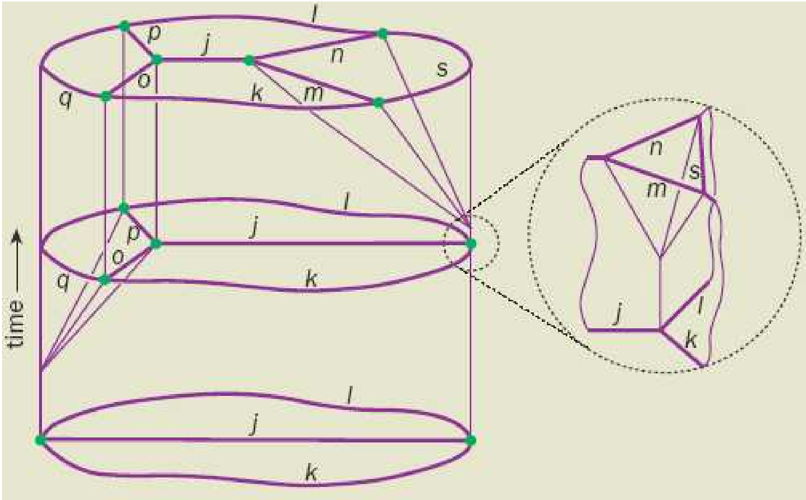

Figure 3.3: *Loop quantum gravity replaces the Newtonian concept of background space with a history of spin networks called a spinfoam. Each link in the network is associated with a quantum number of area called spin, which is measured in units related to the Planck length. Here a θ-shaped spin network (bottom) with three links carrying spins j, k and l evolves in two steps into a spin network carrying spins o, p, q, j, k, l, m, n and s (top). The initial spin network has two nodes where the three links meet, and the vertical lines from these nodes define the edges of the spinfoam. The first vertex which is similar to the vertex of a Feynman diagram is where the left edge branches off, at which point an intermediate spin network with spins o, p, q, j, k and l is formed. The edge on the right branches off in a second interaction vertex, which is enlarged. The "faces" of the spinfoam are the surfaces swept by the links moving in time. The enlargement shows that the vertex is connected to four edges and six faces with associated spins j, k, l, m, n and s. Spinfoams like this one can be thought of as a discretized quantum spacetime.)*

quanta of volume, or elementary grains of space. So far, calculations working strictly within the framework of Loop Quantum Gravity have been unable to determine this quantum of area. But now, thanks to work of Hod [13] and Dreyer [14], an innovative method of calculating the quantum of area have been shown to give an elegant answer: $4 \ln 3$ times the Planck area. This method uses semiclassical ideas from outside Loop Quantum Gravity: the formula for the frequencies of highly damped vibrational modes of Schwarzschild black hole. It is still completely mysterious why this method gives this answer. It could be a misleading coincidence, or it could be an important clue.

The importance of area in quantum gravity has been obvious ever since the early days of black hole thermodynamics. After the arguments of Bekenstein that the entropy of a black hole was proportional to its area, Hawking deter-

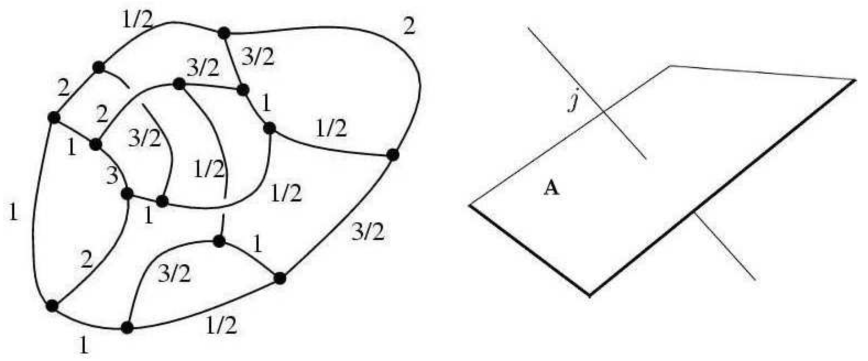

Figure 3.4: Left: *Elementary grains of space represented by the nodes on a spin network. The lines joining the nodes, or adjacent grains of space, are called links. Spins on the links (integer or half-integer numbers) are the quantum number that determine the area of the elementary surfaces separating adjacent grains of space. The quantum number of the nodes, which determine the volume of the grains are not indicated. The spins and the way the come together on the nodes can take on any integer or half-integer value, and they are governed by the same algebra as angular momentum in Quantum mechanics.* Right: *Area corresponding to a surface carried by a link labeled by j.*

mined the constant of proportionality, arriving at the formula

$$S = \frac{A}{4} \tag{3.7}$$

Understanding this formula more deeply has been a challenge ever since. Now, after the introduction of Loop Quantum Gravity, things seems to take a new turn, when Rovelli and Smolin showed that in Loop Quantum Gravity, area is quantized, with the spin networks to describe the geometry of space

Any surface gets its area from spin network edges that puncture it, and an edge labeled by the spin j contributes an area of

$$A = 8\pi l_P^2 \gamma \sqrt{j(j+1)} \qquad j = 0, 1/2, 1, 3/2 \dots \tag{3.8}$$

where γ is a dimensionless parameter called *Barbero-Immirzi* parameter. This parameter parameterizes an ambiguity in the choice of canonically conjugate variables that are used in the quantization. There is no a priori reason to fix the value of this parameter to any particular value.

Given this, it was tempting to attribute the entropy of a black hole to microstates of its event horizon, and to describe these in terms of spin network edges puncturing the horizon. After the pioneering work by Rovelli and Smolin, Krasnov noticed that the horizon of a non-rotating black hole could be described using a field theory called *Chern-Simons theory.*

He began working with Ashtekar and Corichi on using this to compute the entropy of such a black hole. They found out that the geometry of the event horizon is described not only by a list of nonzero spins j_i label the spin network edges that puncture the horizon, but also by a list of numbers m_i which ranges in $(-j_i, j_i)$ in integer steps. The intrinsic geometry of the horizon is flat except at the punctures, and the numbers m_i describes the angle deficit at each puncture. To count the total number of microstates of a black hole of area near A, we must therefore count all lists j_i, m_i for which

$$A \simeq \sum_i 8\pi\gamma\sqrt{j(j+1)} \tag{3.9}$$

It turns out that for a large black hole, the great majority of all microstates come from taking all the spins to be as small as possible. So, we can just count the microstates where all the spins j_i equal 1/2. If there are n punctures, this gives

$$A \simeq 4\pi\gamma n\sqrt{3} \tag{3.10}$$

In a state like this, each number m_i can take just two values at each puncture. Thus if there are n punctures, there are 2^n microstates, and the black hole entropy is

$$S \simeq \ln(2^n) \simeq \frac{A \ln 2}{4\pi\gamma\sqrt{3}} \tag{3.11}$$

In short, we see that entropy is indeed proportional to area, at least for large black holes. However, we only get Hawking's formula $S = A/4$ if we take the Barbero-Immirzi parameter to be

$$\gamma = \frac{\ln 2}{\pi\sqrt{3}} \tag{3.12}$$

On the one hand this is good: it's a way to determine the Barbero-Immirzi parameter, and thus the quantum of area, which works out to

$$8\pi\gamma\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)} = 4\ln 2 \tag{3.13}$$

This makes for a pretty picture in which almost all the spin network edges puncturing the event horizon carry one quantum of area and one qubit of information. One can also check that the same value of γ works for electrically charged black holes and black holes coupled to a dilaton field. On the other hand, it seems annoying that we can only determine the quantum of area with the help of Hawking's semiclassical calculation. The strange value of γ might also make us suspicious of this whole approach.

3.5 Insights form Quasinormal modes

In 1974, Bekenstein had argued that Schwarzschild black holes should have a discrete spectrum of evenly spaced areas. While this law does not hold in the

Loop Quantum Gravity description of black holes, it has some of the same consequences. For example, in 1986 Mukhanov noted that with a law of this sort, the formula $S = A/4$ can only hold exactly if the n^{th} area eigenstate has degeneracy k^n and the spacing between area eigensates is $4\ln k$ for some number $k = 2, 3, 4, \ldots$ He also gave a philosophical argument that the value $k = 2$ is preferred, since then the states in the n^{th} energy level can be described using n qubits.

Many researchers have continued this line of thought in different ways, but in 1998, Hod [13] gave an remarkable argument in favor of $k = 3$. His idea was to determine the quantum of area by looking at the vibrational modes of the Schwarzschild black hole! Hod argues that if classically a system, such as the black hole, can undergo periodic motion at some frequency ω, then in the quantum theory it can emit or absorb quanta of radiation with the corresponding energy. But the energy of a Schwarzschild black hole is just its mass, and this is related to the area of its event horizon by

$$A = 16\pi M^2 \tag{3.14}$$

so when a black hole absorbs one quantum of radiation its area should change by

$$\Delta A = 32\pi M \Delta M = 32\pi M \omega_{tr} \quad (\hbar = 1) \tag{3.15}$$

The frequency ω_{tr} is correlated with the transition between two black hole states, or in our theory, with the appearance or disappearance of a puncture of spin j_{min}. At that time, observed numerically that asymptotic frequency for a Schwarzschild black hole tends to the value $0.0874247/2M$ and he related that value to the transition frequency. By replacing that value to (3.15) he obtained for the quantum of area the numerical value:

$$\Delta A = 4.39444 \tag{3.16}$$

which is quite close to $4\ln 3$. Thus he concluded that $k = 3$ and for the transition frequency (according to the first law of thermodynamics)

$$\omega_{tr} = \omega_R = T_H \ln 3 \tag{3.17}$$

We shall turn now in to more recent developments. In November 2003, Dreyer (see [14]) found an ingenious way to reconcile Hod's conjecture with the Loop Quantum Gravity calculations, within the frame of the statistical-mechanical interpretation of black hole entropy. The calculation due to Ashtekar *et al* used a version of Loop Quantum Gravity where the gauge group is SU(2). This is why so many formulas resemble those familiar from the Quantum mechanics of angular momentum, and this is why the smallest nonzero area comes from a spin network edge labeled by the smallest nonzero spin: $j = 1/2$. But there is also a version of Loop Quantum Gravity with gauge group SO(3), in which the smallest nonzero spin is $j = 1$. In that version the spin networks form a basis of the Hilbert space of the theory. The dimension of that space increases by each puncture that contributes area. It can be shown that if the area is increased by a puncture that its edge has a label j, then the dimension of the Hilbert space increase by a factor of $2j+1$. Thus if the area of the horizon is given by a large number of punctures N of edges that have a spin j_i,$i = 1, 2, \ldots, N$, with some arbitrary degeneracy that is not of interest,then the dimension of that surface is given by

$$(2j_1+1)(2j_2+1)\ldots(2j_N+1) = \prod_{i=1}^{N}(2j_i+1)$$

It has been shown that the microstate with the greatest statistical probability is the puncture with the smallest non-zero spin. Thus the entropy is then just

$$S = N\ln(2j_{min}+1) \tag{3.18}$$

It is easy now to find an expression for the number of punctures N. If we divide the total area A with the area that contributes the puncture with the lowest spin one readily finds for the entropy the following expression

$$S = \frac{A\ln(2j_{min}+1)}{8\pi l_P^2\gamma\sqrt{j_{min}(j_{min}+1)}} \tag{3.19}$$

We must now fix the the Immirzi-Barbero parameter. Again the assumption of an appearance or disappearance of a puncture at the horizon leads to a transition between two states of the black hole that correspond to an area change of

$$\Delta A = A(j)_{min} = 8\pi l_P^2\gamma\sqrt{j_{min}(j_{min}+1)} \tag{3.20}$$

The change ΔM in the mass again equals the energy of the quantum of energy emitted with the characteristic frequency ω_{tr}. Thus, recalling again the expression that we derived for that frequency, we have

$$\begin{aligned}\Delta A &= 32\pi M\Delta M = 32\pi M\omega_{tr} \Rightarrow\\ \Delta A &= 8\pi l_P^2\gamma\sqrt{j_{min}(j_{min}+1)} = 4\ln 3 l_P^2 \Rightarrow\\ \gamma &= \frac{\ln 3}{2\pi\sqrt{j_{min}(j_{min}+1)}}\end{aligned} \tag{3.21}$$

Hence, based on that new value for γ, Dreyer calculated for the entropy

$$S = \frac{A}{4l_P^2}\frac{\ln(2j_{min}+1)}{\ln 3} \tag{3.22}$$

that gives the Bekenstein-Hawking result $S = A/4$, just by replacing $j = 1/2$ by $j = 1$. This new results matches Hod's result! For the new calculated value of the Immirzi parameter

$$\gamma = \frac{\ln 3}{2\pi\sqrt{2}} \tag{3.23}$$

one obtains an $\ln 3$ as the new quantum of area. But ultimately, all that really matters is that when $j = 1$ there are 3 spin states instead of 2. Thus each

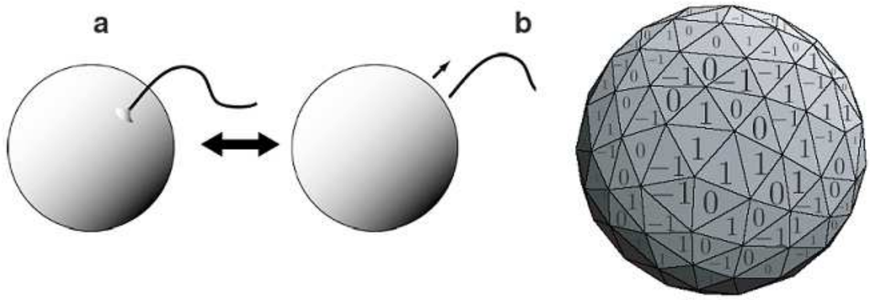

Figure 3.5: (a) *The proposed quantum transition between two black hole states giving rise to the emission of a quantum of energy $\hbar\omega$.* (b) *The emerging "trit of information" picture.*

quantum of area carries a "trit" of information instead of a bit (see Figure 3.5), which is why Dreyer obtains $k = 3$. With the appearance of Dreyer's paper, the suspense became almost unbearable. After all, Hod's observation relied on numerical calculations, so the very next digit of his number might fail to match that of $4 \ln 3$. Luckily, in December 2002, Motl [15] showed that the match is exact by using the analysis of Nollert's continued fraction expansion for the asymptotic frequencies of quasinormal modes.

3.6 Discussion and Outlook

The attempt to quantize gravity seems to be a quite difficult, but also an intriguing task. The formulation of a theory such as the Loop Gravity is not an achievement of the last decade. Actually, the mathematical foundations of the theory are, but not also the fundamental concepts and assumptions. For example, the notion of spin networks and their fundamental role in the theories of quantum gravity was firstly proposed by Roger Penrose in the early 70s. After all, the idea of introducing loops to represent the quantum excitations of the gravitational field arises from an older idea in superconductivity: the quantization of the Faraday's lines, the natural variables of the Yang-Mills theory. The physical representation of the theory seems to be beautiful and fairly satisfactory. What more for a theory that postulates four dimensions and strikes quantum gravity by quantizing spacetime! These arguments become even more significant when considering the fact that Loop Quantum Gravity predicts and confirms properties of objects made from purely gravity, i.e., the black holes.

Hod's conjecture and Dreyer's method for calculating the Immirzi-Barbero introduce a physical process that could give rise to the asymptotic quasinormal frequency. This process can be thought as a growing process of a large black hole which is related to the appearance or disappearance of a puncture at the surface of the horizon, that is, a physical process of conversion of area quanta into matter quanta via the emission of an edge. By equating the mass change corresponding to this area change with the energy $\hbar\omega_{tr} = \hbar\omega_R$ of the quantum with frequency ω_R and using the fact that

$$M\omega_R = \frac{\ln 3}{8\pi}$$

we fix the Immirzi-Barbero parameter. Using the value of that parameter we are able to calculate the entropy of a black hole, which in our case agrees with the Bekenstein-Hawking result provided one chooses $j_{min} = 1$.

It has to be also pointed out the importance of the appearance of ω_R as a transition frequency in the quantum theory. The existence of a universal limit for quasinormal mode frequencies, depending only to macroscopic parameters of the black hole, for non-rotating uncharged black holes is a remarkable fact. An obvious question is whether such a dependence exists also for rotating charged black holes, and if in that case, there is a physical picture within Loop Quantum Gravity that can produce such frequencies.

Indeed, Hod tried to generalize his conjecture to charged and rotating black holes and using as starting point the first law of thermodynamics

$$\delta M = T_H \delta S + \Omega \delta J + \Phi \delta Q \tag{3.24}$$

He argued and proved qualitatively that in a quantum transition of a rotating and charged black hole, the quantum of energy that is emitted has a transition frequency

$$\omega_{tr} = \omega_R \equiv T_H \ln 3 + m\Omega \tag{3.25}$$

where m is the azimuthal eigenvalue of the perturbation[4]. That is, in the asymptotic quasinormal mode spectrum of a charged and rotating black hole no contribution to the real part of the frequency is made from the charge term δQ. Thus, in an numerical investigation of the asymptotic spectrum of an extremal Reissner-Nordström ($Q \to \frac{1}{2}$), we should expect a zero real part. Motl and Neitzke found the following analytic expression for the asymptotic frequencies for a Reissner-Nordstöm black hole

$$e^{\beta\omega} + 2 + 3e^{-\beta_I \omega} = 0 \tag{3.26}$$

In the above relation β is the inverse Hawking temperature, $\beta = 4\pi/(1-k) = 1/T_H$, where k is given by $Q/M = 2\sqrt{k}/(1+k)$ and $\beta_I = -k^2\beta$ in the inverse Hawking temperature of the inner horizon. The formula was later rederived by an independent method. However, the numerical predictions of that formula are a bit peculiar. In particularly, the complex solutions of (3.26) exactly coincides with the numerical results of those derived by Kokkotas and Berti [16]. Kokkotas and Berti computed for the first time the very highly damped quasinormal

[4]Rotation in black holes causes a split in the frequencies of the spectrum. This splitting, which is similar to the Zeeman splitting in the energy levels of an atom due to the external magnetic field, is represented by the azimuthal quantum number m that takes the values $m = -l, \ldots, l$.

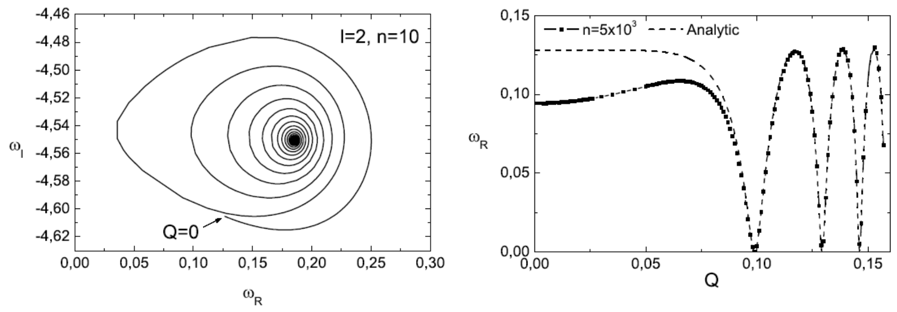

Figure 3.6: Left: *Behavior of the $n = 10$ quasinormal frequency in the complex ω plane. The mode "spirals in" towards its value in the extremal charge limit.* Right: *Real part of the RN frequency as a function of charge for $n = 5\times10^3$. The numerical result is compared with the result of the analytic formula derived by Motl and Neitzke. The analytic formula predicts the oscillatory behavior at large values of charge, but fails to reproduce the $T_H \ln 3$ result in the Schwarzschild limit.*

modes for the Reissner-Nordström and Kerr solution. In the case of charged black holes, they found that quasinormal frequencies show an oscillatory behavior as a function of charge (Figure (**??**)). According to them, as we approach to the extremal limit value $Q \to \frac{1}{2}$ the real part of the frequency approach the value

$$\omega_R \to T_H \ln 3$$

a value that is also reproduced by the real part of the frequency predicted by the analytic formula (3.26).

In the case of the Schwarzschild limit, that is, $Q \to 0$, the numerical results reproduce the expected value $T_H \ln 3$. However, using (3.26) one finds instead (see Figure (**??**))

$$\omega_R \to T_H \ln 5$$

a prediction that, in any case, is hard to reconcile with Hod's interpretation, eq. (3.25), of the asymptotic spectrum.

In the case of Kerr black holes, as we have already stated, rotation introduces a separation of the frequencies that is represented by the azimuthal eigenvalue m. Kokkotas and Berti [**?**] studied again the asymptotic behavior of the quasinormal modes frequencies and they observed, in particularly for the mode $l = m = 2$, that for large n ($n \sim 50$) the real part approaches the value

$$\omega_{l=m=2} = 2\Omega + i2\pi T_H n \tag{3.27}$$

Generally, most modes with $m > 0$ approach the limiting value $\omega_R = m\Omega$ (except of a few exceptions, see Figure 3.7). The imaginary part of all modes with $m > 0$ have a separation $2\pi T_H$, where now T_H is the Hawking temperature of the horizon. We see, however, that the evaluated value for the real part of the

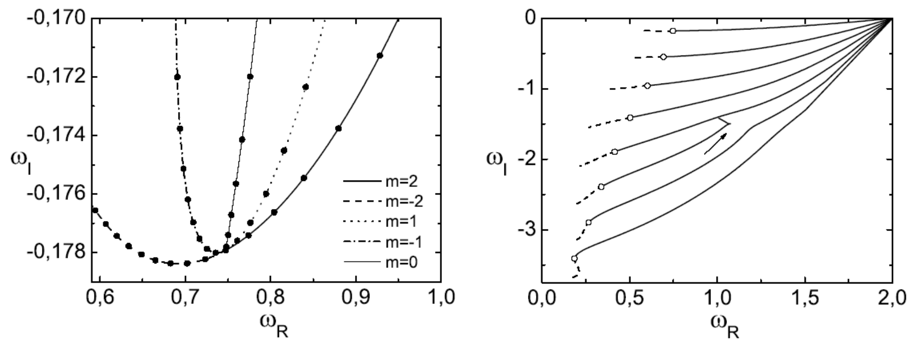

Figure 3.7: Left: *Splitting of the fundamental gravitational mode* $l = 2$*.* Right: *Trajectory of the first eight Kerr quasinormal frequencies with* $m = 2$ *and* $m = -2$ *(dashed lines). We observe the existence of an exceptional modes with* $n = 6$*, that does not tend to the asymptotic frequency* $m\Omega$*.*

asymptotic frequency does not agree with Hod's conjecture. The term $T_H \ln 3$ is totally missing from the derived expression, we observe only the "corrections" $m\Omega$ due to rotation.

As a conclusion, the numerical results for charged and rotating black holes seems to suggest that they do not agree with the behavior predicted by Hod's conjecture, meaning that Hod's conjecture prediction must be wrong or it must be revised and somehow recasted. Similarly, Loop Quantum Gravity has not offered yet any strong arguments about the nature of the asymptotic spectrum of charged and rotating black hole, concluding thus that our physical understanding for highly damped black hole oscillations and their place in quantum gravity is admittedly very poor.

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Appendix A

Spherical Harmonics Normalization

A significant remark is that the ten harmonics, under proper normalization, are complete over the space of symmetric tensor fields, on a 2-sphere and orthonormal. The tensor spherical harmonics satisfy the orthonormality relation

$$\int (Y_{lm}^{\mu\nu})^* \; Y_{\mu\nu}^{l'm'} \sin\theta d\theta d\phi = \frac{1}{2} l(l-1)(l+1)(l+2)\delta_{ll'}\delta_{mm'}$$

where $(Y_{lm}^{\mu\nu})^*$ is the complex conjugate of $Y_{lm}^{\mu\nu}$. The above mentioned relation is valid for both axial and polar tensor spherical harmonics. Furthermore, if $Y_{\mu\nu}^{lm}$ is an even spherical harmonic and $X_{\mu\nu}^{lm}$ an odd one, then it holds the following property

$$\int X_{lm}^{\mu\nu} \; Y_{\mu\nu}^{l'm'} \sin\theta d\theta d\phi = 0$$

which states that the even-parity and odd-parity harmonics are always orthogonal. We list the complete set of normalized tensor spherical harmonics used in the study of the perturbed field equations in Chapter 2:

$$Y_{lm}^{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^2_{lm} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^3_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^4_{lm} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & \partial_\theta & \partial_\phi \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^5_{lm} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta & \partial_\phi \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^6_{lm} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & (1/\sin\theta)\partial_\phi & (-\sin\theta)\partial_\theta \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^7_{lm} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\sin\theta)\partial_\phi & (-\sin\theta)\partial_\theta \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} Y_{lm},$$

$$Y^8_{lm} = \frac{-ir^2}{\sqrt{2l(l+1)(l-1)(l+2)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta)X_{lm} & (\sin\theta)W_{lm} \\ 0 & 0 & * & (\sin\theta)X_{lm} \end{pmatrix},$$

$$Y^9_{lm} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{pmatrix} Y_{lm},$$

$$Y^{10}_{lm} = \frac{r^2}{\sqrt{2l(l+1)(l-1)(l+2)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{lm} & X_{lm} \\ 0 & 0 & * & (\sin^2\theta)W_{lm} \end{pmatrix}$$

where $W_{lm} = (\partial^2_{\theta\theta} - \cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial^2_{\phi\phi})$ and $X_{lm} = 2(\partial^2_{\theta\phi} - \cot\theta\partial_\phi)$.

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