An introduction to relativistic hydrodynamics and magneto-hydrodynamics

José A. Font
Departamento de Astronomía y Astrofísica
Universidad de Valencia (Spain)

Talk given at the Aristotle University of Thessaloniki, Dec. 7th 2005

Outline of the talk

- Introduction & Motivation
- General relativistic hydrodynamics equations (3+1)
- General relativistic (ideal) magneto-hydrodynamics equations
- Numerical solution (hyperbolic systems of conservation laws)
- Applications in relativistic astrophysics

Further information:

J.A. Font, "Numerical hydrodynamics in general relativity", Living Reviews in Relativity (2003) (www.livingreviews.org)

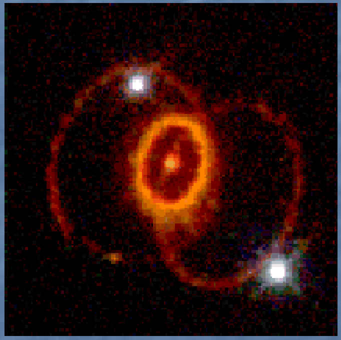
L. Antón, O. Zanotti, J.A. Miralles, J.M. Martí, J.M. Ibáñez, J.A. Font, J.A. Pons, "Numerical 3+1 general relativistic magnetohydrodynamics: a local characteristic approach" (ApJ in press; astro-ph/0506063)

Part 1
Introduction & Motivation

Introduction & Motivation

The natural domain of applicability of general relativistic hydrodynamics (GRHD) and magnetohydrodynamics (GRMHD) is in the field of relativistic astrophysics.

General relativity and relativistic (magneto-)hydrodynamics play a major role in the description of gravitational collapse leading to the formation of compact objects (neutron stars and black holes):



- Stellar core-collapse supernovae (Type II/Ib/Ic)
- Black hole formation (and accretion)
- Coalescing compact binaries (NS/NS, BH/NS, BH/BH)
- gamma-ray bursts
- jet formation (black hole plus thick disc systems)

Time-dependent evolutions of fluid flow *coupled* to the spacetime geometry (Einstein's equations) only possible through accurate, large-scale numerical simulations.

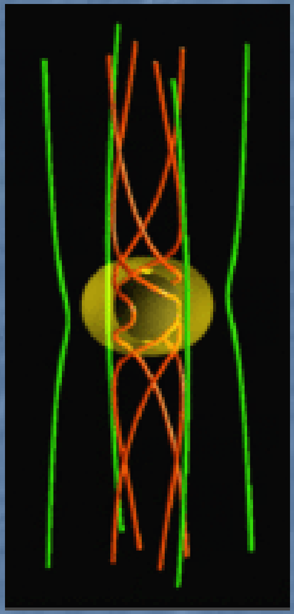
Some scenarios can be described in the test-fluid approximation: GRHD/GRMHD computations in curved backgrounds (highly mature nowadays, particularly GRHD case).

The GRHD/GRMHD equations constitute nonlinear hyperbolic systems.
Solid mathematical foundations and accurate numerical methodology imported from CFD. A "preferred" choice: high-resolution shock-capturing schemes written in conservation form.

In recent years there has been intense work on formulating/solving the MHD equations in general relativistic spacetimes (either background or dynamical).

Pioneers: Wilson (1975), Sloan & Smarr (1985), Evans & Hawley (1988), Yokosawa (1993)

More recently: Koide et al (1998 …), De Villiers & Hawley (2003 …), Baumgarte & Shapiro (2003), Gammie et al (2003), Komissarov (2005), Duez et al (2005), Shibata & Sekiguchi (2005), Anninos et al (2005), Antón et al (2005). Both, artificial viscosity and HRSC schemes developed.

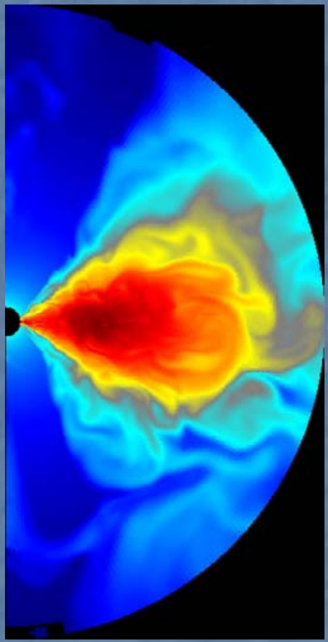


Most of the applications are in the field of black hole accretion and jet formation …

Development of the MRI in a magnetised torus around a Kerr black hole (Gammie, McKinney & Tóth 2003)

Jet formation: the twisting of magnetic field lines around a Kerr black hole. The yellow surface is the ergosphere (Koide et al 2002)

… many others under way (you name it!)



The number of groups working in special relativistic MHD is even larger: Komissarov; Balsara; Koldoba et al; Del Zanna et al; Leisman et al; …

Exact solution of the SRMHD Riemann problem found recently: Romero et al (2005) – particular case; Giacomazzo & Rezzolla (2005) – general case.

Part 2

General relativistic hydrodynamics equations

A brief reminder: Classical (Newtonian) hydrodynamics

Mass conservation (continuity equation)

Let V_t be a volume which moves with the fluid; its mass is given by $m(V_t) = \int_{V_t} \rho(t,\vec{r})\, dV$

From the principle of conservation of mass enclosed within that volume

$$\frac{d}{dt} m(V_t) = \frac{d}{dt}\int_{V_t} \rho(t,\vec{r})\, dV = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Corolary:

$$-\frac{\partial}{\partial t}\int_V \rho\, dV = \int_{\partial V} \rho\vec{v}\cdot d\vec{\Sigma}$$

the variation of the mass enclosed in a fixed volume V is equal to the flux of mass across the surface at the boundary of the volume.

Momentum balance (Euler's equation)

"the variation of momentum of a given portion of fluid is equal to the net force (stresses plus external forces) exerted on it" (Newton's 2nd law):

$$\frac{d}{dt}\int_{V_t} \rho\vec{v}\, dV = -\int_{\partial V_t} p\, d\vec{\Sigma} + \int_{V_t} \vec{G}\, dV = \int_{V_t} \left[\vec{G} - \vec{\nabla}p\right] dV$$

$$\rho\frac{D\vec{v}}{Dt} = \vec{G} - \vec{\nabla}p \quad \Leftrightarrow \quad \rho\vec{a} = \vec{G} - \vec{\nabla}p$$

Energy conservation

Let E be the total energy of the fluid, sum of its kinetic energy and its internal energy:

$$E = E_{\mathrm{K}} + E_{\mathrm{int}} = \frac{1}{2}\int_{V_t} \rho\vec{v}^{\,2}\, dV + \int_{V_t} \rho\varepsilon\, dV$$

Principle of energy conservation: "the variation in time of the total energy of a portion of fluid is equal to the work done per unit time over the system by the stresses (internal forces) and the external forces".

$$\frac{dE}{dt} = \frac{d}{dt}\int_{V_t}\left(\frac{1}{2}\rho\vec{v}^{\,2} + \rho\varepsilon\right)dV = -\int_{\partial V_t} p\vec{v}\cdot d\vec{\Sigma} + \int_{V_t}\vec{G}\cdot\vec{v}\, dV$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\vec{v}^{\,2} + \rho\varepsilon\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho\vec{v}^{\,2} + \rho\varepsilon + p\right)\vec{v}\right] = \rho\vec{g}\cdot\vec{v}$$

$\vec{g}$ is a conservative external force field (e.g. gravitational field):

$$\vec{g} = -\vec{\nabla}\Phi, \;\; \Delta\Phi = 4\pi G\rho$$

The above equations are a nonlinear hyperbolic system of conservation laws:

$$\frac{\partial\vec{u}}{\partial t} + \frac{\partial\vec{f}^{\,i}(\vec{u})}{\partial x^i} = \vec{s}(\vec{u})$$

state vector $\vec{u} = \left(\rho, \rho v^j, e\right)$

flux vector $\vec{f}^{\,i} = \left(\rho v^i, \rho v^i v^j + p\delta^{ij}, (e+p)v^i\right)$

source vector $\vec{s} = \left(0, -\rho\frac{\partial\Phi}{\partial x^j}, -\rho v^i\frac{\partial\Phi}{\partial x^i}\right)$

Hyperbolic equations have finite propagation speed: maximum speed of information given by the largest characteristic curves of the system.

The range of influence of the solution is bounded by the eigenvalues of the Jacobian matrix of the system.

$$A = \frac{\partial\vec{f}^{\,i}}{\partial\vec{u}} \Rightarrow \lambda_0 = v_i, \lambda_{\pm} = v_i \pm c_s$$

General Relativistic Hydrodynamics equations

The general relativistic hydrodynamics equations are obtained from the local conservation laws of the stress-energy tensor, $T^{\mu\nu}$ (the Bianchi identities), and the matter current density J^{μ} (the continuity equation):

$$\nabla_{\mu} T^{\mu\nu} = 0 \qquad \nabla_{\mu} J^{\mu} = 0$$

Equations of motion

As usual ∇_{μ} stands for the covariant derivative associated with the four dimensional spacetime metric $g_{\mu\nu}$. The density current is given by $J^{\mu}=\rho u^{\mu}$, u^{μ} representing the fluid 4-velocity and ρ the rest-mass density in a locally inertial reference frame.

The stress-energy tensor for a non-perfect fluid is defined as:

$$T^{\mu\nu} \equiv \rho(1+\varepsilon)u^{\mu}u^{\nu} + (p - \zeta\Theta)h^{\mu\nu} - 2\eta\sigma^{\mu\nu} + q^{\mu}u^{\nu} + q^{\nu}u^{\mu}$$

where ε is the rest-frame specific internal energy density of the fluid, p is the pressure, and $h^{\mu\nu}$ is the spatial projection tensor, $h^{\mu\nu}=u^{\mu}u^{\nu}+g^{\mu\nu}$. In addition, ζ and η are the shear and bulk viscosity coefficients. The expansion, Θ, describing the divergence or convergence of the fluid world lines is defined as $\Theta=\nabla_{\mu}u^{\nu}$. The symmetric, trace-free, and spatial shear tensor $\sigma^{\mu\nu}$ is defined by:

$$\sigma^{\mu\nu} = \frac{1}{2}\left(\nabla_{\alpha}u^{\mu}h^{\alpha\nu} + \nabla_{\alpha}u^{\nu}h^{\alpha\mu}\right) - \frac{1}{3}\Theta h^{\mu\nu}$$

Finally q^{μ} is the energy flux vector.

In the following we will neglect non-adiabatic effects, such as viscosity or heat transfer, assuming the stress-energy tensor to be that of a perfect fluid:

$$T^{\mu\nu} \equiv \rho h u^{\mu} u^{\nu} + p g^{\mu\nu}$$

where we have introduced the relativistic specific enthalpy, h, defined as:

$$h = 1 + \varepsilon + \frac{p}{\rho}$$

Introducing an explicit coordinate chart the previous conservation equations read:

$$\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}\,\rho u^{\mu}) = 0$$

$$\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}\,T^{\mu\nu}) = \sqrt{-g}\,\Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$$

where the scalar x^0 represents a foliation of the spacetime with hypersurfaces (coordinatised by x^i). Additionally, $\sqrt{-g}$ is the volume element associated with the 4-metric $g_{\mu\nu}$, with g=det($g_{\mu\nu}$), and $\Gamma^{\nu}_{\mu\lambda}$ are the 4-dimensional Christoffel symbols.

The system formed by the equations of motion and the continuity equation must be supplemented with an <u>equation of state</u> (EOS) relating the pressure to some fundamental thermodynamical quantities, e.g.

$$p = p(\rho, \varepsilon)$$

- Ideal fluid EOS: $p = (\Gamma - 1)\rho\varepsilon$
- Polytropic EOS: $p = \kappa\rho^{\Gamma}, \quad \left(\Gamma = 1 + \frac{1}{n}\right)$

In the "test-fluid" approximation where the fluid's self-gravity is neglected, the dynamics of the matter fields is completely governed by the conservation laws of the stress-energy and of the current density, together with the EOS.

In those situations where such approximation does not hold, the previous equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}}$$

Einstein's equations are covariant, i.e. independent of a coordinate system (absence of a preferential reference frame; equivalence principle).

The freedom in the choice of the reference frame has led to different formulations of the EE. A concept of "time" is introduced, whose level surfaces can be spacelike (3+1) or null (CIVP).

The evolution equations are obtained by projecting EE on to the hypersurfaces.

The projections w.r.t. the normal vector correspond to the so-called constraint equations.

In the 3+1 formulation of GR Einstein's equations can be formulated as an initial value (Cauchy) problem: evolution equations for the 3-metric and extrinsic curvature, and constraint equations (Hamiltonian and momentum) to be satisfied on every time slice.

Einstein's equations in the 3+1 formulation (γ_{ij}, K_{ij})

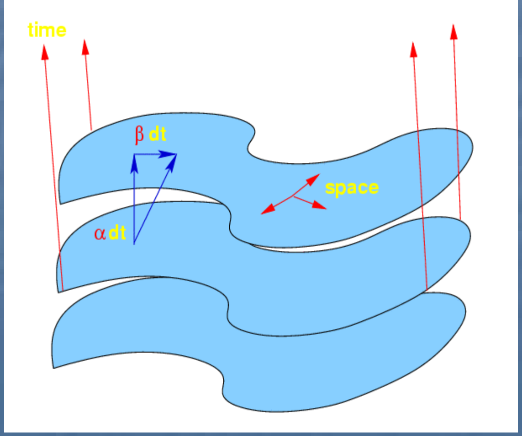
The most widely used approach to formulate and solve Einstein's equations in Numerical Relativity is the so-called Cauchy or 3+1 formulation.

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Spacetime is foliated with a set of non-intersecting spacelike hypersurfaces Σ. Within each surface distances are measured with the spatial 3-metric γ_{ij}. The extrinsic curvature tensor K_{ij} is also introduced. This tensor describes how the spacelike hipersurfaces are "embedded" in spacetime.

$$K_{ij} \equiv -\gamma_i^k \gamma_j^l \nabla_k n_l$$

There are two kinematical variables which describe the evolution between each hypersurface: the lapse function α which describes the rate of proper time along a timelike unit vector normal to the hypersurface, and the shift vector β^i, spacelike vector which describes the movement of coordinates within the hypersurface.



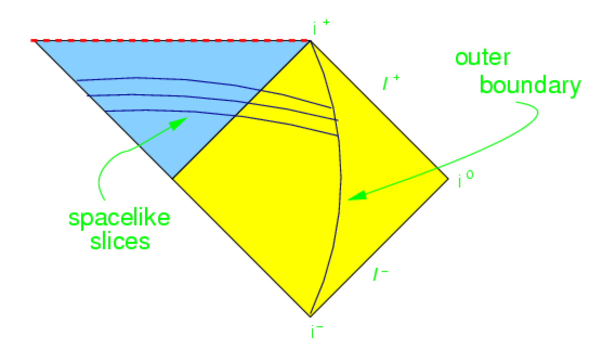
$$\partial_t = \alpha \vec{n} + \beta^i \partial_i$$

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right), \quad n_\mu = (-\alpha, 0, 0, 0)$$

Metric:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

(Numerical) General Relativity: Which portion of spacetime shall we foliate?



3+1 (Cauchy) formulation

Lichnerowicz (1944); Choquet-Bruhat (1962); Arnowitt, Deser & Misner (1962); York (1979)

Standard choice for most Numerical Relativity groups.

Spatial hypersurfaces have a **finite** extension.

(original formulation of the 3+1 equations)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

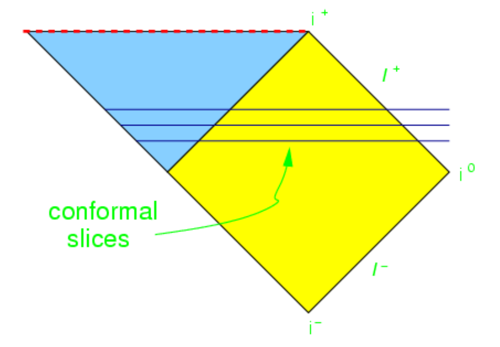
$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2 K_{im} K^m_j \right) + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{jm} \nabla_i \beta^m - 8\pi T_{ij}$$

$$R + K^2 - K_{ij} K^{ij} - 16\pi\alpha^2 T^{00} = 0$$

$$\nabla_i \left(K^{ij} - \gamma^{ij} K \right) - 8\pi S^j = 0$$

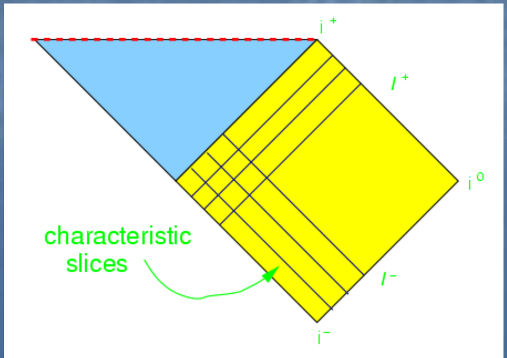
Reformulating these equations to achieve numerical stability is one of the arts of numerical relativity.

Conformal formulation: Spatial hypersurfaces have **infinite** extension (Friedrich et al).



Characteristic formulation (Winicour et al).

Hypersuperfaces are light cones (incoming/outgoing) with **infinite** extension.



Evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha\left(R_{ij} + K K_{ij} - 2K_{im} K_j^m\right)$$
$$+ \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{jm} \nabla_i \beta^m$$
$$-8\pi\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij} S\right) - 4\pi\alpha\rho\gamma_{ij}$$

Constraint equations:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\nabla_j K^{ij} - \gamma^{ij}\nabla_j K = 8\pi S^j$$

Definitions:

∇_i Covariant derivative w.r.t. sopatial metric

$$R_{ij} = \partial_n \Gamma^n_{ij} - \partial_j \Gamma^n_{in} + \Gamma^n_{mn}\Gamma^m_{ij} - \Gamma^n_{jm}\Gamma^m_{in}$$ Ricci tensor

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{in}\left(\partial_k \gamma_{nj} + \partial_j \gamma_{nk} - \partial_n \gamma_{jk}\right)$$ Christoffel symbols

$R = R_{ij}\gamma^{ij}$ Scalar curvature

$K = K_{ij}\gamma^{ij}$ Trace of extrinsic curvature

$$\left.\begin{aligned} \rho &\equiv T^{\mu\nu} n_\mu n_\nu = \rho h W^2 - P \\ S^i &\equiv -\perp^i_\mu T^{\mu\nu} n_\nu = \rho h W^2 v^i \\ S_{ij} &\equiv \perp^\mu_i \perp^\nu_j T_{\mu\nu} = \rho h W^2 v_i v_j + \gamma_{ij} P \\ S &= \rho h W^2 v_i v^i + 3P \end{aligned}\right\}$$ Matter terms

Cauchy problem (IVP) for the 3+1 formulation of Einstein's equations:

- Prescribe γ_{ij}, K_{ij} at t=0 subject to the constraint equations.
- Specify coordinates via α and β
- Evolve data using evolution equations for γ_{ij}, K_{ij}.

Bianchi identities guarantee that if the constraints are satisfied at t=0, they'll be satisfied at t>0; that is, the evolution equations satisfy the constraint equations.

3+1 GR Hydro equations - formulations

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\,\rho u^\mu) = 0$$

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\,T^{\mu\nu}) = \sqrt{-g}\,\Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

Different formulations exist depending on:

1. The choice of time-slicing: the level surfaces of x^0 can be spatial (3+1) or null (characteristic)
2. The choice of physical (primitive) variables (ρ, ε, u^i ...)

Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology are more recent:

- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS
- Papadopoulos & Font (2000): covariant, general EOS

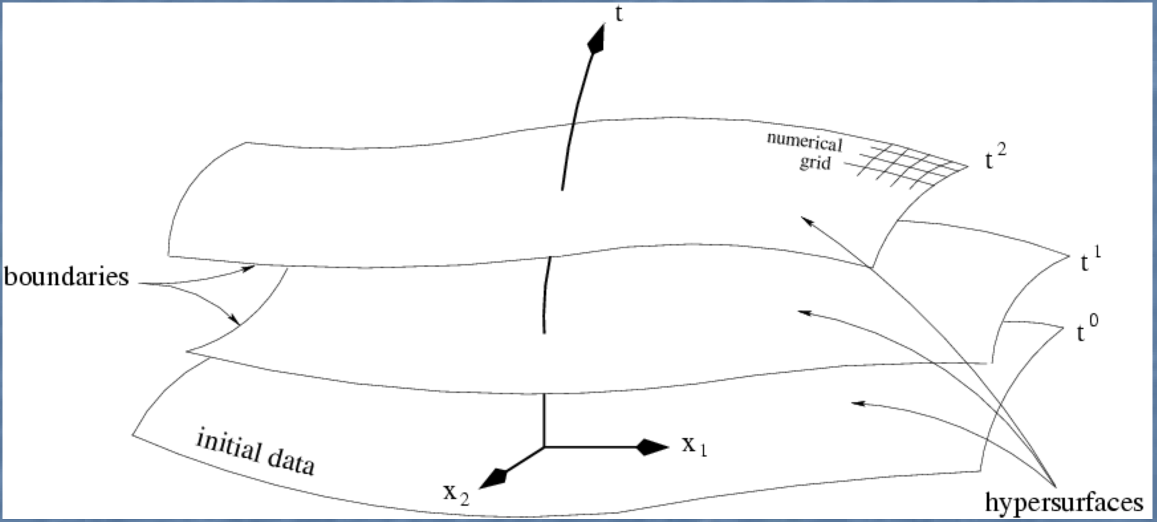
Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system, translating to relativistic hydrodynamics existing tools of CFD.

This procedure departs from earlier approaches, most notably in avoiding the need for artificial dissipation terms to handle discontinuous solutions as well as implicit schemes as proposed by Norman & Winkler (1986).

3+1 GR Hydro equations – Eulerian observer

Foliate the spacetime with *t=const* spatial hypersurfaces Σ_t

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$



Let n be the unit timelike 4-vector orthogonal to Σ_t such that

$$n = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i)$$

Eulerian observer: at rest in a given hypersurface, moves from Σ_t to $\Sigma_{t+\Delta t}$ along the normal to the slice:

$$v \equiv -\frac{n \cdot \partial_i}{n \cdot u} \qquad v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

Definitions: *u* : fluid's 4-velocity, *p* : isotropic pressure, *ρ* : rest-mass density
ε : specific internal energy density, *e=ρ(1+ε)* : energy density

The extension of modern HRSC schemes from classical fluid dynamics to relativistic hydrodynamics was accomplished in three steps:

1. Casting the GRHD equations as a system of conservation laws.
2. Identifying the suitable vector of unknowns.
3. Building up an approximate Riemann solver (or high-order symmetric scheme).

Replace the *"primitive variables"* in terms of the *"conserved variables" :*

$$\vec{w} = \left(\rho, v^i, \varepsilon\right) \rightarrow \begin{cases} D \equiv \rho W \\ S_j \equiv \rho h W^2 v_j \\ E \equiv \rho h W^2 - p \end{cases}$$

$$W^2 \equiv 1/(1 - v^j v_j)$$

Lorentz factor

$$h \equiv 1 + \varepsilon + \frac{p}{\rho}$$

specific enthalpy

First-order flux-conservative hyperbolic system

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial \sqrt{\gamma}\,\vec{u}(\vec{w})}{\partial t} + \frac{\partial \sqrt{-g}\,\vec{f}^{\,i}(\vec{w})}{\partial x^i}\right) = \vec{s}(\vec{w})$$

Banyuls et al, ApJ, **476**, 221 (1997)

Font et al, PRD, **61**, 044011 (2000)

$$\vec{u}(\vec{w}) = \left(D, S_j, E - D\right)$$ state vector

$$\vec{f}^{\,i}(\vec{w}) = \left(D\left(v^i - \frac{\beta^i}{\alpha}\right), S_j\left(v^i - \frac{\beta^i}{\alpha}\right) + p\delta^i_j, E - D\left(v^i - \frac{\beta^i}{\alpha}\right) + p v^i\right)$$ fluxes

$$\vec{s}(\vec{w}) = \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^{\delta}_{\mu\nu} g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^{0}_{\mu\nu}\right)\right)$$ sources

Recovering special relativistic and Newtonian limits

General Relativity

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\,\rho W}{\partial t}+\frac{\partial\sqrt{-g}\,\rho W v^i}{\partial x^i}\right)=0$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\,\rho h W^2 v^j}{\partial t}+\frac{\partial\sqrt{-g}\left(\rho h W^2 v^i v^j+p\delta^{ij}\right)}{\partial x^i}\right)=T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^{\delta}_{\mu\nu}g_{\delta j}\right)$$

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\left(\rho h W^2-p-\rho W\right)}{\partial t}+\frac{\partial\sqrt{-g}\left(\rho h W^2-\rho W\right)v^i}{\partial x^i}\right)=\alpha\left(T^{\mu 0}\frac{\partial\ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^{0}_{\mu\nu}\right)$$

$$\Downarrow$$

Minkowski

$$\frac{\partial\rho W}{\partial t}+\frac{\partial\rho W v^i}{\partial x^i}=0$$

$$\frac{\partial\rho h W^2 v^j}{\partial t}+\frac{\partial\left(\rho h W^2 v^i v^j+p\delta^{ij}\right)}{\partial x^i}=0$$

$$\frac{\partial\left(\rho h W^2-p-\rho W\right)}{\partial t}+\frac{\partial\left(\rho h W^2-\rho W\right)v^i}{\partial x^i}=0$$

Newton

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v^i}{\partial x^i}=0$$

$$\frac{\partial\rho v^j}{\partial t}+\frac{\partial\left(\rho v^i v^j+p\delta^{ij}\right)}{\partial x^i}=0$$

$$\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2+p\right)v^i}{\partial x^i}=0$$

HRSC schemes based on approximate Riemann solvers use the local characteristic structure of the hyperbolic system of equations. For the previous system, this information was presented in Banyuls et al (1997).

The eigenvalues (characteristic speeds) are all real (but not distinct, one showing a threefold degeneracy), and a complete set of right-eigenvectors exist. The above system satisfies, hence, the definition of hiperbolicity.

Eigenvalues (along the *x* direction)

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)}$$

$$\lambda_\pm = \frac{\alpha}{1-v^2c_s^2}\left(v^x\left(1-c_s^2\right) \pm c_s\sqrt{\left(1-v^2\right)\left[\gamma^{xx}\left(1-v^2c_s^2\right)-v^xv^x\left(1-c_s^2\right)\right]}\right) - \beta^x$$

Right-eigenvectors

$$\vec{r}_{0,1} = \begin{bmatrix} \frac{\kappa}{hW} \\ v_x \\ v_y \\ v_z \\ 1-\frac{\kappa}{hW} \end{bmatrix} \quad \vec{r}_{0,2} = \begin{bmatrix} Wv_y \\ h\left(\gamma_{xy}+2W^2v_xv_y\right) \\ h\left(\gamma_{yy}+2W^2v_yv_y\right) \\ h\left(\gamma_{zy}+2W^2v_zv_y\right) \\ Wv_y\left(2hW-1\right) \end{bmatrix} \quad \vec{r}_{0,3} = \begin{bmatrix} Wv_z \\ h\left(\gamma_{xz}+2W^2v_xv_z\right) \\ h\left(\gamma_{yz}+2W^2v_yv_z\right) \\ h\left(\gamma_{zz}+2W^2v_zv_z\right) \\ Wv_z\left(2hW-1\right) \end{bmatrix} \quad \vec{r}_\pm = \begin{bmatrix} 1 \\ hW\left(v_x - \frac{v^x-\Lambda_\pm^x}{\gamma^{xx}-v^x\Lambda_\pm^x}\right) \\ hWv_y \\ hWv_z \\ \frac{hW\left(\gamma^{xx}-v^xv^x\right)}{\gamma^{xx}-v^x\Lambda_\pm^x}-1 \end{bmatrix}$$

Special relativistic limit (along x-direction)

$$\lambda_0 = v^x \quad \text{(triple)}$$

$$\lambda_\pm = \frac{1}{1 - v^2 c_s^2}\left(v^x \left(1 - c_s^2\right) \pm c_s \sqrt{\left(1 - v^2\right)\left[1 - v^x v^x - \left(v^2 - v^x v^x\right) c_s^2\right]}\right)$$

coupling with transversal components of the velocity
(important difference with Newtonian case)

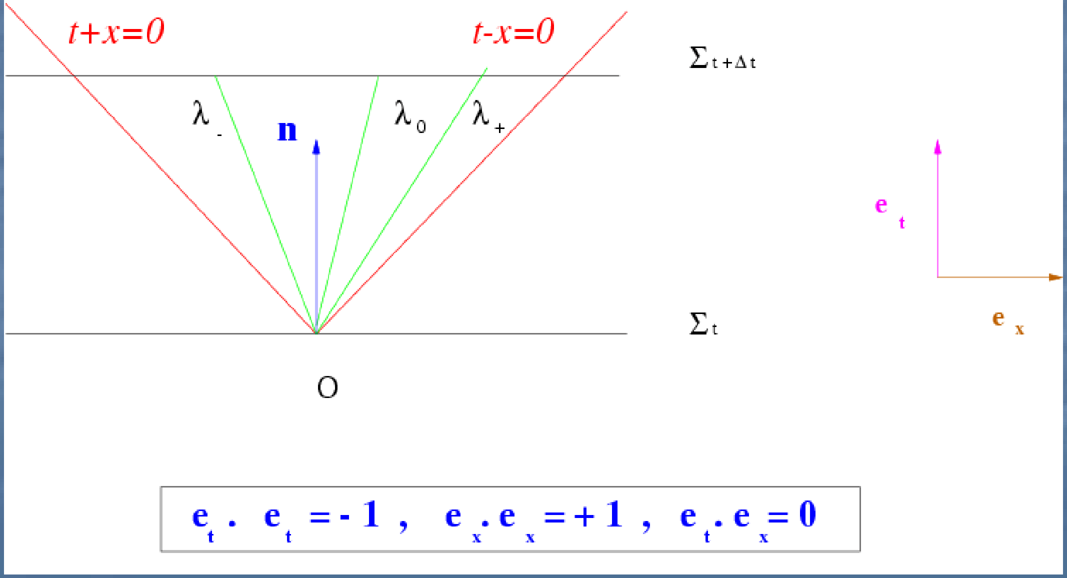
Even in the purely 1D case: $\vec{v} = \left(v^x, 0, 0\right) \Rightarrow \lambda_0 = v^x, \quad \lambda_\pm = \dfrac{v^x \pm c_s}{1 \pm v^x c_s}$

$\Updownarrow$

Recall Newtonian (1D) case:

$$\lambda_0 = v^x, \lambda_\pm = v^x \pm c_s$$

For causal EOS the sound cone lies within the light cone



Part 3
General relativistic (ideal) magneto-hydrodynamics equations

3+1 General Relativistic (Ideal) Magnetohydrodynamics equations (1)

GRMHD: Dynamics of relativistic, electrically conducting fluids in the presence of magnetic fields.

Ideal GRMHD: Absence of viscosity effects and heat conduction in the limit of infinite conductivity (perfect conductor fluid).

The stress-energy tensor includes the contribution from the perfect fluid and from the magnetic field b^{μ} measured by the observer comoving with the fluid.

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\mathrm{PF}} + T^{\mu\nu}_{\mathrm{EM}}$$

$$T^{\mu\nu}_{\mathrm{PF}} \equiv \rho h u^{\mu} u^{\nu} + p g^{\mu\nu}$$

$$T^{\mu\nu}_{\mathrm{EM}} \equiv F^{\mu\lambda} F^{\nu}_{\lambda} - \frac{1}{4} g^{\mu\nu} F^{\lambda\delta} F_{\lambda\delta} = \left(u^{\mu} u^{\nu} + \frac{1}{2} g^{\mu\nu} \right) b^2 - b^{\mu} b^{\nu}$$

$$\boxed{T^{\mu\nu} = \rho h^{*} u^{\mu} u^{\nu} + p^{*} g^{\mu\nu} - b^{\mu} b^{\nu}}$$

with the definitions:

$$b^2 = b^{\nu} b_{\nu}$$

$$p^{*} = p + \frac{b^2}{2}$$

$$h^{*} = h + \frac{b^2}{\rho}$$

$$F^{\mu\nu} = -\eta^{\mu\nu\lambda\delta} u_{\lambda} b_{\delta}$$

$$F^{\mu\nu} u_{\nu} = 0$$

Ideal MHD condition: electric four-current must be finite.

$$J^{\mu} = \rho_{\mathrm{q}} u^{\mu} + \sigma F^{\mu\nu} u_{\nu} \quad (\sigma \to \infty)$$

3+1 General Relativistic (Ideal) Magnetohydrodynamics equations (2)

1. Conservation of mass: $\nabla_\mu (\rho u^\mu) = 0$
2. Conservation of energy and momentum: $\nabla_\mu T^{\mu\nu} = 0$
3. Maxwell's equations: $\nabla_\mu {}^*F^{\mu\nu} = 0 \quad , \quad {}^*F^{\mu\nu} = \frac{1}{W}\left(u^\mu B^\nu - u^\nu B^\mu\right)$
   - Induction equation: $\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\vec{B}\right) = \vec{\nabla}\times\left[\left(\alpha\vec{v} - \vec{\beta}\right)\times\vec{B}\right]$
   - Divergence-free constraint: $\vec{\nabla}\cdot\vec{B} = 0$

Adding all up: first-order, flux-conservative, hyperbolic system of balance laws

+ constraint (divergence-free condition)

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\,\vec{u}}{\partial t} + \frac{\partial\sqrt{-g}\,\vec{f}^{\,i}}{\partial x^i}\right) = \vec{s} \qquad \frac{\partial\sqrt{\gamma}\,B^i}{\partial x^i} = 0$$

sources

state vector

$$\vec{u} = \begin{pmatrix} D \\ S_j \\ \tau \\ B^k \end{pmatrix}$$

fluxes

$$\vec{f}^{\,i} = \begin{pmatrix} D\tilde{v}^i \\ \rho h^* W^2 v_j \tilde{v}^i + p^*\delta^i_j - b^i b_j \\ \rho h^* W^2 \tilde{v}^i + p^*\beta^i/\alpha - D\tilde{v}^i - \alpha b^0 b^i \\ \tilde{v}^i B^k - \tilde{v}^k B^i \end{pmatrix}$$

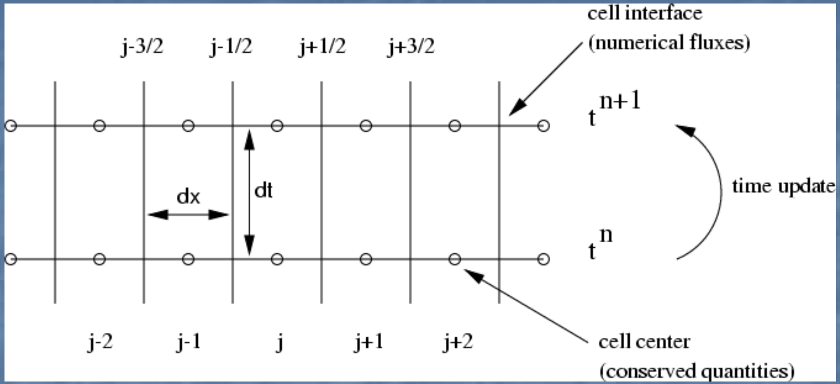
$$\vec{s} = \begin{pmatrix} 0 \\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{pmatrix}$$

Part 4
Numerical solution
(hyperbolic systems of conservation laws)

A standard implementation of a HRSC FD scheme

1. Time update: Conservation form algorithm

$$\vec{u}_j^{\,n+1} = \vec{u}_j^{\,n} - \frac{\Delta t}{\Delta x}\left(\hat{\vec{f}}^{\,n}_{j+1/2} - \hat{\vec{f}}^{\,n}_{j-1/2} \right)$$



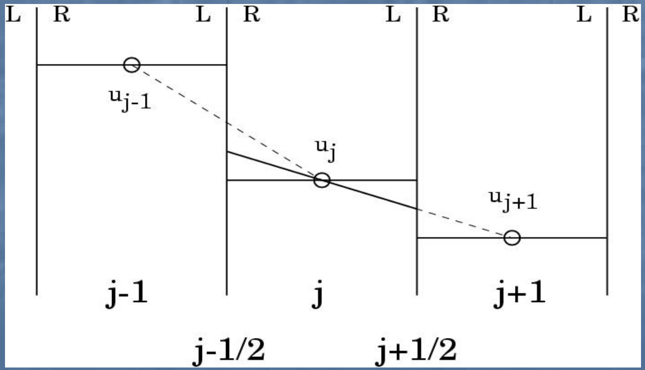
In practice: 2nd or 3rd order time accurate, conservative Runge-Kutta schemes (Shu & Osher 1989; MoL)

2. Cell reconstruction: Piecewise constant (Godunov), linear (MUSCL, MC, van Leer), parabolic (PPM, Colella & Woodward 1984), hyperbolic (Marquina 1992), ENO, etc **interpolation procedures** of state-vector variables from cell centers to cell interfaces.

3. Numerical fluxes: Approximate Riemann solvers (Roe, HLLE, Marquina). Explicit use of the spectral information of the system

$$\hat{\vec{f}}_i = \frac{1}{2}\left[\vec{f}_i(w_R) + \vec{f}_i(w_L) - \sum_{n=1}^{5} \left|\tilde{\lambda}_n\right| \Delta\tilde{\omega}_n \tilde{R}_n \right]$$

$$\mathrm{U}(w_R) - \mathrm{U}(w_L) = \sum_{n=1}^{5} \Delta\tilde{\omega}_n \tilde{R}_n$$

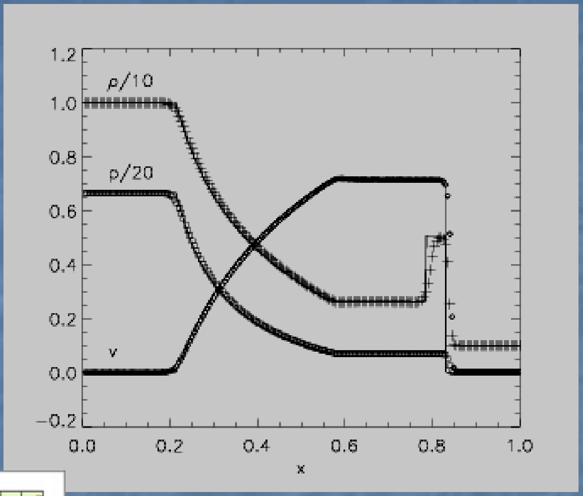


MUSCL minmod recosnstruction (piecewise linear)

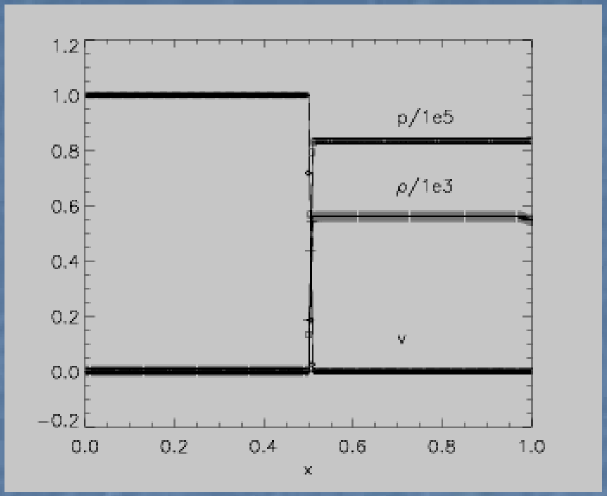
HRSC schemes for the GR hydrodynamics equations

- Stable and sharp discrete shock profiles
- Accurate propagation speed of discontinuities
- Accurate resolution of multiple nonlinear structures: discontinuities, raraefaction waves, vortices, etc

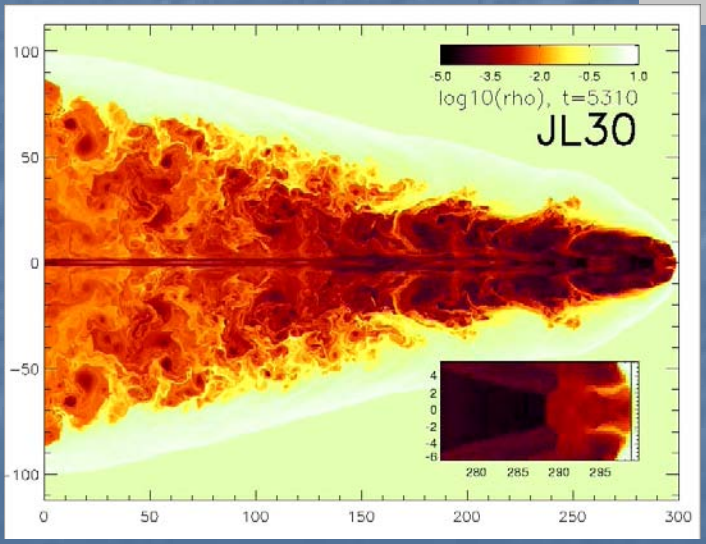
Relativistic shock reflection



Shock tube test

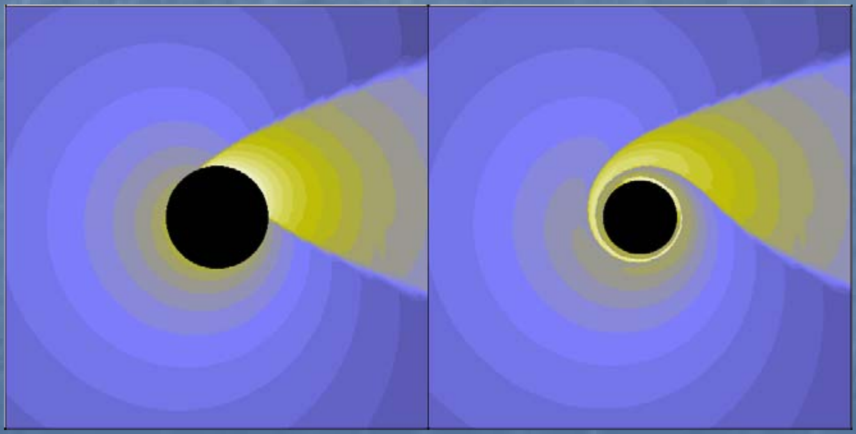


V=0.99999c (W=224)



Simulation of a extragalactic relativistic jet

Scheck et al, MNRAS, 331, 615 (2002)

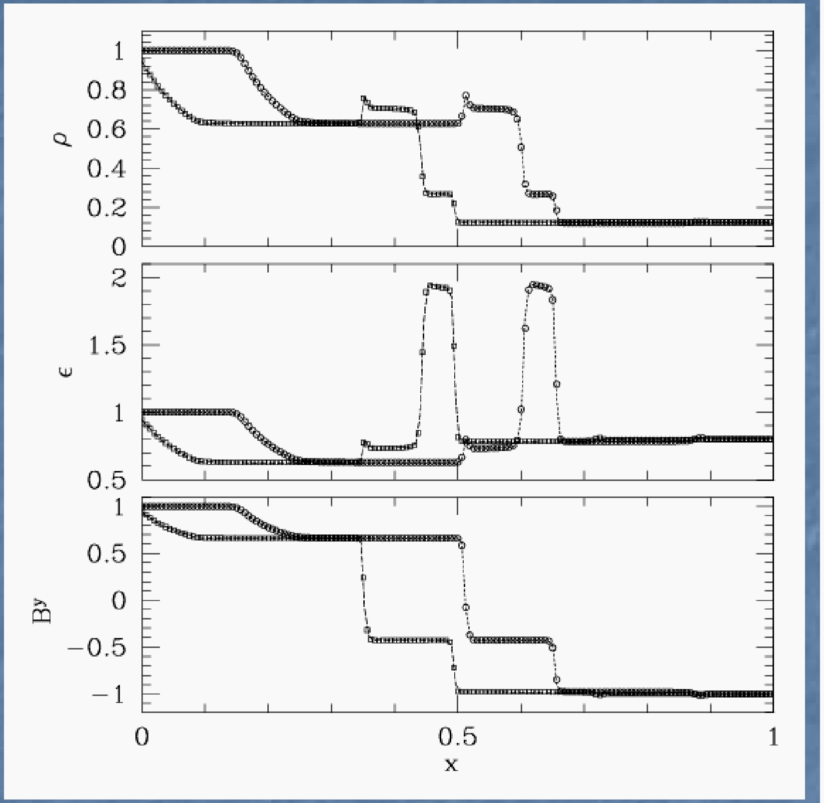


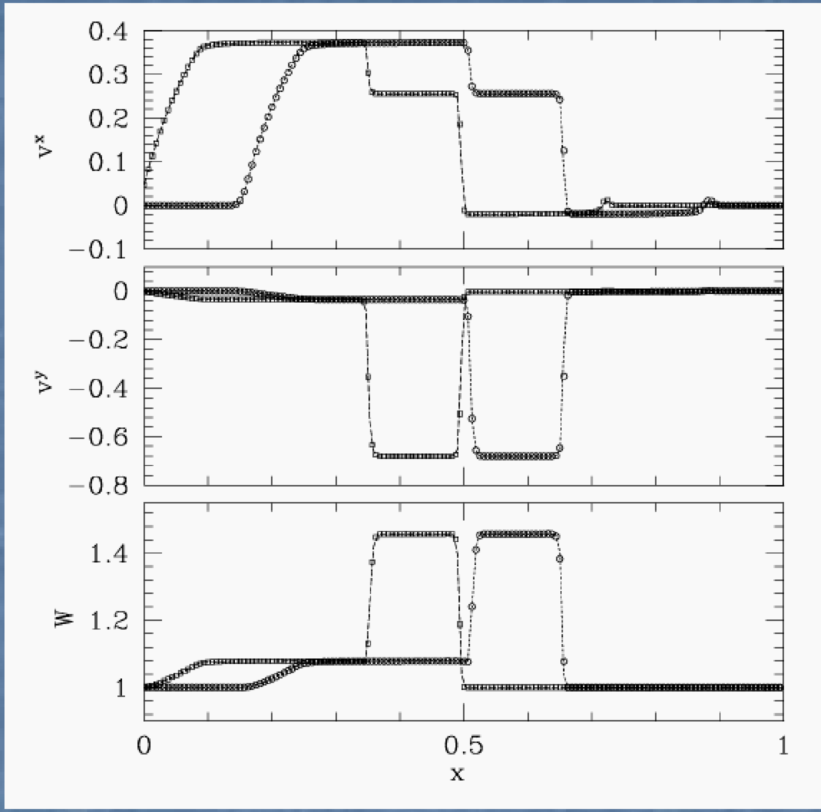
Wind accretion onto a Kerr black hole (a=0.999M)

Font et al, MNRAS, 305, 920 (1999)

GRMHD equations: shock tube tests

1D Relativistic Brio-Wu shock tube test (van Putten 1993, Balsara 2001)



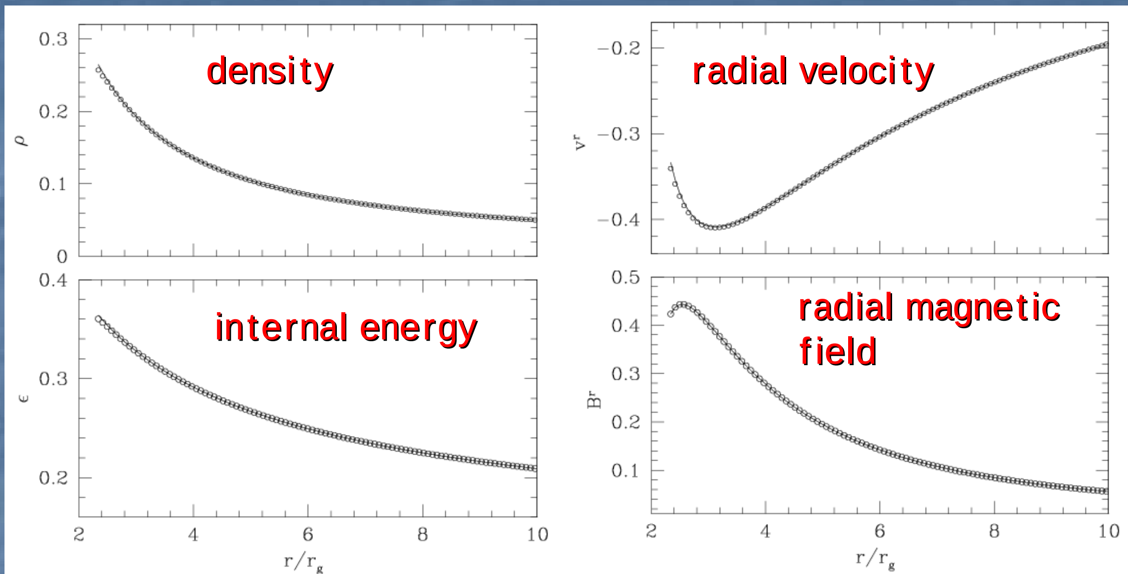


Dashed line: wave structure in Minkowski spacetime at time t=0.4
Open circles: nonvanishing lapse function (2), at time t=0.2
Open squares: nonvanishing shift vector (0.4), at time t=0.16

HLL solver
1600 zones
CFL 0.5

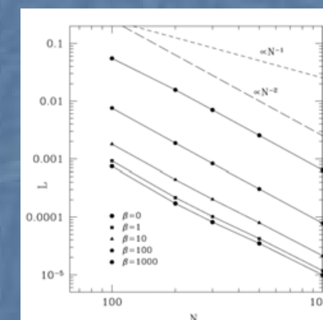
Agreement with previous authors (Balsara 2001) regarding wave locations, maximum Lorentz factor achieved, and numerical smearing of the solution.

GRMHD equations: code tests in strong gravity (black holes)

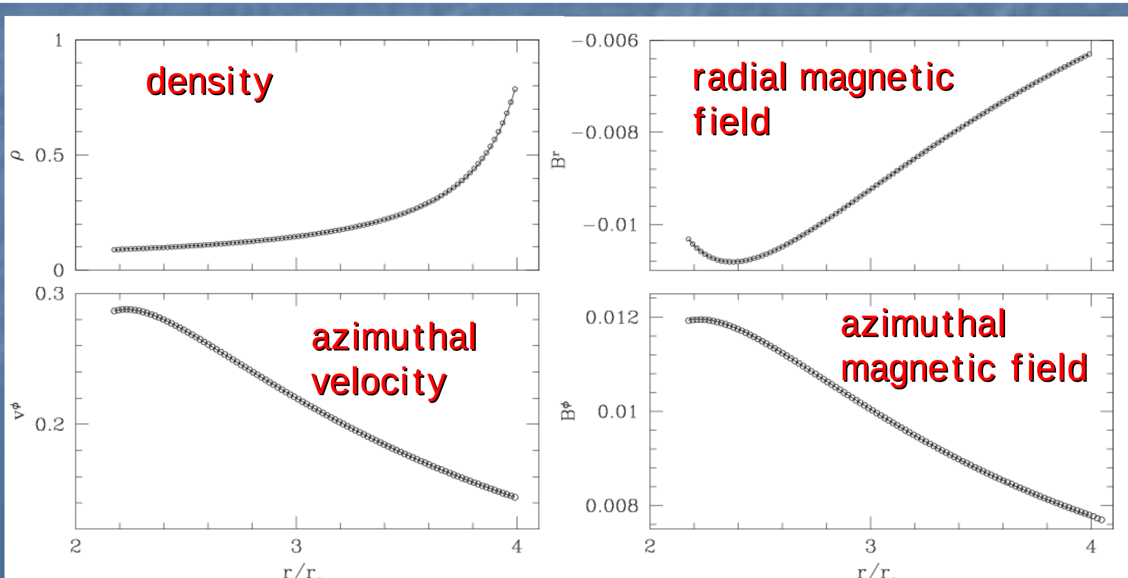


Magnetised spherical accretion onto a Schwarzschild BH

Test difficulty: keep stationarity of the solution. Used in the literature (Gammie et al 2003, De Villiers & Hawley 2003)



2nd order convergence



Magnetised equatorial Kerr accretion (Takahashi et al 1990, Gammie 1999)

Test difficulty: keep stationarity of the solution (algebraic complexity augmented, Kerr metric)

Used in the literature (Gammie et al 2003, De Villiers & Hawley 2003)

Part 5

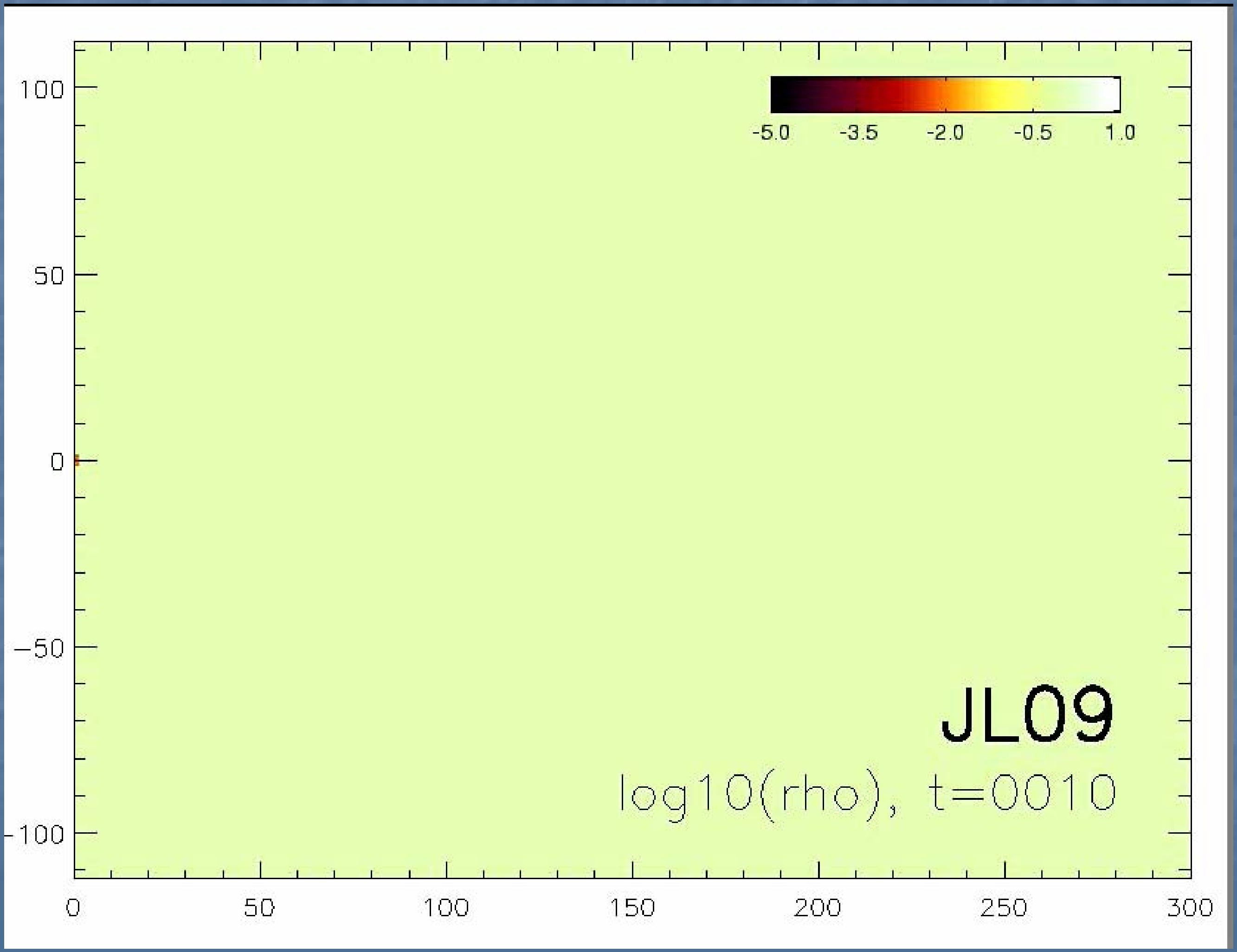
Applications in relativistic astrophysics

Applications of the GRHD/GRMHD equations in relativistic astrophysics

1. <u>Heavy ion collisions</u> (SR limit): Clare & Strottman 1986, Wilson & Mathews 1989, Rischke et al 1995a,b.
2. <u>Simulations of relativistic jets (SR limit):</u> Martí et al 1994, 1995, 1997, Gómez et al 1995, 1997, 1998, Aloy et al 1999, 2003, Scheck et al 2002, Leismann et al 2005.
3. <u>GRB models:</u> Aloy et al 2000, Zhang, Woosley & MacFadyen 2003, Aloy, Janka & Müller 2004.
4. <u>Gravitational stellar core collapse</u>: Dimmelmeier, Font & Müller 2001, 2002a,b, Dimmelmeier et al 2004, Cerdá-Durán et al 2004, Shibata & Sekiguchi 2004, 2005.
5. <u>Gravitational collapse and black hole formation:</u> Wilson 1979, Dykema 1980, Nakamura et al 1980, Nakamura 1981, Nakamura & Sato 1982, Bardeen & Piran 1983, Evans 1984, 1986, Stark & Piran 1985, Piran & Stark 1986, Shibata 2000, Shibata & Shapiro 2002, Baiotti et al 2005, Zink et al 2005.
6. <u>Pulsations and instabilities of rotating relativistic stars</u>: Shibata, Baumgarte & Shapiro, 2000; Stergioulas & Font 2001, Font, Stergioulas & Kokkotas 2000, Font et al 2001, 2002, Stergioulas, Apostolatos & Font 2004, Shibata & Sekiguchi 2003, Dimmelmeier, Stergioulas & Font 2005.
7. <u>Accretion on to black holes</u>: Font & Ibáñez 1998a,b, Font, Ibáñez & Papadopoulos 1999, Brandt et al 1998, Papadopoulos & Font 1998a,b, Nagar et al 2004, Hawley, Smarr & Wilson 1984, Petrich et al 1989, Hawley 1991.
8. <u>Disk accretion</u>: Font & Daigne 2002a,b, Daigne & Font 2004, Zanotti, Rezzolla & Font 2003, Rezzolla, Zanotti & Font 2003, Zanotti et al 2005.
9. <u>GRMHD simulations of BH accretion disks</u>: Yokosawa 1993, 1995, Igumenshchev & Belodorov 1997, De Villiers & Hawley 2003, Hirose et al 2004. Gammie et al 2003.
10. <u>Jet formation</u>: Koide et al 1998, 2000, 2002, 2003, McKinney & Gammie 2004, Komissarov 2005.
11. <u>Binary neutron star mergers</u>: Miller, Suen & Tobias 2001, Shibata, Taniguchi & Uryu 2003, Evans et al 2003, Miller, Gressman & Suen 2004, Wilson, Mathews & Marronetti 1995, 1996, 2000, Nakamura & Oohara 1998, Shibata 1999, Shibata & Uryu 2000, 2002.

Example: Long-term evolution of a relativistic, hot, leptonic (e^+/e^-) jet up to 6.3×10^6 years

Scheck et al. MNRAS, 331, 615-634 (2002)



Example: Nonlinear instabilities in relativistic jets

Perucho et al. ApJ (2004)

2D "slab jet" Cartesian coordinates. Lorentz factor 5. Periodic boundary conditions. 256 zones per beam radius.

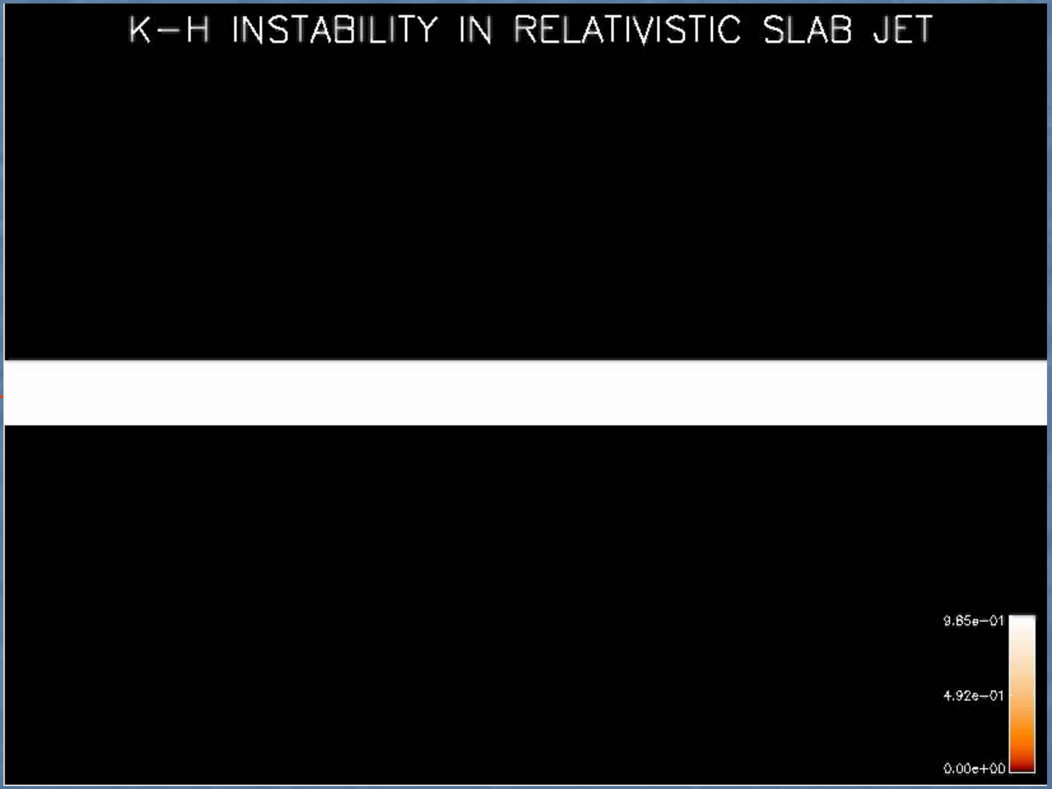
Initial model perturbed with 4 symmetric perturbations and 4 antisymmetric perturbations.

Initial location of the beam

The evolution of the linear phase agrees with analytic results from perturbation theory.

Development of nonlinearities visible (Kelvin-Helmholtz instability). Once they saturate a quasi-equilibrium (turbulent) state is reached.

Time evolution of the jet mass fraction



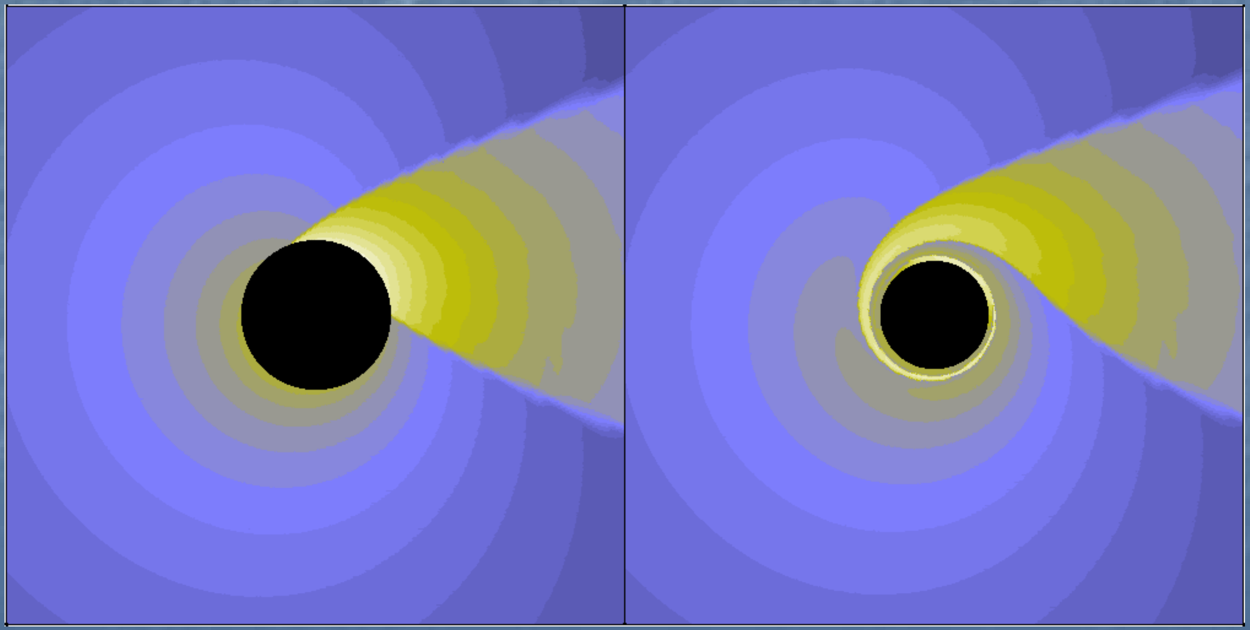
Black: external medium. White: jet

Example: Relativistic wind accretion onto a Kerr black hole

- Binary system: black hole + giant star (O,B spectral type). Strong stellar wind. No Roche lobe overflow (common envelope evolution, Thorne-Zytkow objects).
- Isolated compact objects accreting from the interstellar medium.
- Compact objects in stellar clusters, AGNs and QSOs.

Wind (Bondi-Hoyle-Lyttleton) accretion onto black holes: hydrodynamics in general relativity.

Difficulties: 1) Strong gravitational fields, 2) Ultrarelativistic flows and shock waves.



Kerr-Schild | Boyer-Lindquist

Wind accretion on to a Kerr black hole

(a=0.999M)

Font et al, MNRAS, 305, 920 (1999)

Roe's Riemann solver.

Isocontours of the log of the density.

Relativistic wind from left to right.

Black hole spinning counter-clockwise.

Example: Relativistic rotating core collapse simulation

For movies of additional models visit:

www.mpa-garching.mpg.de/rel_hydro/axi_core_collapse/movies.shtml

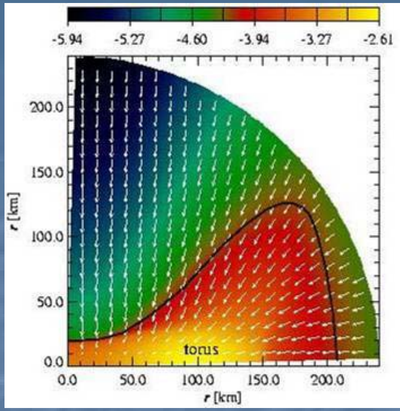
MAX PLANCK INSTITUTE FOR ASTROPHYSICS
GARCHING, GERMANY
http://www.mpa-garching.mpg.de

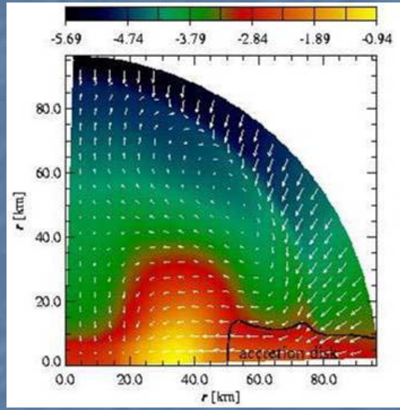
General Relativistic Collapse of Rotating Stellar Cores in Axisymmetry

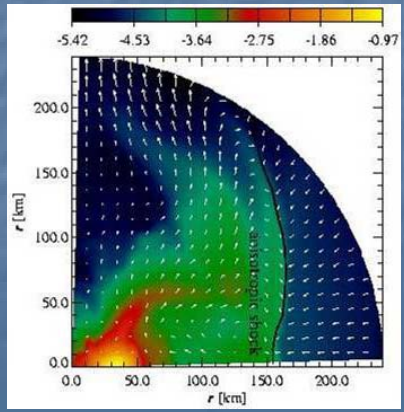
Harald Dimmelmeier
José A. Font
Ewald Müller

References:

- Dimmelmeier, H., Font, J. A., and Müller, E., *Astron. Astrophys.*, 388, 917–935 (2002), astro-ph/0204288.
- Dimmelmeier, H., Font, J. A., and Müller, E., *Astron. Astrophys.*, submitted (2002), astro-ph/0220489.







Larger central densities in relativistic models
Similar gravitational radiation amplitudes (or smaller in the GR case)

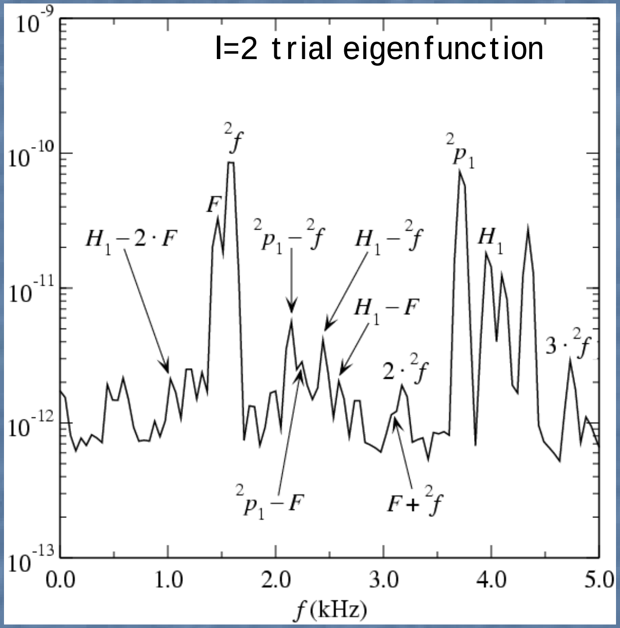
GR effects do not improve the chances for detection (at least in axisymmetry): Only a Galactic supernova (10 kpc) would be detectable by the first generation of gravitational wave laser interferometers.

Waveform catalogue: **www.mpa-garching.mpg.de/rel_hydro/wave_catalogue.shtml**

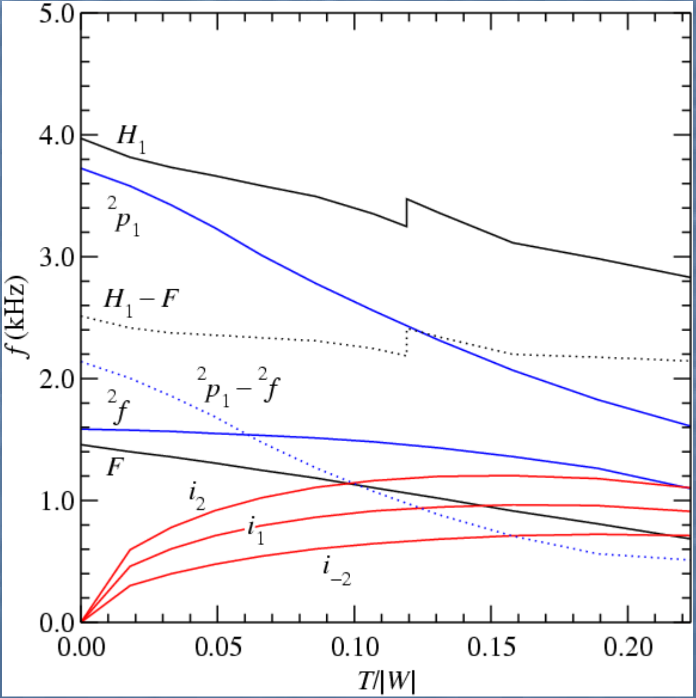
Example: nonlinear pulsations of rotating neutron stars

Dimmelmeier, Stergioulas & Font, MNRAS, submitted (2005)

GRHD simulations show the appearance of nonlinear harmonics of the linear pulsation modes: linear sums and differences of linear mode frequencies.



The presence of nonlinear harmonics makes possible 3-mode couplings when the star is rotating, as different modes are affected in different ways by rotation.



3-mode couplings could *potentially* lead to resonance effects or (parametric) instabilities: significant amount of energy from one mode could be transfered to other modes.

Best case scenario: Pulsational energy from the quasi-radial mode (stored during core bounce) transfered to stronger radiating nonradial modes.

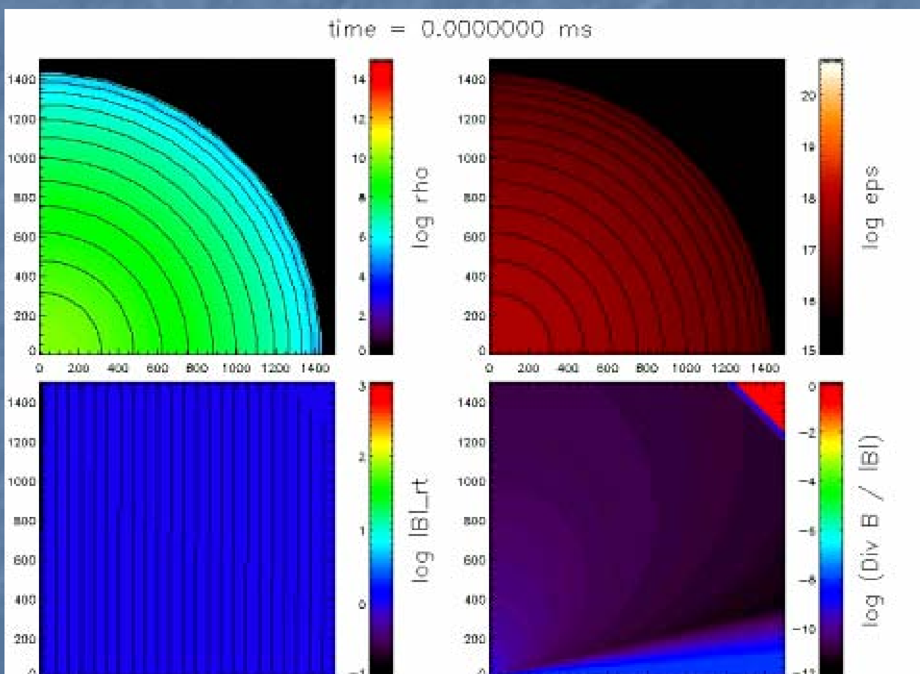
Example: GRMHD, spherical core collapse simulation

As a first step towards relativistic magnetized core collapse simulations we employ the test (passive) field approximation for weak magnetic field.

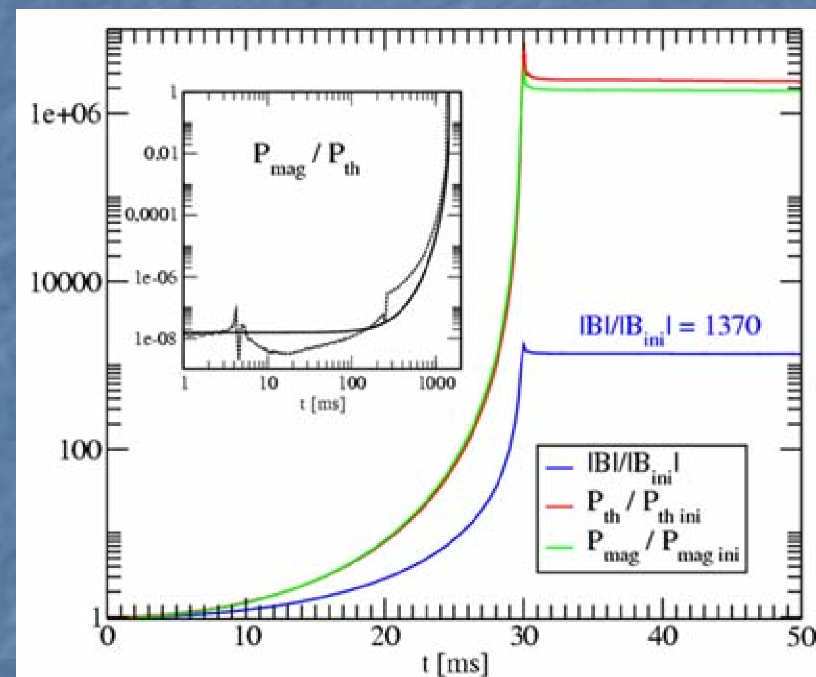
• magnetic field attached to the fluid (does not backreact into the Euler-Einstein eqns.).

• eigenvalues (fluid + magnetic field) reduce to the fluid eigenvalues only.

HLL solver + PPM, Flux-CT, 200x10 zones



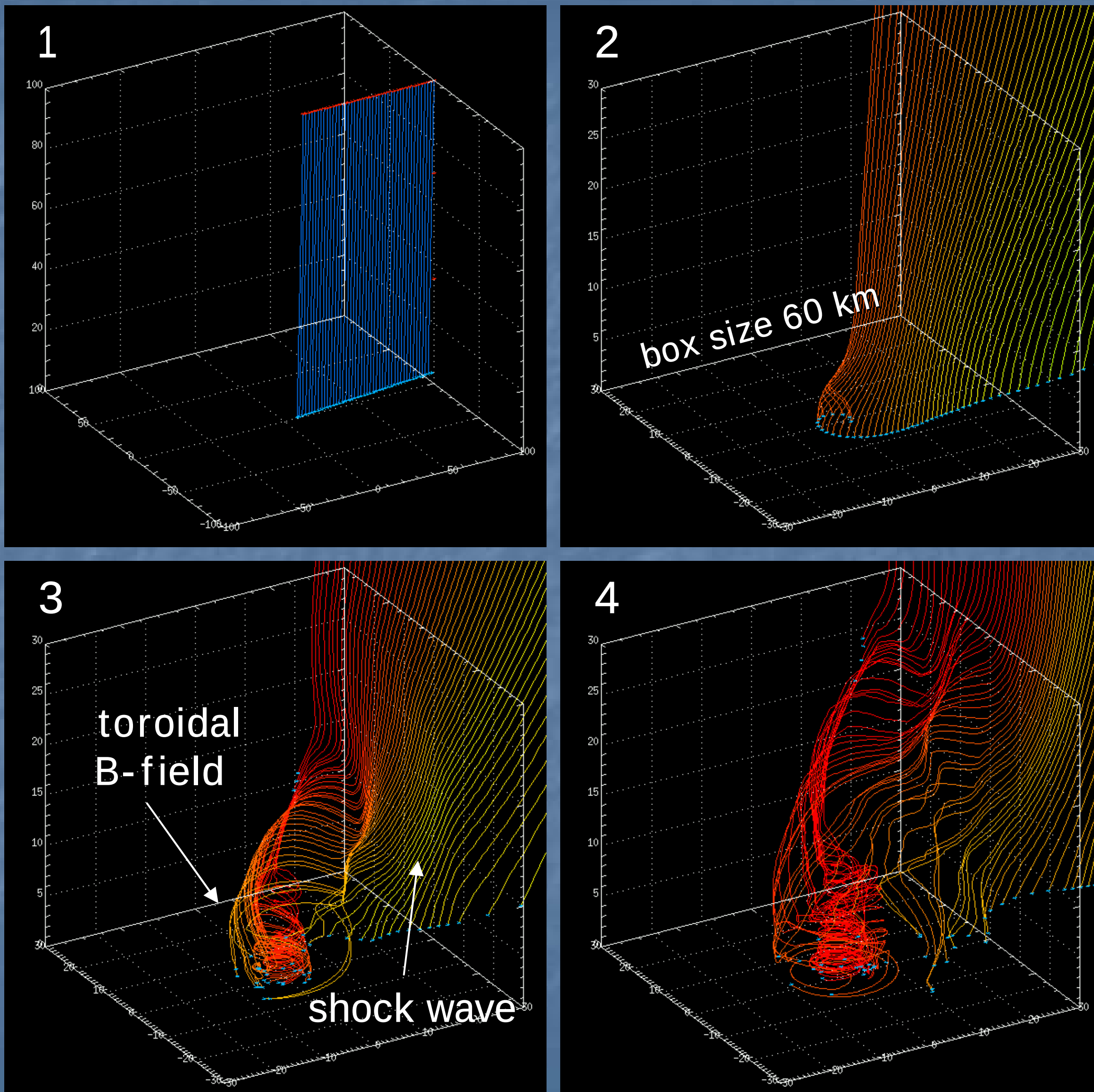
The divergence-free condition is fulfilled to good precision during the simulation.



The amplification factor of the initial magnetic field during the collapse is 1370.

PhD thesis of Pablo Cerdá-Durán, U. Valencia (2006)

Example: Rotational magnetised core collapse - field lines evolution



PhD thesis of Pablo Cerdá-Durán, U. Valencia (2006)

Initial model: A1B3G5 (Dimmelmeier et al 2002) + homogeneous, poloidal (test) magnetic field.

Equation of the magnetic field lines:

$$\frac{d\vec{x}}{d\lambda} = \vec{n} \qquad \vec{n} = \frac{\vec{B}}{|\vec{B}|}$$

$$\longrightarrow$$

$$r = r_0 + \int n^r d\lambda$$

$$\theta = \theta_0 + \int \frac{n^\theta}{r} d\lambda$$

$$\phi = \phi_0 + \int \frac{n^\phi}{r \sin\theta} d\lambda$$

Colour code: blue (low magnetic flux), red (high magnetic flux)

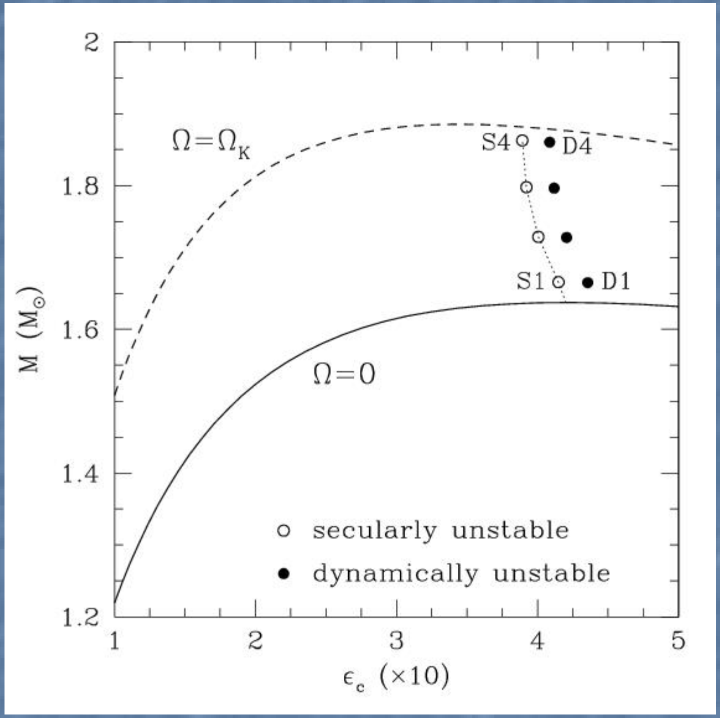
Differential rotation drives the twisting of the magnetic field lines in the nascent PNS, and the appearance of a toroidal component in the magnetic field.

Example: 3D relativistic simulations of rotating NS collapse to a Kerr black hole

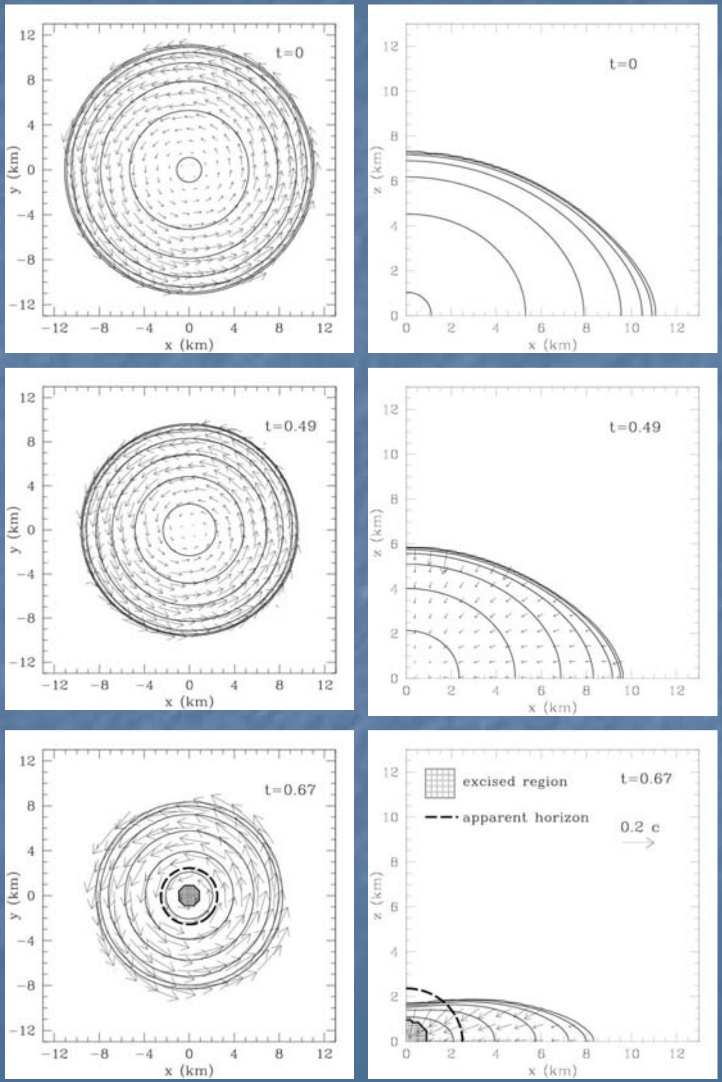
Baiotti, Hawke, Montero, Löffler, Rezzolla, Stergioulas, Font & Seidel, Phys. Rev. D, 2005, **75**, 024035

Simulation code Whisky developed at AEI, SISSA, AUTH, UV (www.eu-network.org)

Gravitational mass of secularly and dynamically unstable initial models vs central energy density



Evolution of model D4



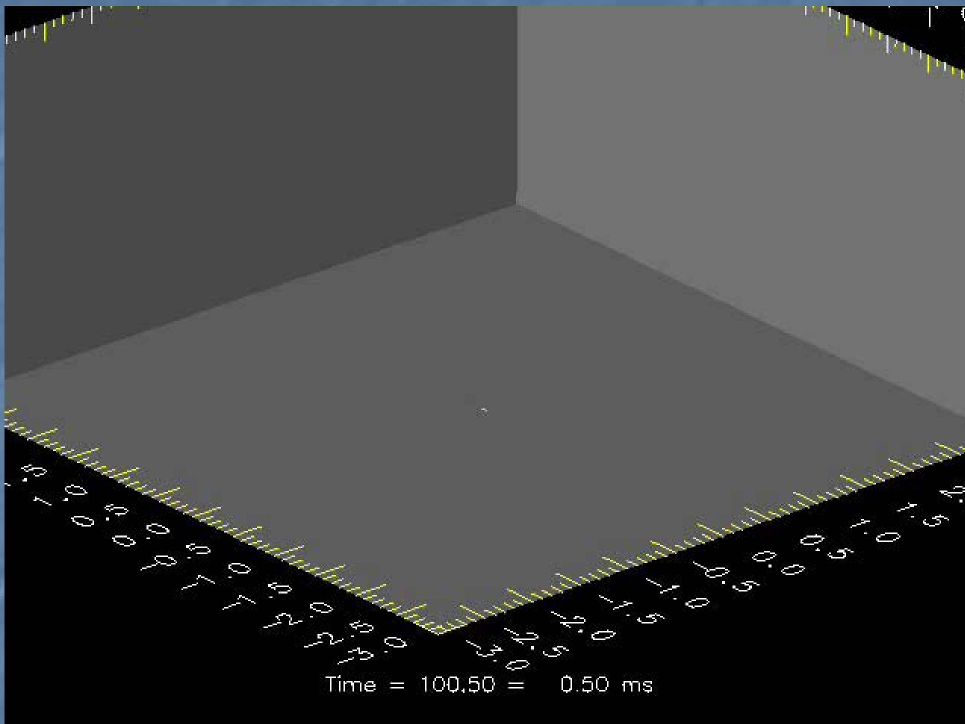
Solid line: sequence of nonrotating models

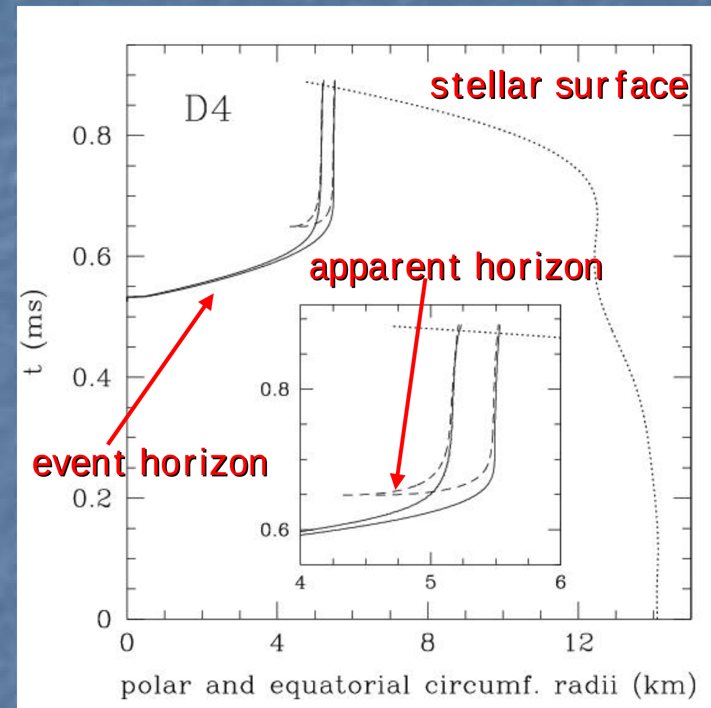
Dashed line: models rotating at the mass-shedding limit

Dotted line: sequence of models at the onset of secular instability to axisymmetric perturbations

Calculating apparent and event horizons

Grey surface: event horizon. White surface: apparent horizon. Circles: horizon generators



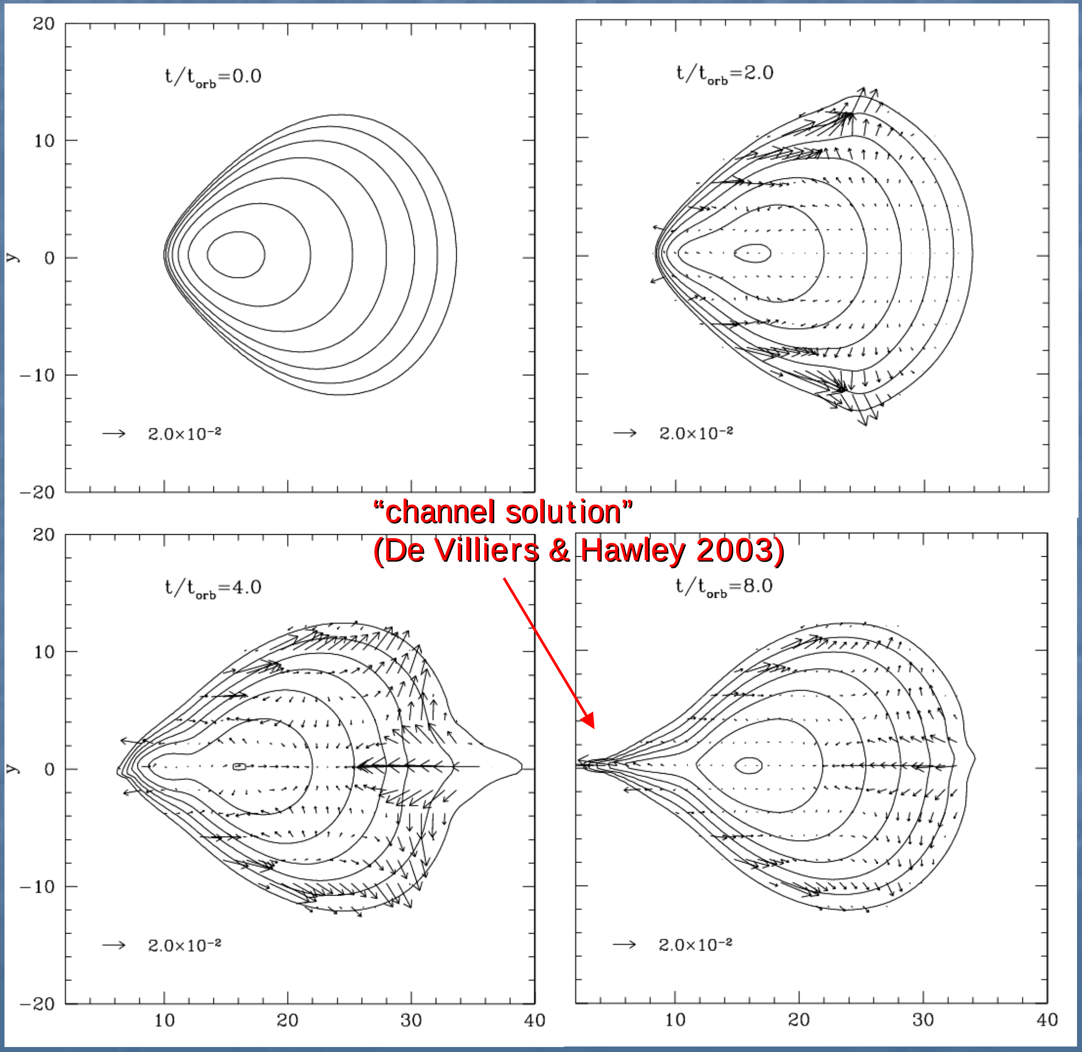


Calculations and visualization by P. Diener (AEI/LSU)

- As the collapse proceeds, trapped surfaces form (photons cannot leave).
- Most relevant surfaces are the *apparent horizon* (outermost of the trapped surfaces) and the *event horizon* (global null surface).
- AH can be computed at any time (zero expansion of a photon congruence). EH requires the construction of the whole spacetime.

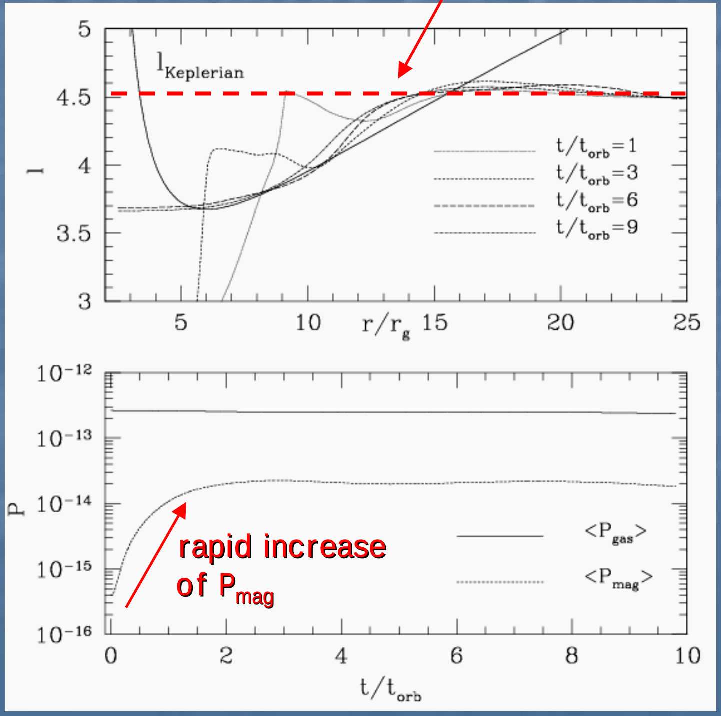
Example: GRMHD equations, magnetised torus (Antón et al 2005)

Evolution of a magnetised accretion torus around a Schwarzschild black hole.
An ad-hoc poloidal magnetic field added to the hydrodynamical disc model.



Development of the magnetorotational instability.

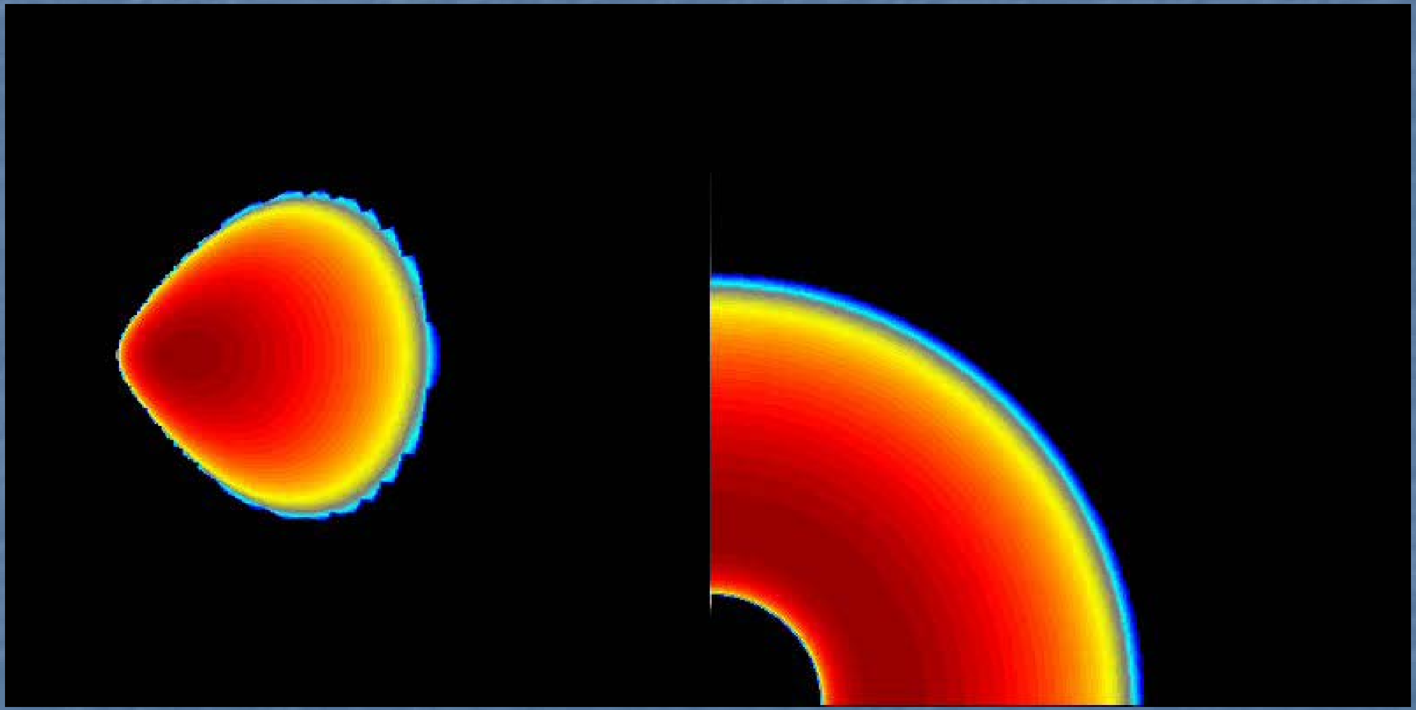
Magnetic field + differentially rotating Keplerian disk $\rightarrow$ MRI : generation of effective viscosity and angular momentum transport outwards through MHD turbulence.



General Relativistic Magnetohydrodynamic Simulations of Accretion Tori

Schwarzschild BH, development of the magnetorotational instability.

Magnetic field + differentially rotating Keplerian disk $\rightarrow$ MRI : generation of effective viscosity and angular momentum transport outwards through MHD turbulence.



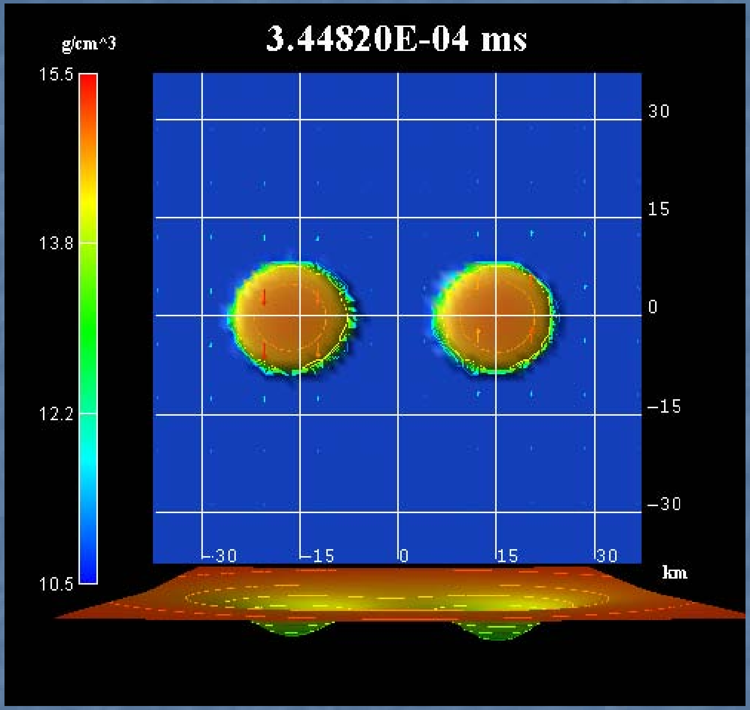
For further animations visit:
http://www.astro.virginia.edu/~jd5v/

De Villiers & Hawley (2003)

Example: Binary neutron star coalescence. Simulations with realistic EOS

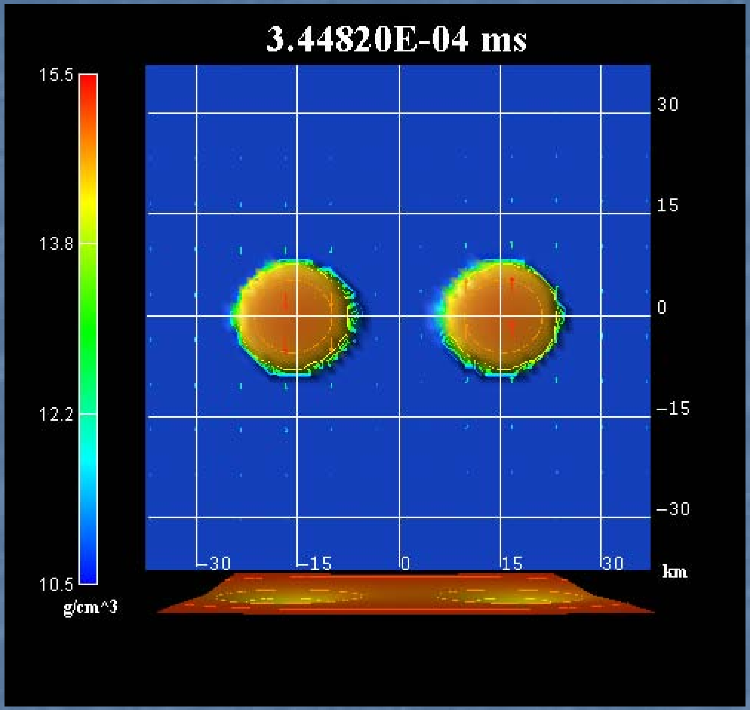
Shibata, Taniguchi & Uryu, 2005, Phys. Rev. D, 71, 084021

Shibata & Font, 2005, Phys. Rev. D, 72,



$1.25M_{sun}$ - $1.35M_{sun}$ case

Formation of a differentially rotating hypermassive neutron star



$1.4M_{sun}$ - $1.4M_{sun}$ case

Prompt formation of a rotating black hole

Quasi-periodic, large amplitude gravitational waves are emitted, with frequencies between 3-4 kHz during ~ 100 ms (after which the NS collapses to a BH).

Those waves could be detected by LIGO-II up to distances of ~ 100 Mpc.

Summary of the talk

- Conservative formulations of the GRHD/GRMHD equations currently available.

- High-resolution shock-capturing numerical schemes based on the wave structure of those hyperbolic systems developed in recent years.

- GRHD simulations in relativistic astrophysics routinely performed nowadays within the so-called "test-fluid" approximation.

- Important advances also achieved for the GRMHD case, but the full development still awaits for a thorough numerical exploration.

- Long-term stable formulations of the full Einstein equations (or accurate enough approximations) have been proposed by several Numerical Relativity groups.

- Stable, coupled evolutions (GRHD/GRMHD + Einstein equations) are becoming possible in 3D, allowing for the study of interesting relativistic astrophysics scenarios (e.g. gravitational collapse, accretion, binary neutron star mergers).