Numerical relativistic hydrodynamics

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Plan of the Talk

- Astrophysical motivation
- General Relativistic Hydrodynamics
- Numerical Methods
  - High-Resolution Shock-Capturing schemes
- Simulations of relativistic flows
  - Runaway instability of thick accretion disks
  - Stellar core-collapse
  - Rotating relativistic stars

Further information: J.A. Font "Numerical hydrodynamics in general relativity"
Living Reviews in Relativity, 3, 1, (2000) www.livingreviews.org

Astrophysical motivation

General relativity plays a major role in the description of compact objects

Core-collapse supernovae

BH formation (critical phenomena) and accretion

Coalescing binaries (NS/NS, BH/NS, BH/BH)

Except the (vacuum) BH/BH system, all systems contain matter fields.

Time-dependent evolutions of fluid flow *coupled* to geometry is only possible through accurate, large-scale numerical simulations. Some scenarios can be described in the test-fluid approximation: hydrodynamical computations in (static) curved backgrounds (highly mature nowadays).

<u>The (GR) hydrodynamic equations constitute a non-linear hyperbolic system</u>. Solid mathematical foundations and accurate numerical methodology imported from CFD. A “preferred” choice: high-resolution shock-capturing schemes written in conservation form.

3+1 General Relativistic Hydrodynamics equations (1)

$$\nabla_\mu (\rho u^\mu) = 0 \qquad [1]$$
$$\nabla_\mu T^{\mu\nu} = 0 \qquad [4]$$
$$p = p(\rho, \varepsilon) \qquad [1]$$

Equations of motion:

local conservation laws of density current (continuity equation) and stress-energy (Bianchi identities)

Perfect fluid stress-energy tensor

$$T^{\mu\nu} \equiv \rho h u^\mu u^\nu + p g^{\mu\nu}$$

Introducing an explicit coordinate chart:

$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\,\rho u^\mu) = 0$$
$$\frac{\partial}{\partial x^\mu}(\sqrt{-g}\,T^{\mu\nu}) = \sqrt{-g}\,\Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

Different formulations exist depending on:

1. The choice of time-slicing: the level surfaces of x^0 can be spatial (3+1) or null (characteristic)
2. The choice of physical (primitive) variables (ρ, ε, u^i …)

Wilson (1972) wrote the system as a set of advection equation within the 3+1 formalism. Non-conservative.

Conservative formulations well-adapted to numerical methodology are more recent:

- Martí, Ibáñez & Miralles (1991): 1+1, general EOS
- Eulderink & Mellema (1995): covariant, perfect fluid
- Banyuls et al (1997): 3+1, general EOS
- Papadopoulos & Font (2000): covariant, general EOS

3+1 General Relativistic Hydrodynamics equations (2)

$$\nabla_\mu(\rho u^\mu) = 0 \qquad [1]$$

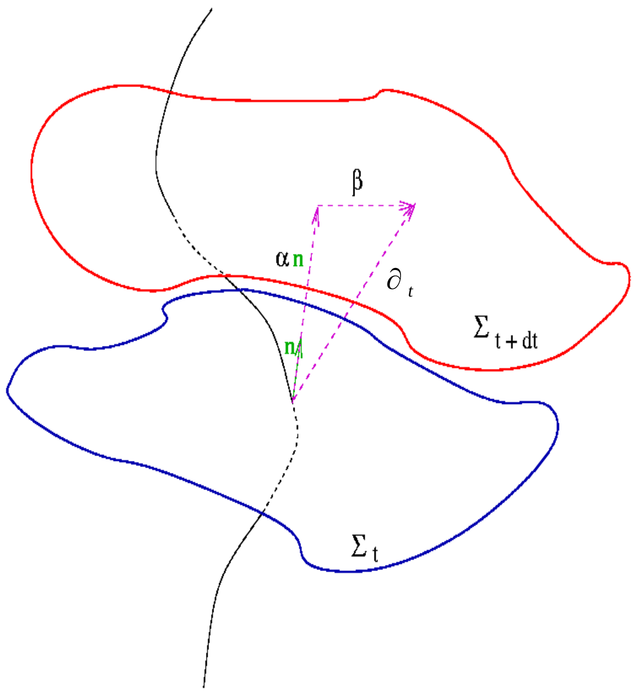
$$\nabla_\mu T^{\mu\nu} = 0 \qquad [4]$$

$$p = p(\rho, \varepsilon) \qquad [1]$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \qquad [10]$$

Einstein's equations

Foliate the spacetime in *t=const* spatial hypersurfaces Σ_t



$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Let n be the unit timelike 4-vector orthogonal to Σ_t such that

$$\mathrm{n} = \frac{1}{\alpha}(\partial_t - \beta^i \partial_i)$$

Eulerian observers

$$\mathrm{v} \equiv -\frac{\mathrm{n} \cdot \partial_i}{\mathrm{n} \cdot \mathrm{u}} \qquad v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

u: fluid's 4-velocity, p: isotropic pressure, ρ : rest-mass density, ε : specific internal energy density, $e=\rho(1+\varepsilon)$: energy density

3+1 General Relativistic Hydrodynamics equations (3)

Replace the *"primitive variables"* in terms of the *"conserved variables" :*

$$\vec{w} = \left(\rho, v^i, \varepsilon\right) \rightarrow \begin{cases} D \equiv \rho W \\ S_j \equiv \rho h W^2 v_j \\ E \equiv \rho h W^2 - p \end{cases} \qquad W^2 \equiv 1/(1 - v^j v_j) \qquad h \equiv 1 + \varepsilon + \frac{p}{\rho}$$

First-order flux-conservative hyperbolic system

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial \sqrt{\gamma}\,\vec{u}(\vec{w})}{\partial t} + \frac{\partial \sqrt{-g}\,\vec{f}^{\,i}(\vec{w})}{\partial x^i}\right) = \vec{s}(\vec{w})$$

Banyuls et al, ApJ, **476**, 221 (1997)

Font et al, PRD, **61**, 044011 (2000)

$\vec{u}(\vec{w}) = \left(D, S_j, E - D\right)$ is the vector of conserved variables

$$\vec{f}^{\,i}(\vec{w}) = \left(D\left(v^i - \frac{\beta^i}{\alpha}\right), S_j\left(v^i - \frac{\beta^i}{\alpha}\right) + p\delta^i_j, E - D\left(v^i - \frac{\beta^i}{\alpha}\right) + p v^i\right)$$

fluxes

$$\vec{s}(\vec{w}) = \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\mu\nu} g_{\delta j}\right), \alpha\left(T^{\mu 0}\frac{\partial \ln \alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\mu\nu}\right)\right)$$

sources

Nonlinear hyperbolic systems of conservation laws (1)

Let us consider the system of *p* equations of conservation laws

$$\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \vec{f}_j(\vec{u})}{\partial x_j} = 0 \quad (\vec{s}(\vec{u}))$$

$$\vec{u} = (u_1, \cdots, u_p) \quad \text{state vector}$$

$$\vec{f}_j(\vec{u}) = (f_{1j}, \cdots, f_{pj}) \quad \text{fluxes}$$

Formally this system expresses the conservation of the state vector. Let *D* be an arbitrary domain of R_d and let $\vec{n} = (n_1, \cdots, n_d)$ be the outward unit normal to the boundary of *D*. Then,

$$\frac{d}{dt}\int_D \vec{u}\, d\vec{x} + \sum_{j=1}^{d} \int_{\partial D} \vec{f}_j(\vec{u}) n_j\, d\vec{S} = 0$$

In most situations one considers the so-called initial value problem (IVP), i.e. the solution of the above system with the initial condition $\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x})$

A key property of hyperbolic systems is that features in the solution propagate at the characteristic speeds given by the eigenvalues of the Jacobian matrices.

The characteristic variables are constant along the characteristic curves

$$\frac{dx}{dt} = \lambda_k(\vec{u}(x,t)),\ k = 1, \cdots, p$$

These curves give information about the propagation of the initial data, which formally permits to reconstruct the future solution for the IVP.

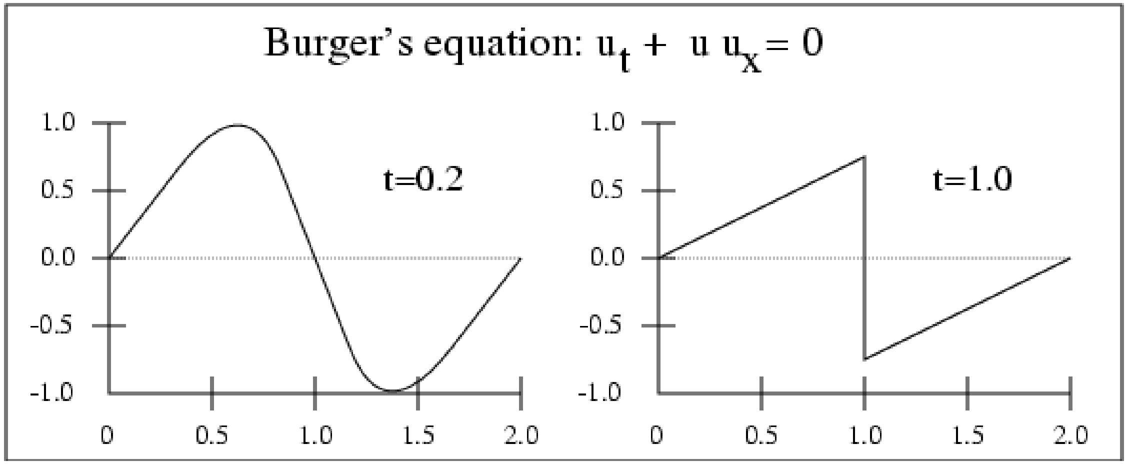
Nonlinear hyperbolic systems of conservation laws (2)

Continuous and differentiable solutions that satisfy the IVP pointwise are called *classical solutions.*

We seek *generalized solutions* that satisfy the integral form of the conservation system, which are classical solutions where they are continuous and have a finite number of discontinuities: *weak solutions.*

The class of all weak solutions is too wide in the sense that there is no uniqueness for the IVP.

For nonlinear systems classical solutions do not exist in general even for smooth initial data. Discontinuities develop after a finite time.



A numerical scheme should guarantee convergence to the *physically admissible solution:* limit solution when $\varepsilon \to 0$ of the "viscous version" of the IVP:

$$\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \vec{f}_j(\vec{u})}{\partial x_j} = \varepsilon \sum_{j=1}^{d} \frac{\partial^2 \vec{u}}{\partial x_j^2}$$

Nonlinear hyperbolic systems of conservation laws (3)

Mathematically, physical solutions are characterized by the so-called entropy condition (the entropy of any fluid element should increase when running into a discontinuity)

The characterization of the entropy-satisfying solutions for scalar equations follows Oleinik (1963), whereas for systems of conservation laws was developed by Lax (1972).

For hyperbolic systems of conservation laws, schemes written in conservation form guarantee that the convergence (if it exists) is to one of the weak solutions of the original system of equations (Lax-Wendroff theorem 1960).

A scheme written in conservation form reads:

$$\vec{u}_j^{n+1} = \vec{u}_j^{n} - \frac{\Delta t}{\Delta x}\left(\hat{\vec{f}}(\vec{u}_{j-r}^{n}, \vec{u}_{j-r+1}^{n}, \cdots, \vec{u}_{j+q}^{n}) - \hat{\vec{f}}(\vec{u}_{j-r-1}^{n}, \vec{u}_{j-r}^{n}, \cdots, \vec{u}_{j+q-1}^{n})\right)$$

where $\hat{\vec{f}}$ is a consistent numerical flux function: $\hat{\vec{f}}(\vec{u}, \vec{u}, \cdots, \vec{u}) = \vec{f}(\vec{u})$

Nonlinear hyperbolic systems of conservation laws (4)

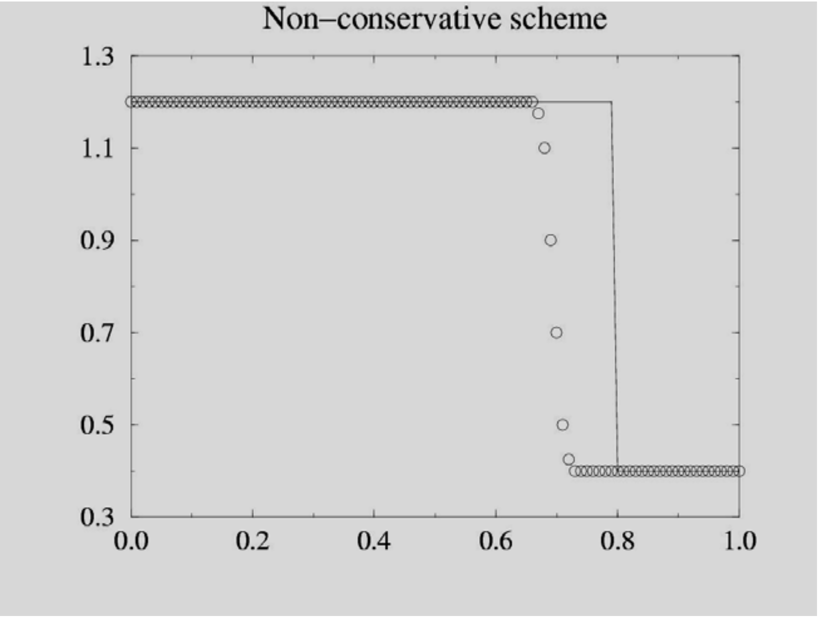
Example: Burger's equation with discontinuous initial data $\frac{\partial u}{\partial t} + \frac{\partial (u^2/2)}{\partial x} = 0$

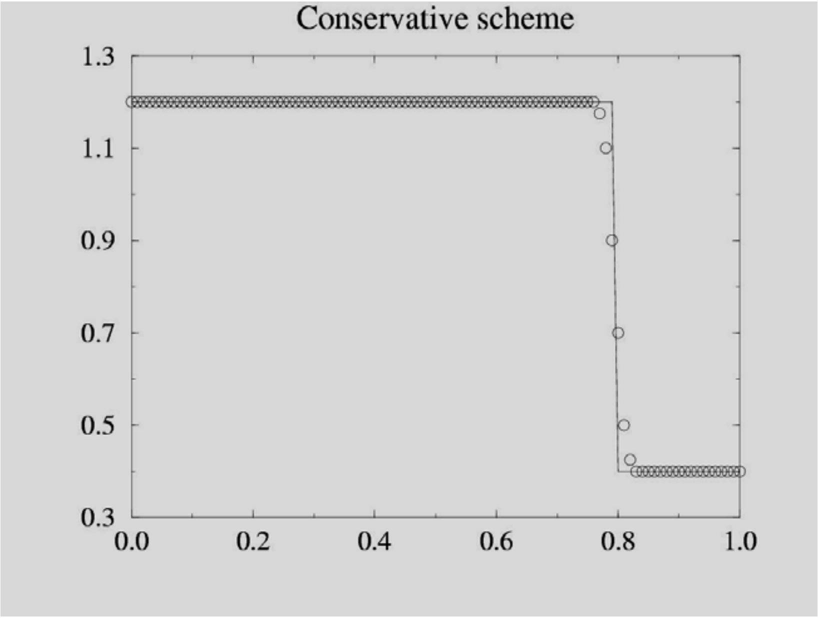
can be discretized by a conservative upwind scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_j^n)^2 - \frac{1}{2}(u_{j-1}^n)^2\right)$$

or using a non-conservative upwind scheme:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n)$$





Nonlinear hyperbolic systems of conservation laws (5)

The Lax-Wendroff theorem does not state whether the method converges. Some form of stability is required to guarantee convergence, as for linear problems (Lax equivalence theorem 1956).

The notion of total-variation stability has proven very successful. Powerful results have only been obtained for scalar conservation laws.

The conservation form of the scheme is ensured by starting with the integral version of the PDE in conservation form. By integrating the PDE within a spacetime computational cell $[x_{j-1/2}, x_{j+1/2}]\times[t^n, t^{n+1}]$ the numerical flux function is an approximation to the time-averaged flux across the interface:

$$\hat{\vec{f}}_{j+1/2} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \vec{f}(\vec{u}(x_{j+1/2}, t))dt$$

The flux integral depends on the solution at the numerical interfaces $\vec{u}(x_{j+1/2}, t)$ during the time step

Key idea: a possible procedure is to calculate $\vec{u}(x_{j+1/2}, t)$ by *solving Riemann problems* at every cell interface (Godunov)

$$\vec{u}(x_{j+1/2}, t) = \vec{u}(0; \vec{u}_j^n, \vec{u}_{j+1}^n)$$

Riemann solution for the left and right states along the ray *x/t=0*.

The Riemann problem

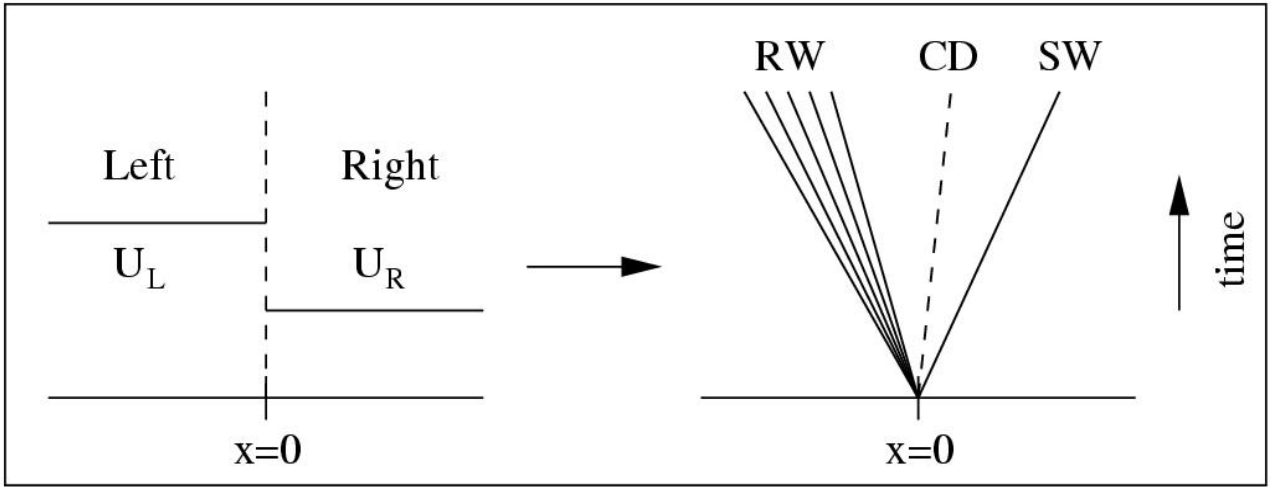
A Riemann problem is an IVP with discontinuous initial data:

$$\vec{u}_0 = \begin{cases} \vec{u}_L & \text{if } x < 0 \\ \vec{u}_R & \text{if } x > 0 \end{cases}$$

The Riemann problem is invariant under similarity transformations:

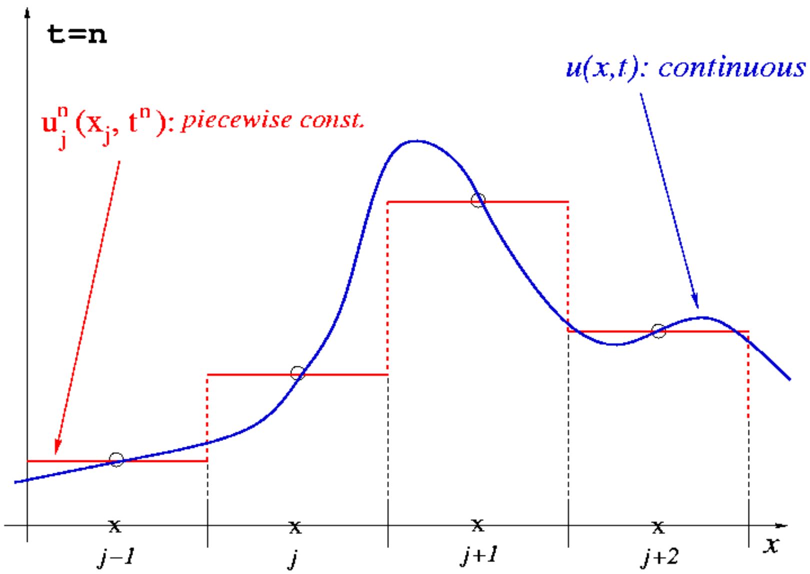
$$(x,t) \to (ax, at) \quad a > 0$$

The solution is constant along the straight lines *x/t=constant*, and, hence, self-similar. It consists of constant states separated by rarefaction waves (continuous self-similar solutions of the differential equations), shock waves, and contact discontinuities (Lax 1972).



The incorporation of the exact solution of Riemann problems to compute the numerical fluxes is due to **Godunov** (1959)

When a Cauchy problem described by a set of continuous PDEs is solved in a discretized form the numerical solution is piecewise constant (collection of local Riemann problems).

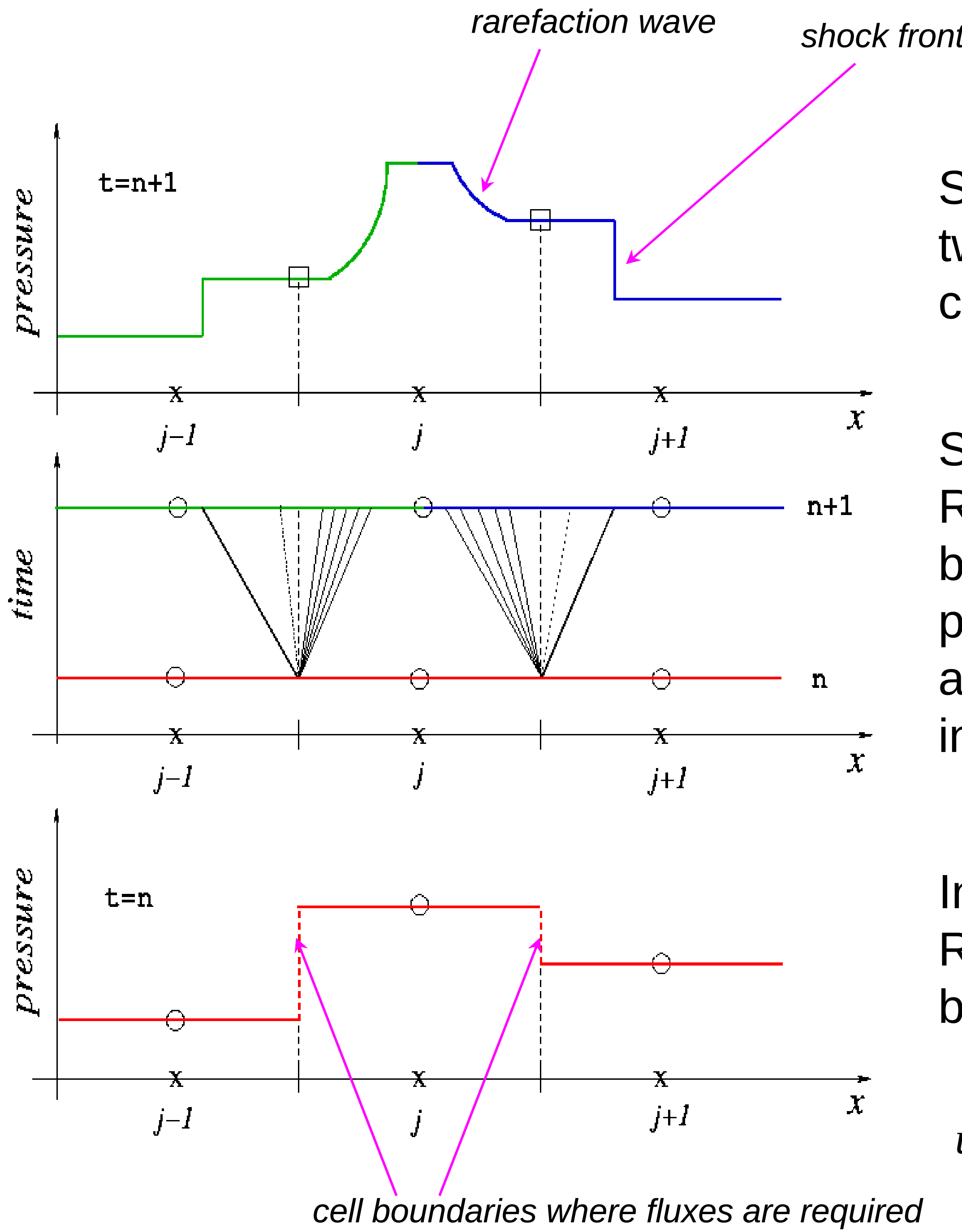


This is particularly problematic when solving the hydrodynamic equations (either Newtonian or relativistic) for compressible fluids.

Their hyperbolic, nonlinear character produces discontinuous solutions in a finite time (shock waves, contact discontinuities) even from smooth initial data!

Any FD scheme must be able to handle discontinuities in a satisfactory way.

1. **1st order accurate schemes (Lax-Friedrich):** Non-oscillatory but inaccurate across discontinuities (excessive diffusion)
2. **(standard) 2nd order accurate schemes (Lax-Wendroff):** Oscillatory across discontinuities
3. 2nd order accurate schemes with artificial viscosity
4. Godunov-type schemes (upwind High Resolution Shock Capturing schemes)



Solution at time n+1 of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions

Initial data at time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

$$\vec{u}_j^{\,n+1} = \vec{u}_j^{\,n} - \frac{\Delta t}{\Delta x}\left(\hat{\vec{f}}^{\,n}_{j+1/2} - \hat{\vec{f}}^{\,n}_{j-1/2} \right)$$

Approximate Riemann solvers

In Godunov's method the structure of the Riemann solution is "lost" in the cell averaging process (1st order in space).

The exact solution of a Riemann problem is computationally expensive particularly in multidimensions and for complicated EoS.

Relativistic multidimensional problems: coupling of all flow velocity components through the Lorentz factor.

- Shocks: increase in the number of algebraic jump (RH) conditions.
- Rarefactions: solving a system of ODEs.

This motivated the development of approximate (linearized) Riemann solvers.

They are based in the exact solution of Riemann problems corresponding to a new system of equations obtained by a suitable linearization of the original one (quasi-linear form). The spectral decomposition of the Jacobian matrices is on the basis of all solvers.

Approach followed by an important subset of shock-capturing schemes, the so-called ***Godunov-type methods*** (Harten & Lax 1983; Einfeldt 1988).

Special Relativistic Riemann Solvers and Flux Formulae

• Roe-type SRRS	→	Martí, Ibáñez & Miralles, 1991
• HLLE SRRS	→	Schneider et al, 1993
• Exact SRRS	→	Martí & Müller, 1994; Pons et al, 2000
• Two-shock approximation	→	Balsara, 1994
• ENO SRRS	→	Dolezal & Wong, 1995
• Roe GRRS	→	Eulderink & Mellema, 1995
• Upwind SRRS	→	Falle & Komissarov, 1996
• Glimm SRRS	→	Wen, Panaitescu & Laguna, 1997
• Iterative SRRS	→	Dai & Woodward, 1997
• Marquina's FF	→	Donat et al, 1998

Martí & Müller, 1999

Living Reviews in Relativity **www.livingreviews.org**

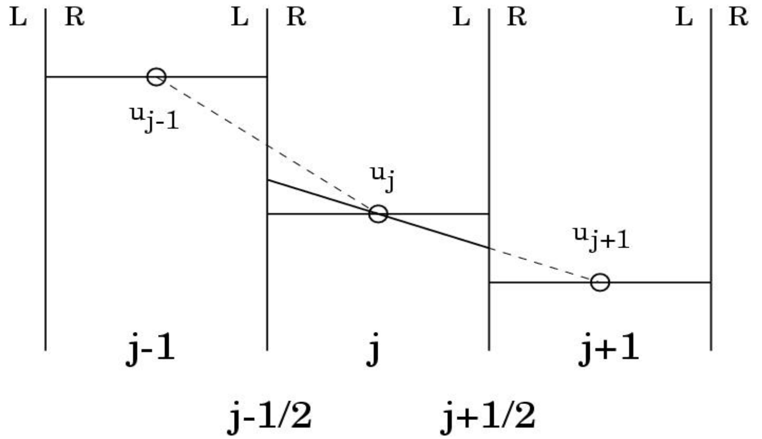
A standard implementation of a HRSC scheme

1. Time update: Conservation form algorithm

$$\vec{u}_j^{\,n+1} = \vec{u}_j^{\,n} - \frac{\Delta t}{\Delta x}\left(\hat{\vec{f}}^{\,n}_{j+1/2} - \hat{\vec{f}}^{\,n}_{j-1/2} \right)$$

In practice: 2nd or 3rd order time accurate, conservative Runge-Kutta schemes (Shu & Osher 1989)

2. Cell reconstruction: Piecewise constant (Godunov), linear (MUSCL, MC, van Leer), parabolic (PPM, Colella & Woodward) interpolation procedures of state-vector variables from cell centers to cell interfaces.



3. Numerical fluxes: Approximate Riemann solvers (Roe, HLLE, Marquina). Explicit use of the spectral information of the system

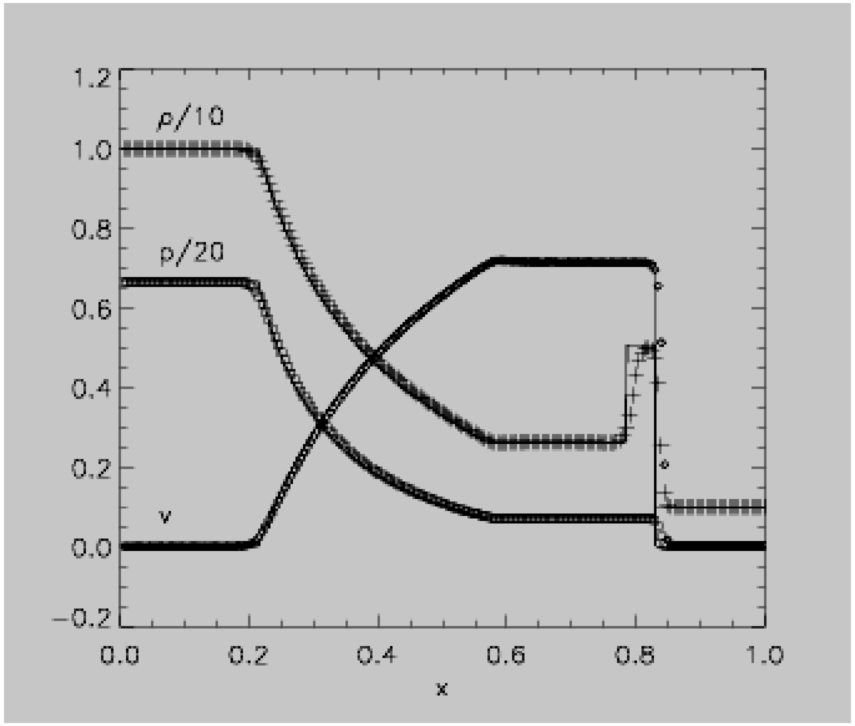
$$\hat{\vec{f}}_i = \frac{1}{2}\left[\vec{f}_i(w_R) + \vec{f}_i(w_L) - \sum_{n=1}^{5} \left|\tilde{\lambda}_n\right| \Delta\tilde{\omega}_n \tilde{R}_n \right]$$

$$\mathrm{U}(w_R) - \mathrm{U}(w_L) = \sum_{n=1}^{5} \Delta\tilde{\omega}_n \tilde{R}_n$$

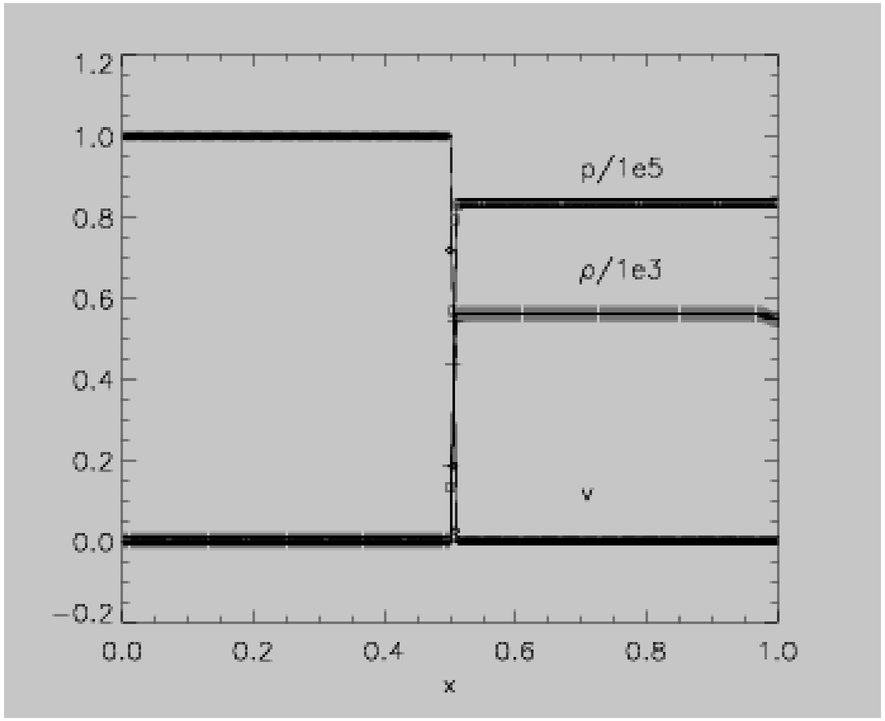
Stable and sharp discrete shock profiles

Accurate propagation speed of discontinuities

Shock tube test

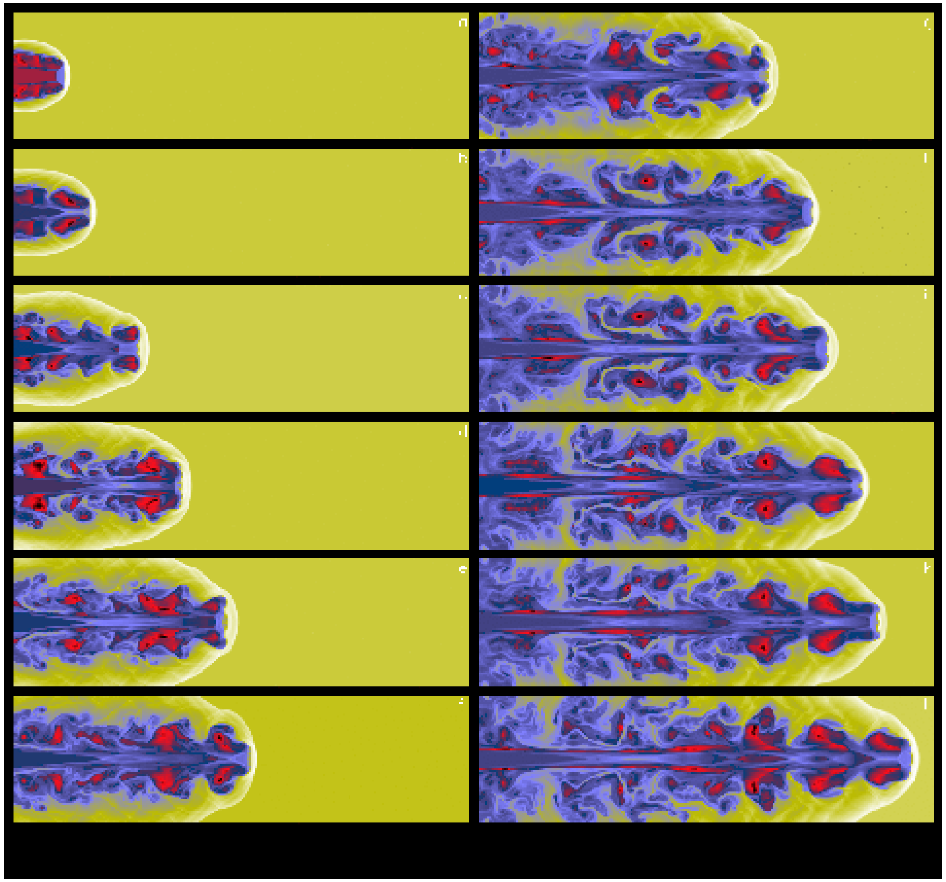


Relativistic shock reflection



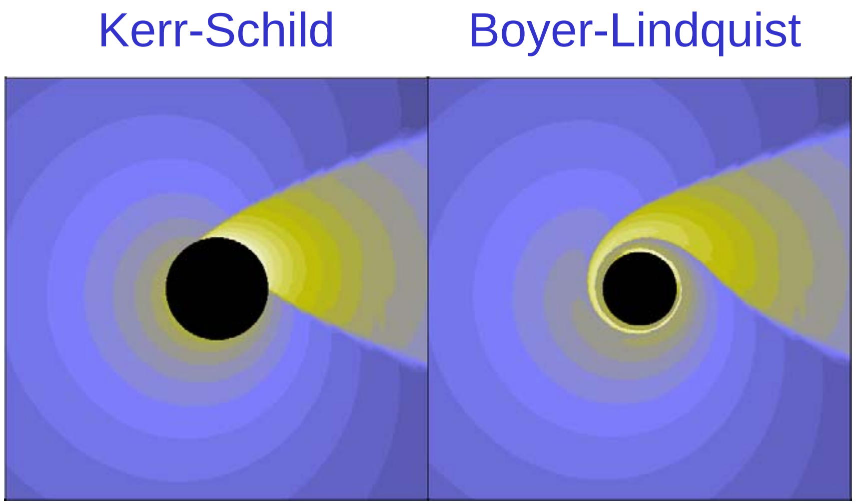
V=0.99999c (W=224)

Accurate resolution of multiple non-linear structures discontinuities, rarefaction waves, vortices,etc



Simulation of a extragalactic relativistic jet

Martí et al, Astrophys. J. 479, 151 (1997)



Wind accretion on to a Kerr black hole

(a=0.999M)

Font et al, MNRAS, 305, 920 (1999)

The runaway instability of thick accretion disks around black holes

- Thick accretion disks (tori) are present in many astronomical objects: AGNs, X-ray binaries, microquasars, central engine of GRBs.
- In a black hole + thick disk system the gas flows in an effective (gravitational + centrifugal) potential whose structure is similar to that of a close binary.
- The Roche torus has a cusp-like inner edge at the Lagrange point L_1 where mass transfer driven by the radial pressure gradient is possible.
- These systems may undergo a runaway instability (Abramowicz et al 1983): due to accretion from the disk the BH mass and spin increase and the gravitational field changes. Two evolutions are feasible:

1. The cusp moves inwards toward the BH, the mass transfer slows down. *Stable.*
2. The cusp moves deeper inside the disk material, the mass transfer speeds up. *Unstable.*

Most existing studies in the literature are stationary

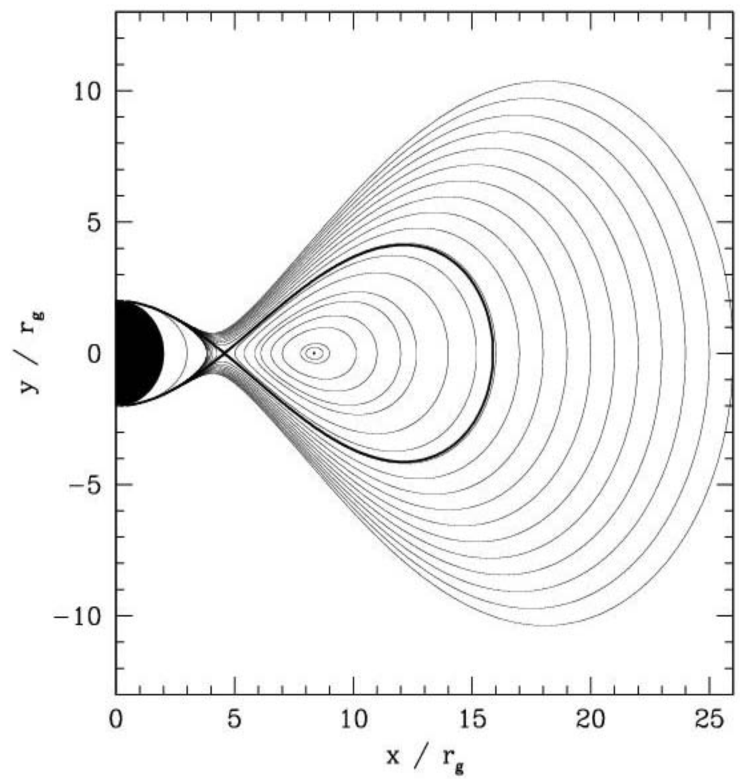
Recent time-dependent hydrodynamic simulations in GR available (Font & Daigne (2002); Zanotti, Rezolla & Font (2003).

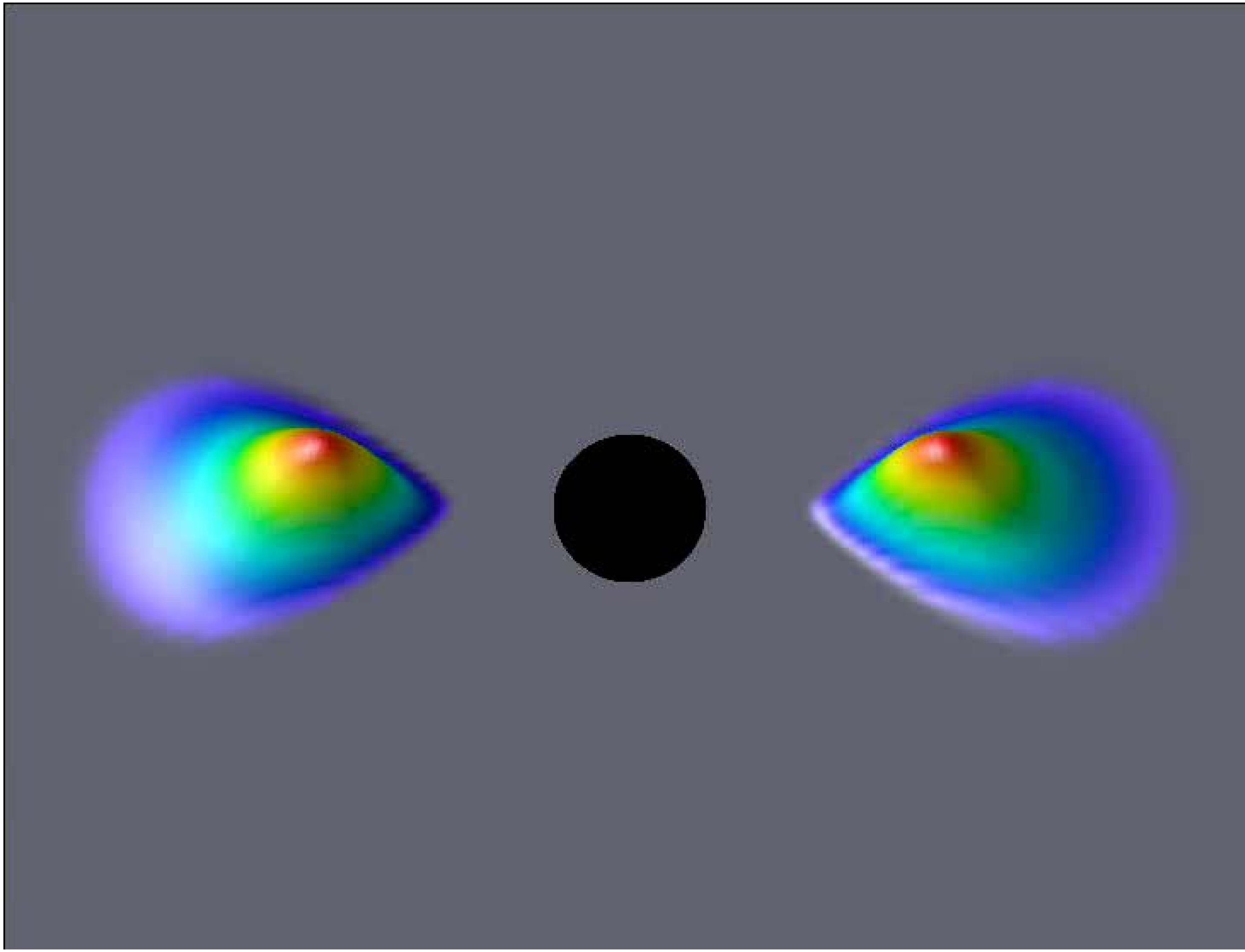
• Stationary, relativistic, constant angular momentum thick disc (Fishbone & Moncrief 1976; Kozlowski et al 1978).

• Kerr spacetime + Relativistic Euler (force balance) Bernouilli-type equation + non-Keplerian circular motion.

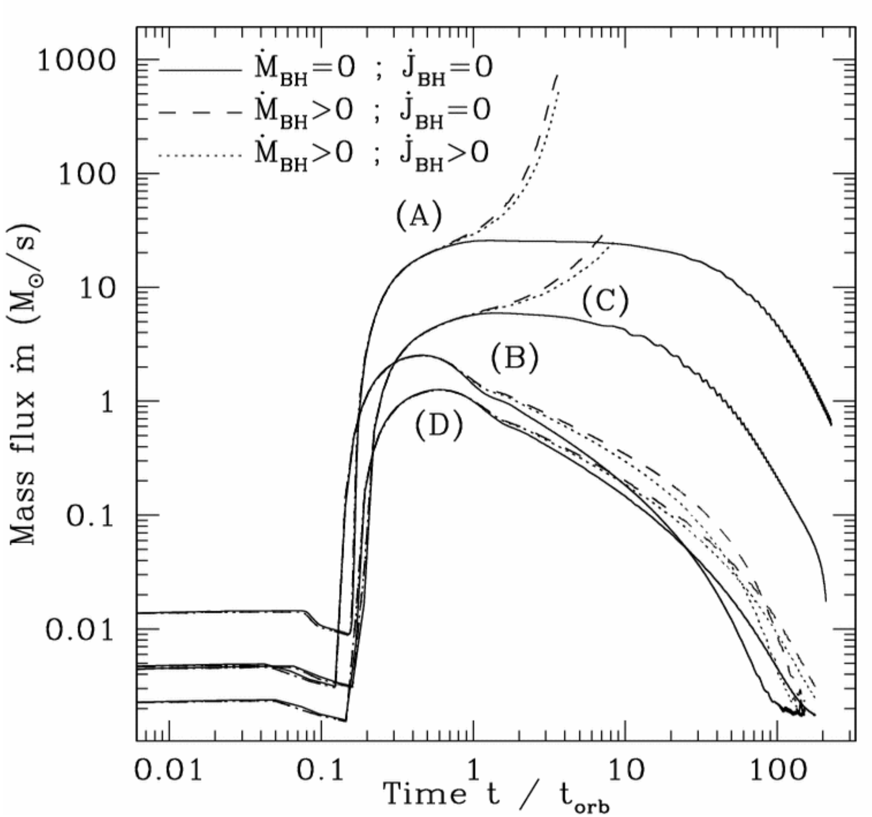
• Hydrodynamics: Test fluid approximation.

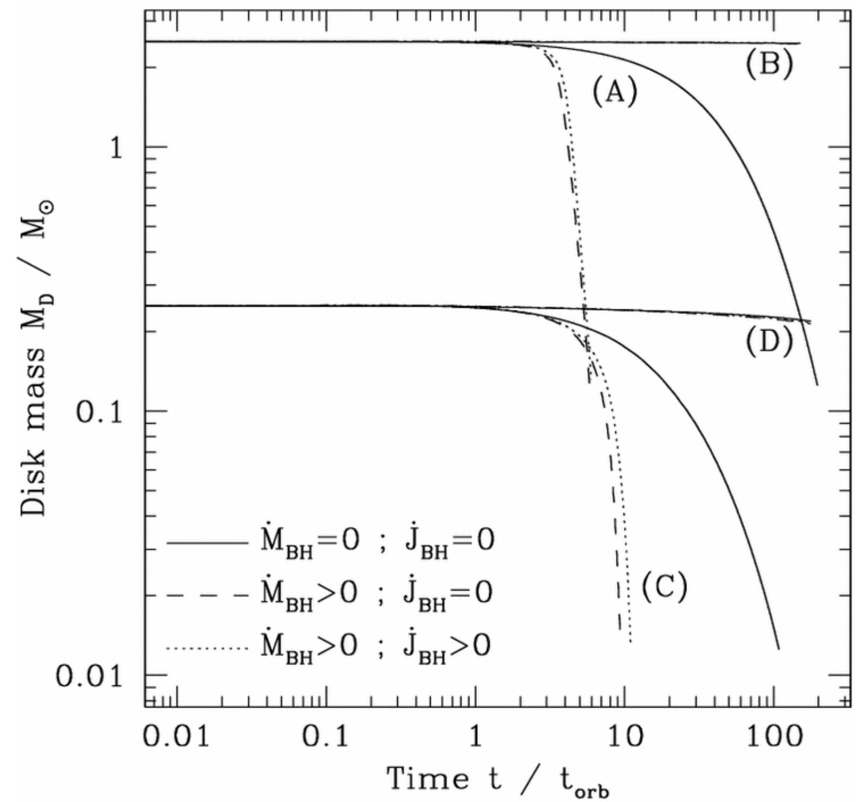
• Spacetime "dynamics": BH mass and spin increase determined by the mass and angular momentum accretion rates across the event horizon (sequence of exact Kerr BHs of varying mass and spin.) Fine for small disk-to-hole mass ratios.





Runaway instability – stable vs unstable evolutions





For a black hole of growing mass and spin the accretion process becomes rapidly *unstable when the disk's angular momentum distribution is constant* in agreement with stationary models).

Runaway instability: sudden loss of the mass of the disk at late times and rapid increase of the mass of the black hole.

Relativistic Rotational Core Collapse

Dimmelmeier, Font & Müller, ApJ, 560, L163 (2001); A&A, 388, 917 (2002a); A&A, 393, 523 (2002b)

Goals

extend to GR previous results on Newtonian rotational core collapse (Zwerger & Müller 1997)

determine the importance of relativistic effects on the collapse dynamics (angular momentum)

compute the associated gravitational radiation (waveforms)

Model assumptions

axisymmetry and equatorial plane symmetry

(uniformly or differentially) rotating 4/3 polytropes in equilibrium as initial models (Komatsu, Eriguchi & Hachisu 1989). Central density 10^{10} g cm^{-3} and radius 1500 km. Various rotation profiles and rotation rates

simplified EoS: P = Ppoly + Pth (neglect complicated microphysics and proper treatment of shocks)

constrained system of the Einstein equations (IWM conformally flat condition)

Max Planck Institute for Astrophysics
Garching, Germany

http://www.mpa-garching.mpg.de

General Relativistic Collapse of Rotating Stellar Cores in Axisymmetry

Harald Dimmelmeier
José A. Font
Ewald Müller

References:

- Dimmelmeier, H., Font, J. A., and Müller, E., *Astron. Astrophys.*, 388, 917–935 (2002), astro-ph/0204288.
- Dimmelmeier, H., Font, J. A., and Müller, E., *Astron. Astrophys.*, submitted (2002), astro-ph/0220489.

<u>HRSC scheme:</u>

PPM + Marquina flux-formula

<u>Solid line:</u> relativistic simulation

<u>Dashed line:</u> Newtonian

Larger central densities in relativistic models

Similar gravitational radiation amplitudes (or smaller in the GR case)

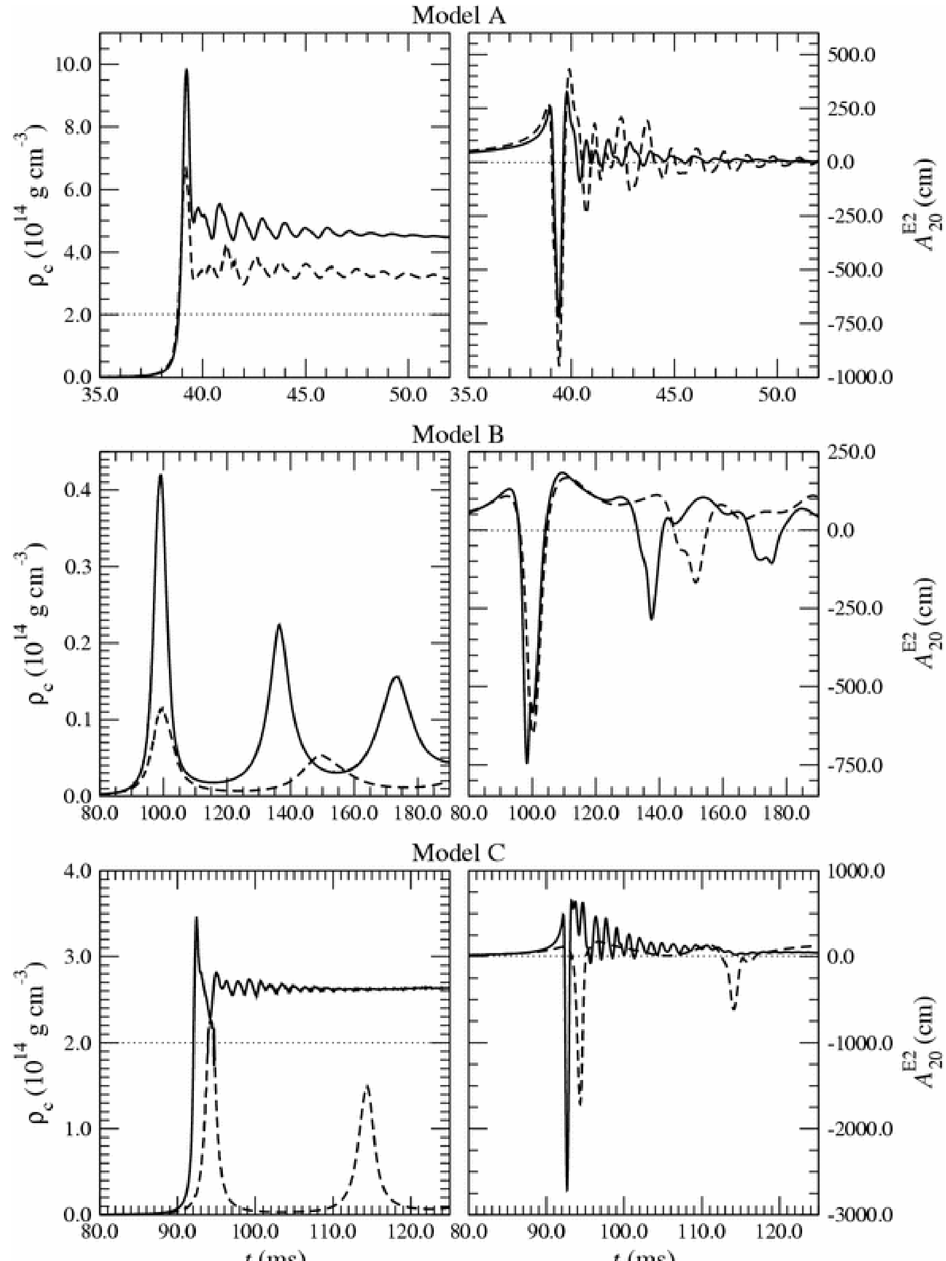
GR effects do not improve the chances for detection (at least in axisymmetry)

Central Density   Gravitational Waveform

Type I “regular”

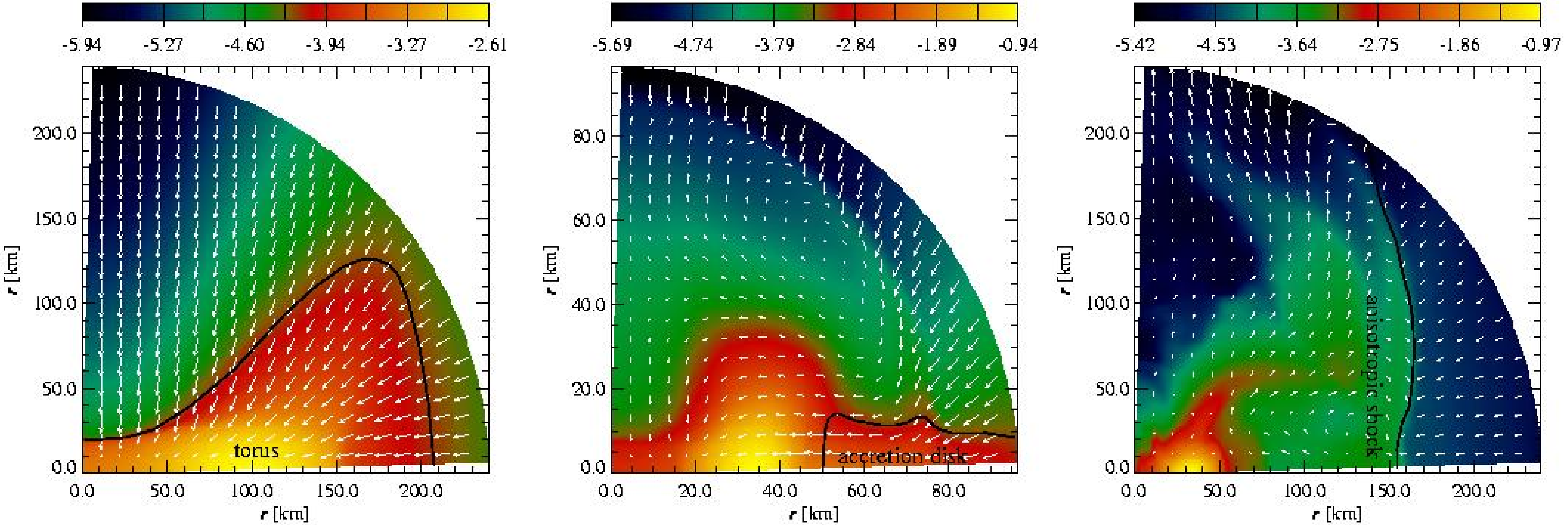
Type II “multiple bounce”

“transition”



Rapidly Rotating Models

Fast and extremely differential rotation, rapid collapse (~ 30 ms)

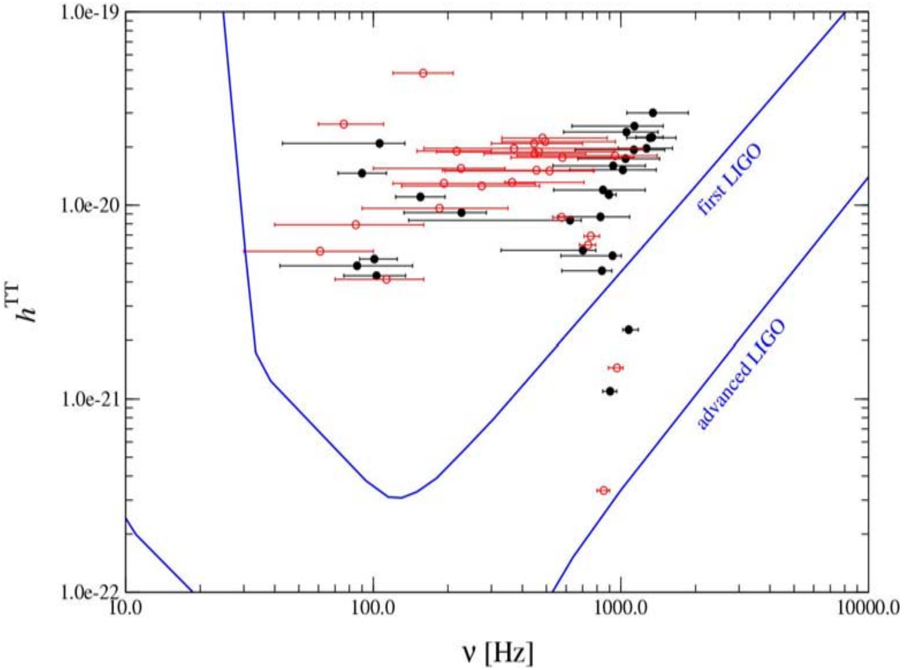


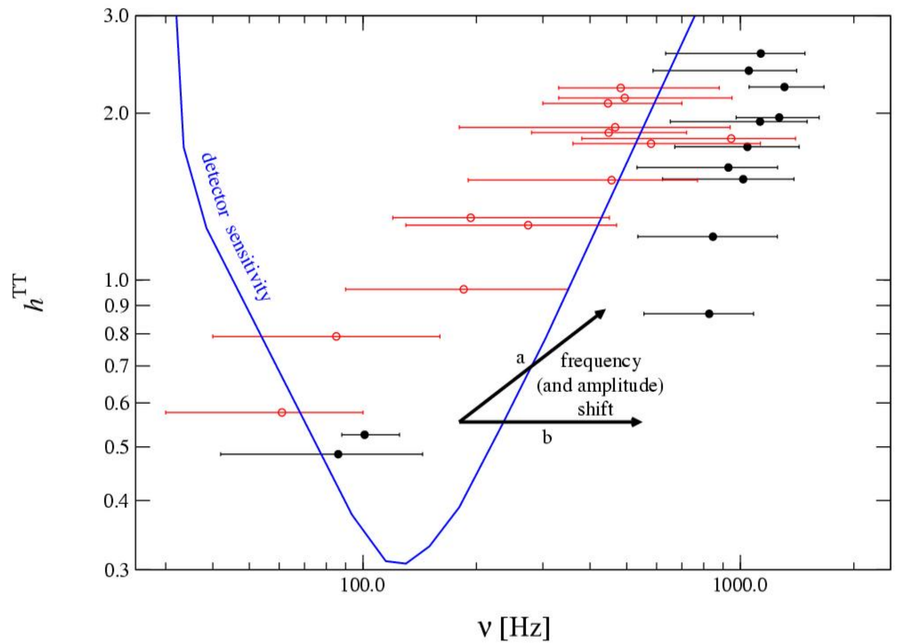
- Initial model has toroidal density shape. Torus becomes more pronounced during contraction.
- PNS is surrounded by a disk-like structure, which is accreted.
- Bar instabilities are likely to develop on dynamical timescales.
- After bounce, a strongly anisotropic shock front forms.

Gravitational Wave Signals

www.mpa-garching.mpg.de/Hydro/RGRAV/index.html

Influence of relativistic effects on the signals: Investigate amplitude-frequency diagram





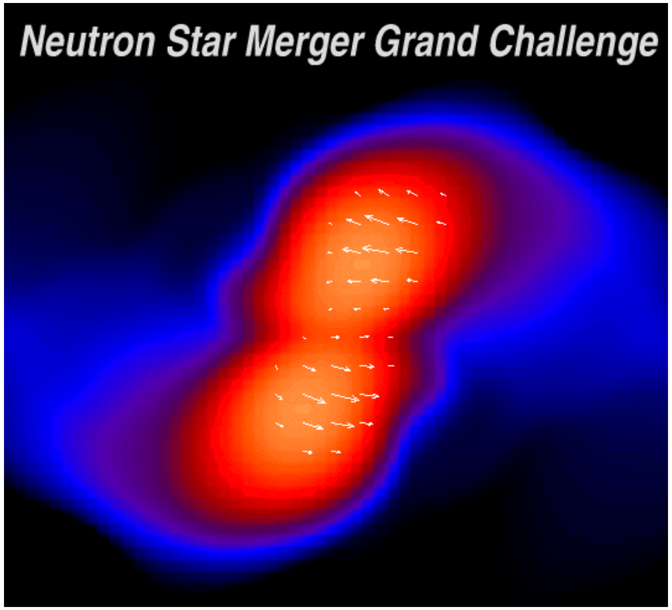
Spread of the 26 models does not change much

Signal of a galactic supernova detectable

On average: Amplitude → Frequency ↑

If close to detection threshold: Signal could fall out of the sensitivity window!

NASA NS/NS Grand Challenge and the **CACTUS** *code*



"Multipurpose code for 3D relativistic astrophysics and gravitational wave astronomy: application to coalescing neutron star binaries" `wugrav.wustl.edu`

Coupled system of Einstein and relativistic hydrodynamics equations

GR hydrodynamics code **MAHC** written mainly by M. Miller (WashU)

Simulations performed using the **CACTUS** code `www.cactuscode.org`

Developed in an international Numerical Relativity collaboration (AEI/NCSA/WashU)

Open source code: freely available to the scientific community

References: Alcubierre et al, PRD, 62, 044034 (2000);
Font et al, PRD, 61, 044011 (2000); 65, 084024 (2002)

*EU-TMR Network on Sources of Gravitational Radiation and the **WHISKY** code*

Whisky: a 3D, parallel code in Cartesian coordinates, which solves the GR hydrodynamics equations using HRSC schemes **www.eu-network.org**

(Main) institutions involved in the code development:

AEI (Golm, Germany), SISSA (Trieste, Italy), University of Thessaloniki (Greece), University of Valencia (Spain)

SPACETIME EVOLUTION

Cartesian 3D unigrid and with FMR (fixed mesh refinement). Implementation of AMR is in progress

Iterative Crank-Nicholson scheme (2nd order accurate in space and time), but also RK, Leapfrog, etc.

HYDRODYNAMICS

Different approximate Riemann solvers implemented: HLLE, Roe, Marquina.

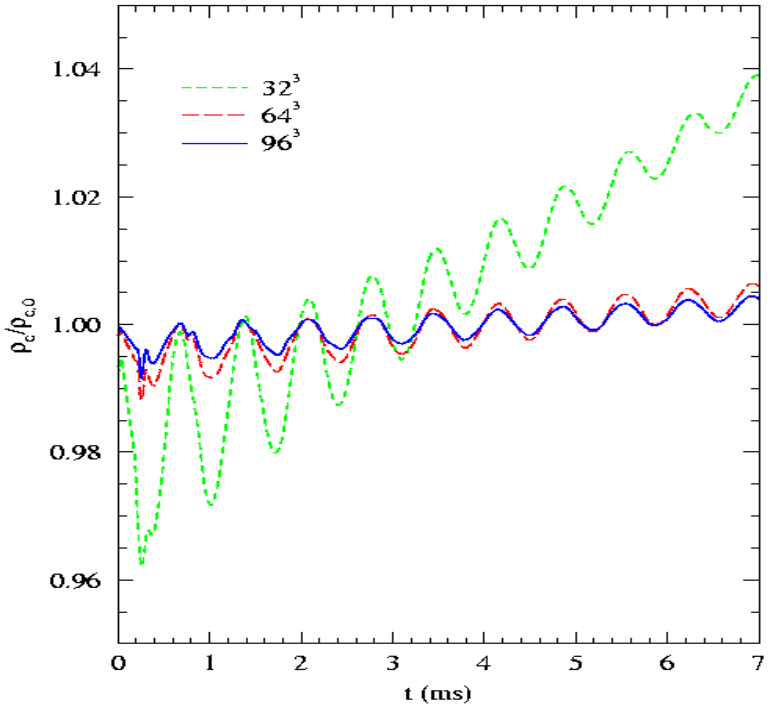
Slope limiters: MUSCL, MC and PPM.

Optimal results are obtained with Marquina's flux formula and the MC slope limiter (2nd order away from local extrema)

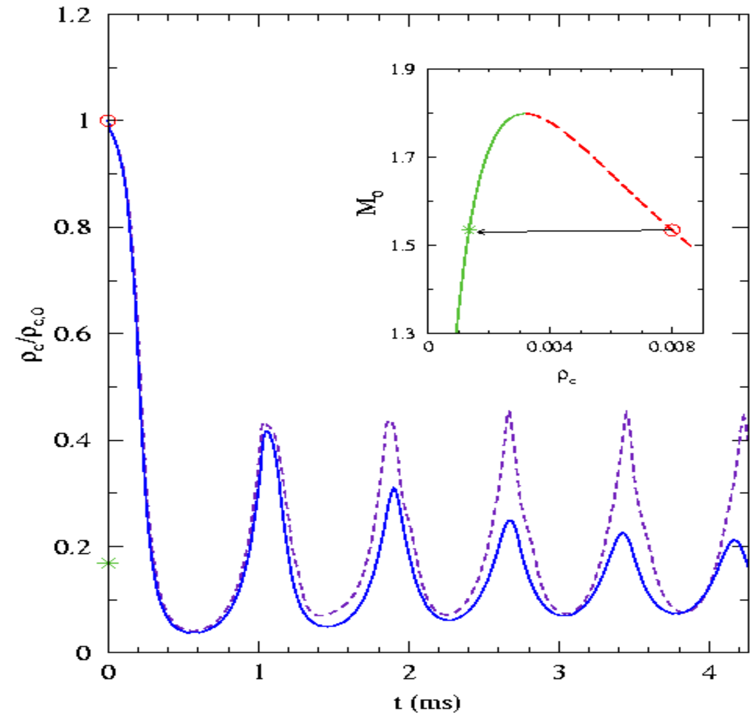
Polytropic and realistic EoS

Coupled time evolutions of polytropic spherical stars

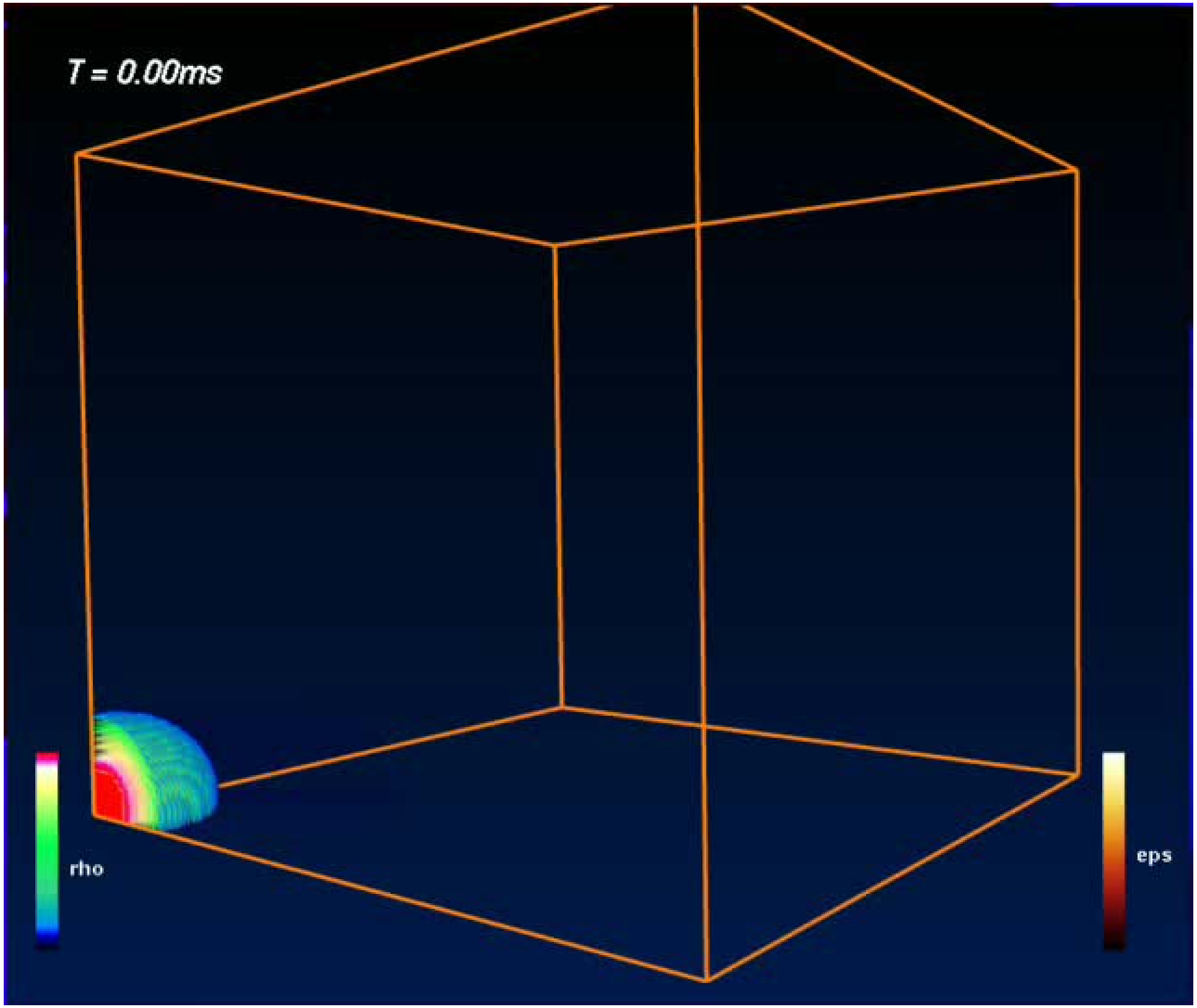
Central density of a stable model



"Migration" of an unstable model to the stable branch



Font et al, PRD, 65, 084024 (2002)

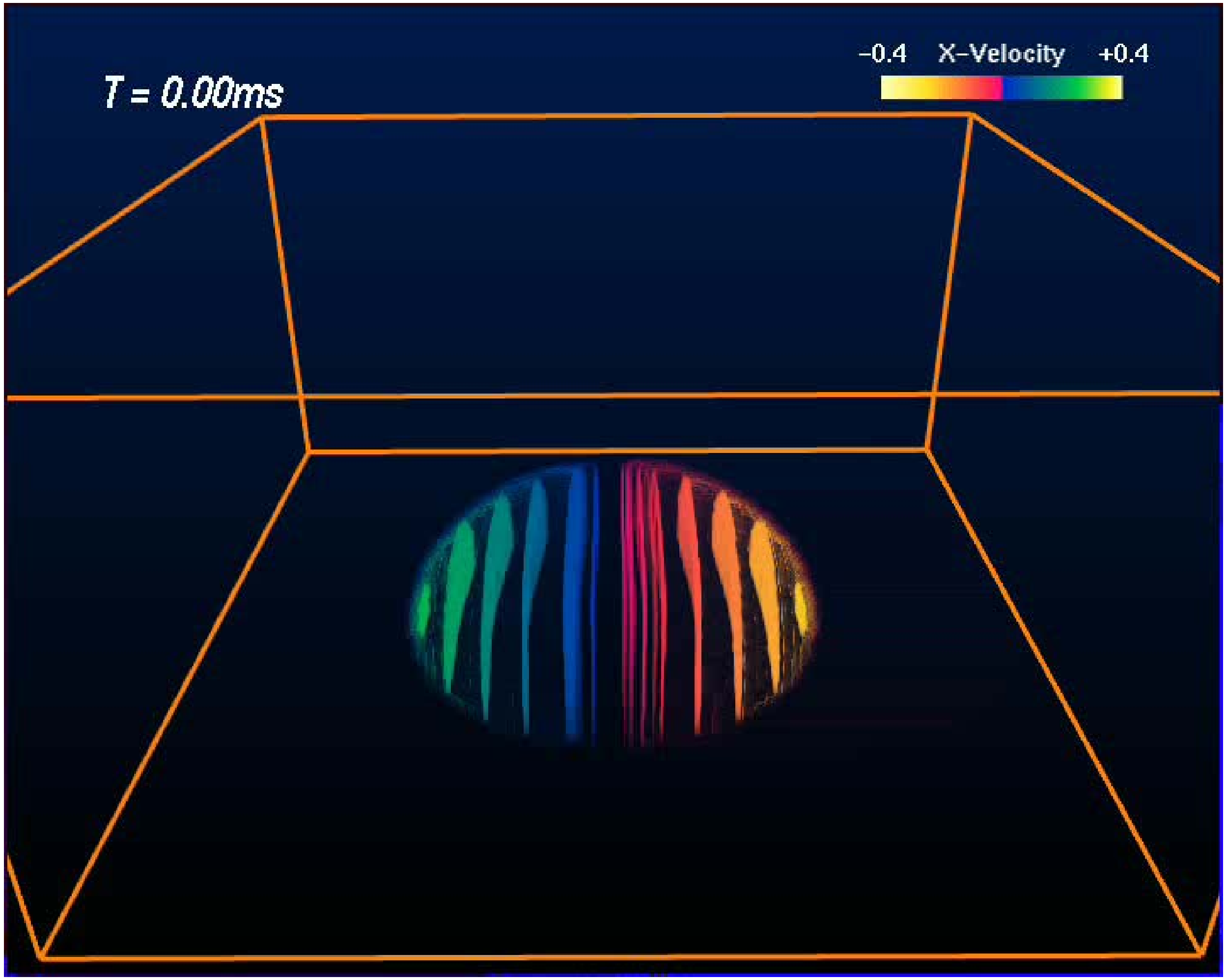

T = 0.00ms
rho
eps

Stable evolutions of rapidly rotating stars

Isocontours of v^x along the x-direction

$$\frac{\Omega}{\Omega_K} = 0.92$$

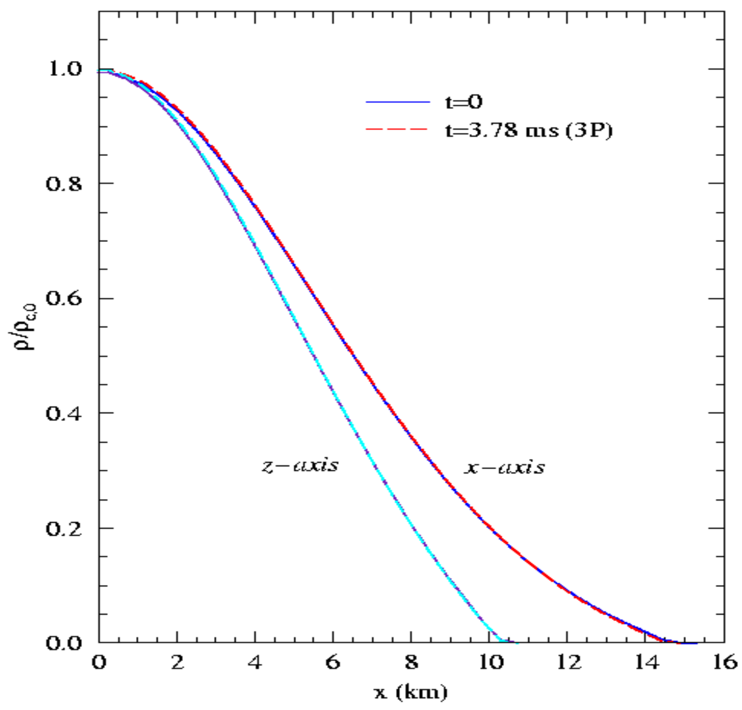
$$\frac{R_{pole}}{R_{equat}} = 0.7$$



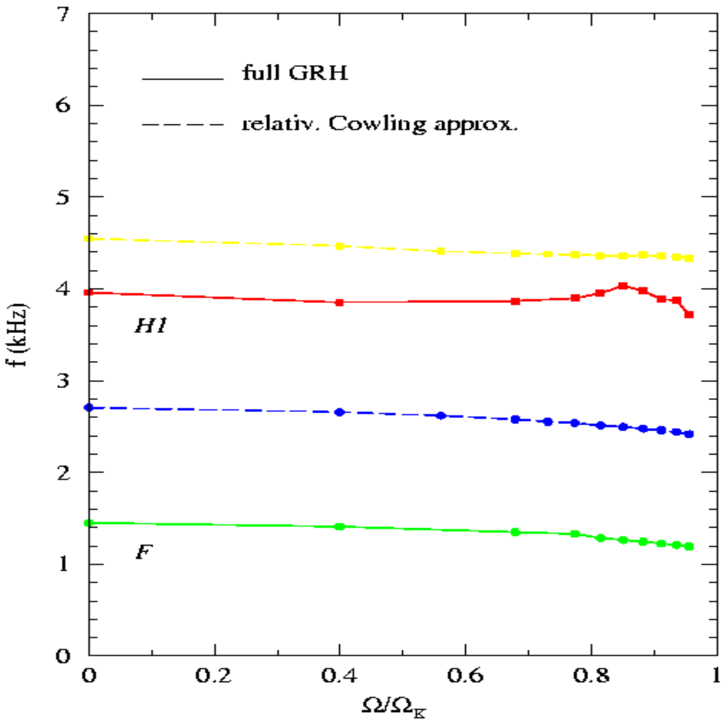
Stable Rapidly Rotating Relativistic Star

Consider a stable Γ=2 star, rapidly rotating at 92% Ω_K, and r_p/r_e=0.7

Profile of the rest-mass density



Power spectrum of central density



These frequencies are still to be computed using perturbation theory!

Conclusions

- The **hydrodynamic equations** (either Newtonian or relativistic) constitute a **non-linear hyperbolic system**, perhaps the archetypal example of hyperbolic model in astrophysics.

- There exist solid mathematical foundations and accurate numerical methodology imported from classical CFD and recently extended to Relativistic Astrophysics.

- An emerging, "preferred" choice: high-resolution shock-capturing schemes based upon Riemann solvers, and written in conservation form.

- Nowadays, computational general relativistic astrophysics is an increasingly important field of research in Numerical Relativity worldwide: gravitational stellar collapse, black hole formation, coalescence of compact binaries, gamma-ray bursts, …

- In particular, hydrodynamical simulations in (static) curved backgrounds – *test-fluid approximation* – are routinely performed with satisfactory levels of accuracy.