Numerical Simulations of MHD turbulence at low to moderate magnetic Reynolds number

Bernard Knaepen*
(Thessaloniki, October 2003)

(*) Université Libre de Bruxelles, Belgium
Plan

- Description of turbulence (hydrodynamics)
  - Exemples
  - Phenomenology
  - Overview of Large–Eddy Simulations (LES)

- MHD
  - LES in the quasi-static approximation

- Prospects
“Definition” of Turbulence (I)

Reynolds number:

\[ Re = \frac{VL}{\mu} \]  
(non-dimensional number)

If \( Re \) is small

\[ Re = 1.54 \]

(Flow: Van Dyke M. An Album of Fluid Motion)
“Definition” of Turbulence (II)

\[ Re = 9.6 \]
\[ Re = 13.1 \]
\[ Re = 26 \]
\[ Re = 140 \]
\[ Re = 1800 \]

(Photo: Van Dyke M. An Album of Fluid Motion)
Turbulence: exemples (II)

Cyclone (Earth)  Red spot (Jupiter)

http://www.nasa.gov

http://www.nasa.gov
Turbulence: exemples (III)

Convection (Sun)

http://www.nasa.gov

Star formation (gaz cloud)

http://www.nasa.gov
Navier–Stokes equations

Incompressible flow (non MHD):

\[
\partial_t u_i + u_j \partial_j u_i = -\partial_i (p/\rho) + \nu \Delta u_i, \quad \partial_i u_i = 0
\]

+ initial condition: \( u_i(x, t = 0) = u_i^0(x) \)

+ boundary conditions

Millenium Prize: http://www.claymath.org
Kolmogorov phenomenology (1941)

\[ u(k) = \int u(x)e^{-i k \cdot x} \, dk \]

\[ E(k) = K \varepsilon^{2/3} k^{-5/3} \]
Direct numerical simulations (DNS)

- Step 1: grid generation
- Step 2: Navier–Stokes equations discretization

\[ \partial_t u_i + u_j \partial_j u_i = -\partial_i \left( \frac{p}{\rho} \right) + \nu \Delta u_i \]

\[ \partial_t u_i = F(x_k) + O(\delta^n) \]

(\(\delta/L \ll 1\))
Direct numerical simulations (DNS)

Is turbulence a solved problem? → NO

- Not satisfying from the theoretical point of view.
- Inapplicable for most flows encountered in nature or industrial applications.

\[ \# \text{ of grid points} \sim Re^{9/4} \]

- Commercial aircraft: \( Re = 10^8 \) → 10\(^{18}\) grid points
- “Teraflops” computer \( (10^{12} / s) \) → 1s of flight \( \approx 1000 \text{ years of computing time} \)

But:

DNS provide accurate and non-intrusive measurements for flows at low Reynolds numbers and are efficient to validate and test models and approximate methods.
Large–Eddy Simulations (LES)

\[ f, \bar{f} \]

![Graph showing complete and resolved signals with a function mapping from \( f(x) \) to \( \bar{f}(x) \)]

\[ E(k) \quad \text{(log)} \]

Diagram explaining resolved and subgrid scales:
- Resolved scales
- Subgrid scales

Mathematical expression:
\[ F : f(x) \rightarrow \bar{f}(x) \]
Large–Eddy Simulations (II)

Navier–Stokes equations:
\[
\partial_t u_i + u_j \partial_j u_i = -\partial_i (p/\rho) + \nu \Delta u_i
\]

\[
u_j \partial_j u_i \neq \overline{u_j \partial_j u_i} \quad \Rightarrow \quad [F, P] \neq 0 \quad P : \text{product}
\]

\[
\partial_t \overline{u_i} + \overline{u_j} \partial_j \overline{u_i} = -\partial_i (\overline{p/\rho}) + \nu \Delta \overline{u_i} + (\overline{u_j \partial_j u_i} - \overline{u_j \partial_j u_i})
\]

Filtered Navier–Stokes equations (LES):

\[
\partial_t \overline{u_i} + \overline{u_j} \partial_j \overline{u_i} = -\partial_i (\overline{p/\rho}) + \nu \Delta \overline{u_i} - \partial_j \tau_{ij}
\]

\[
\tau_{ij} = [F, P](u_i, u_j) : \text{subgrid–scale stress tensor}
\]

\[
\tau_{ij} = \overline{u_i u_j} - \overline{u_i} \overline{u_j}
\]
Large-Eddy Simulations (III)

\[ \tau_{ij} \text{ needs to be modelled to close the equations} \]

\[ \tau_{ij}^M = \tau_{ij}^M (\overline{u}_i) \approx \tau_{ij} \]

- How to model \( \tau_{ij} \)? \( \rightarrow \) eddy viscosity

\[ \tau_{ij} = -2\nu_t S_{ij} \quad \text{with} \quad S_{ij} = \frac{1}{2}(\partial_i \overline{u}_j + \partial_j \overline{u}_i) \]

\[ \nu \rightarrow \nu + \nu_t \quad \text{(homogeneous turbulence)} \]
Kolmogorov model

\[ [\nu_t] = L^2 T^{-1} \]
\[ [\epsilon] = L^2 T^{-3} \]
\[ \overline{\Delta} \sim \frac{1}{k_{max}} \]
\[ [\overline{\Delta}] = L \]

\[ \nu_t = C_{\nu} \epsilon^{1/3} \overline{\Delta}^{4/3} \]

(Kolmogorov model)
Smagorinski model

\[
\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i = -\partial_i (\bar{p}/\rho) + \nu \Delta \bar{u}_i - \partial_j \tau_{ij}
\]

\[\epsilon_\tau = u_i \partial_j \tau_{ij} = -S_{ij} \tau_{ij}\]

\[= -S_{ij} (-2C_\nu \epsilon^{1/3} \Delta^{4/3} S_{ij}) = 2C_\nu \epsilon^{1/3} \Delta^{4/3} (S_{ij} S_{ij})\]

Hypothesis: \(\epsilon_\tau = \epsilon\)

\[\epsilon = \Delta^2 \left[C_\nu (2S_{ij} S_{ij})\right]^{3/2}\]

\[\tau_{ij} = -2C \Delta^2 \sqrt{2S_{ij} S_{ij} S_{ij}}\]

\((C = C_\nu^{3/2})\)

\(C\) is the Smagorinski constant (only parameter)
Magnetohydrodynamics

\[ \partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + b_j \partial_j b_i + \nu \Delta u_i \]
\[ \partial_t b_i = -u_j \partial_j b_i + b_j \partial_j u_i + \eta \Delta b_i \]
\[ \partial_i u_i = 0 \]
\[ \partial_i b_i = 0 \]

\[ \vec{B}^{ext} \] (imposed)
Motivations

Figure I.5  Magnetic damping.  
(Davidson, 2001)
Magnetic Reynolds number

\[ \partial_t b_i = -u_j \partial_j b_i + b_j \partial_j u_i + \eta \Delta b_i \]

\[ Re_m = \frac{u L}{\eta} \]

- \( Re_m \ll 1 \): Liquid metals
- \( Re_m \sim 1 \): Hypersonic flight
- \( Re_m \gg 1 \): Astrophysics

\[ Re_m \ll 1 : \text{ Quasi-Static approximation} \]
Quasi–Static approximation

\[ \partial_t b_i = -u_j \partial_j b_i + b_j \partial_j u_i + B_{j}^{ext} \partial_j u_i + \eta \Delta b_i \]

\[ 0 = B_{j}^{ext} \partial_j u_i + \eta \Delta b_i \]

\[ \partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + \frac{1}{(\mu \rho)} b_j \partial_j b_i + \frac{1}{(\mu \rho)} B_{j}^{ext} \partial_j b_i + \nu \Delta u_i \]

\[ \partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + \left( \frac{B_z^{ext}}{\mu \eta \rho} \right)^2 \Delta^{-2} \partial_z \partial_z u_i + \nu \Delta u_i \]

Quasi–Static approximation
QS approximation: Large–Eddy Simulation

\[ \partial_t u_i = -\partial_i (\bar{p}/\rho) - u_j \partial_j u_i + \frac{(B^\text{ext}_z)^2}{\mu \eta \rho} \Delta^{-2} \partial_z \partial_z u_i + \nu \Delta u_i \]

Intensity parameter:

\[ N = \frac{(B^\text{ext}_z)^2 L}{\mu \eta \rho v} \]

N=1: magnetic effects ~ non-linear effects

N=10: magnetic effects ~ 10 x non-linear effects

LES:

\[ \partial_t \bar{u}_i = -\partial_i (\bar{p}/\rho) - \bar{u}_j \partial_j \bar{u}_i + \frac{(B^\text{ext}_z)^2}{\mu \eta \rho} \Delta^{-2} \partial_z \partial_z \bar{u}_i + \nu \Delta \bar{u}_i - \partial_j \tau_{ij} \]

\[ \tau_{ij} = \bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j \]
Dynamic procedure

\[ \tau_{ij} = -2C \Delta^2 \sqrt{2S_{ij}S_{ij}S_{ij}} \quad \text{(Smagorinski)} \]

One parameter: Smagorinski constant \( C \)

Self-similarity in the inertial range

\[ \tau_{ij} \hat{\quad} , \tau_{ij}^{-} \quad \text{same } C \]

\[ C = C(\overline{u}_i) \]

Dynamical Model

\[ \tau_{ij} = -2C(\overline{u}_i) \Delta^2 \sqrt{2S_{ij}S_{ij}S_{ij}} \]
Decaying turbulence

(Photo: Van Dyke M. An Album of Fluid Motion)

$E_k$ vs. $t$

$512^3$

$B = 0$

$N = 1$

$N = 10$

BK, Moin (2003)
DNS versus LES

- Four graphs showing the energy dissipation $E_k$ over time $t$ for different resolutions $N=1$ and $N=10$, with and without model.
- The modelled results are indicated with solid lines, the filtered DNS results with diamonds, and the LES results with dashed lines.
- The resolution $512^3$ is compared to $32^3$.
Energy Spectra N=1

$E_k$

$E(k)$

$32^3$

$N=1$

$t$

$k$

No Model
LES
Filt. DNS

No Model
LES
Filt. DNS

No Model
LES
Filt. DNS

No Model
LES
Filt. DNS
Energy Spectra $N=10$
Dynamic constant

![Graph showing the dependence of C on t for different values of B and N. The graph has a y-axis labeled C ranging from 0.000 to 0.020 and an x-axis labeled t ranging from 0.0 to 3.5. There are three curves, one for B = 0, one for N = 1, and one for N = 10.](image)
Energy contours $N=1$

Initial flow

Filtered DNS $32^3$

LES $32^3$

$E_k$

$N=1$

$32^3$

$t$

No Model

LES

Filt. DNS
Energy Contours $N=10$

Initial flow

Filtered DNS $32^3$

LES $32^3$
Conclusions

- Turbulence is a complex process for which very few analytical results are known.
- Numerical methods exist but they require theoretical model development in order to explore realistic flows.
- Large-Eddy Simulations are a powerful tool for simulating flows at a lower computational cost.
- Large-Eddy Simulations are widely used for engineering flows in complex geometries.
- Only limited work has been devoted to this method in the case of MHD turbulence.