ULB

Numerical Simulations of MHD turbulence at low to moderate magnetic Reynolds number

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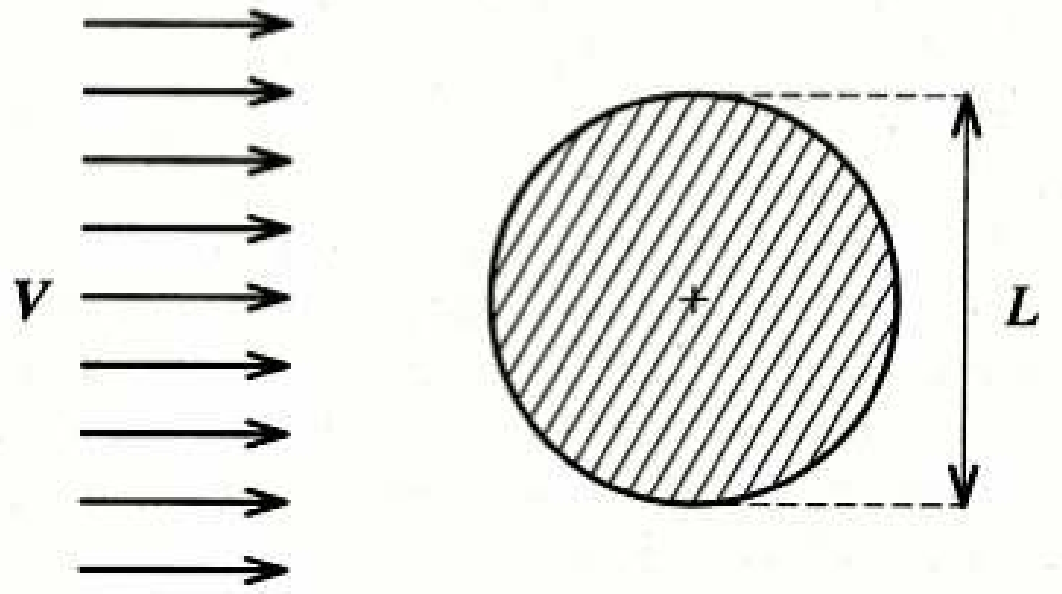
Plan

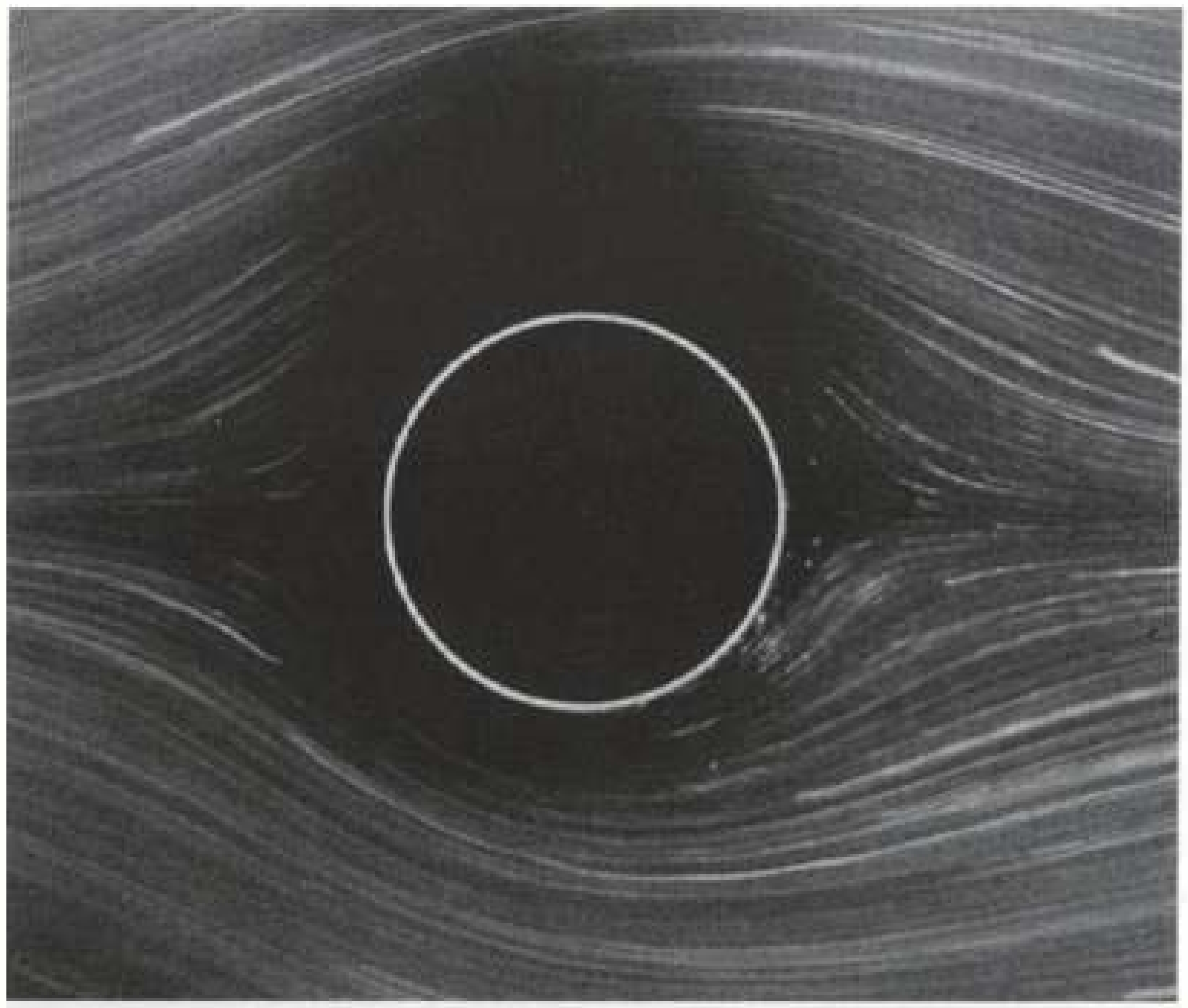
- Description of turbulence (hydrodynamics)
  - Exemples
  - Phenomenology
  - Overview of Large-Eddy Simulations (LES)
- MHD
  - LES in the quasi-static approximation
- Prospects

"Definition" of Turbulence (I)



Reynolds number:

$$Re = \frac{VL}{\mu}$$ (non-dimensional number)

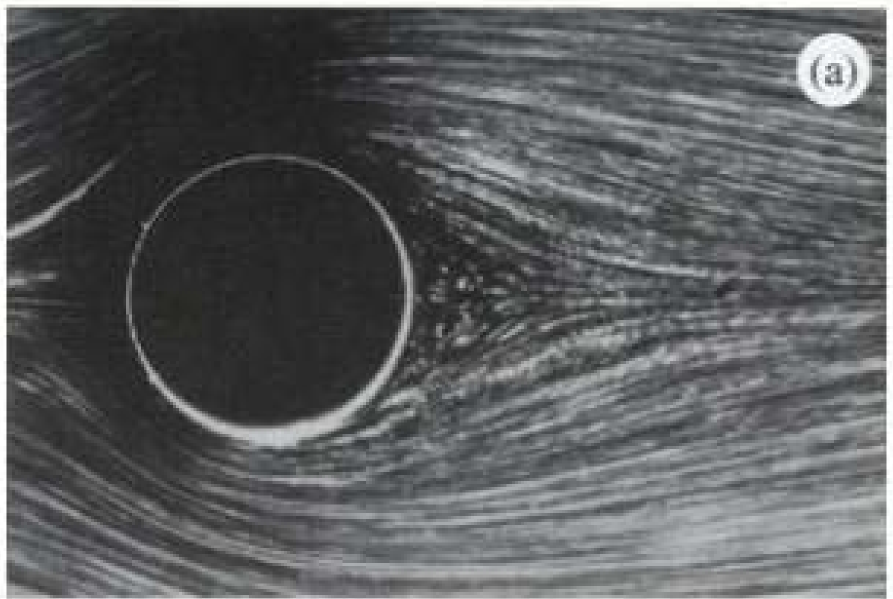

(Photo:Van Dyke M.An Album of Fluid Motion)

If Re is small

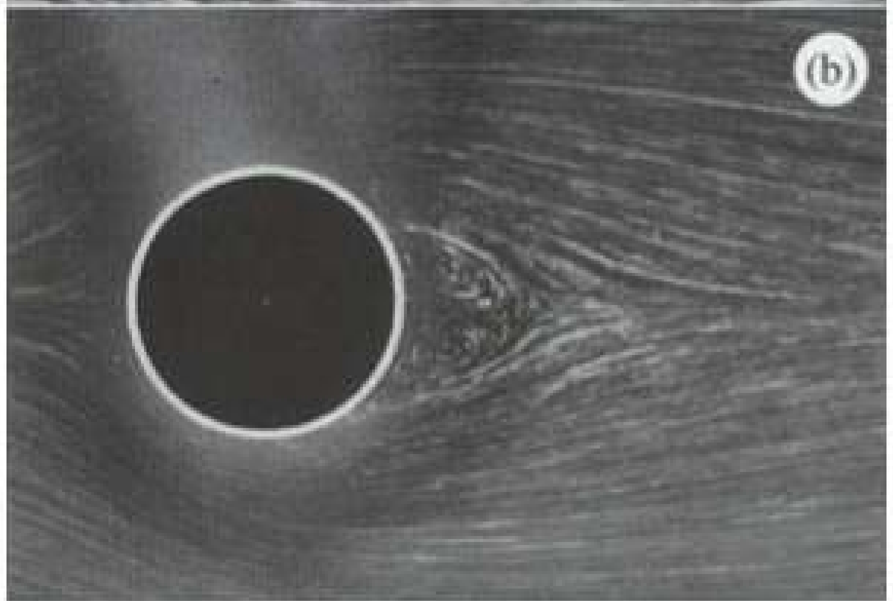
→ laminar flow

$Re = 1.54$

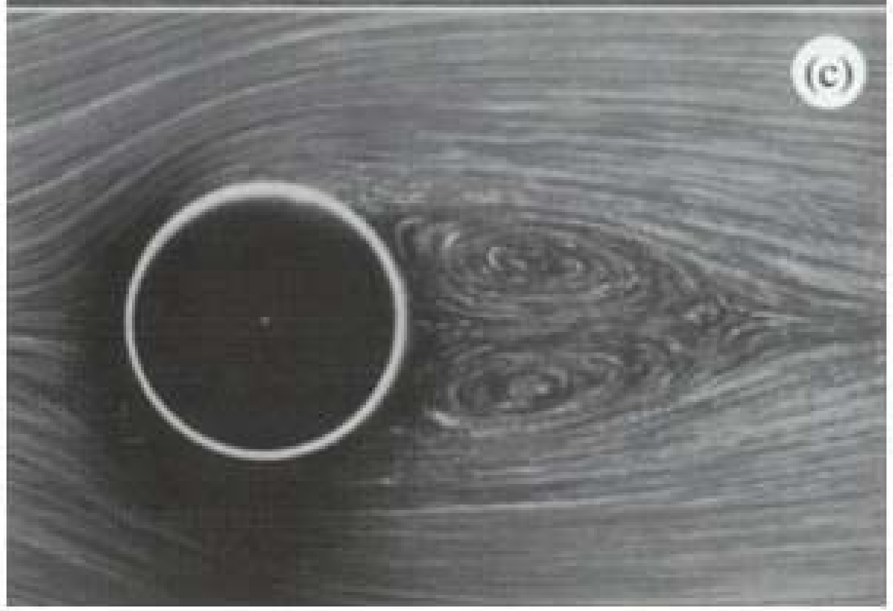
“Definition” of Turbulence (II)



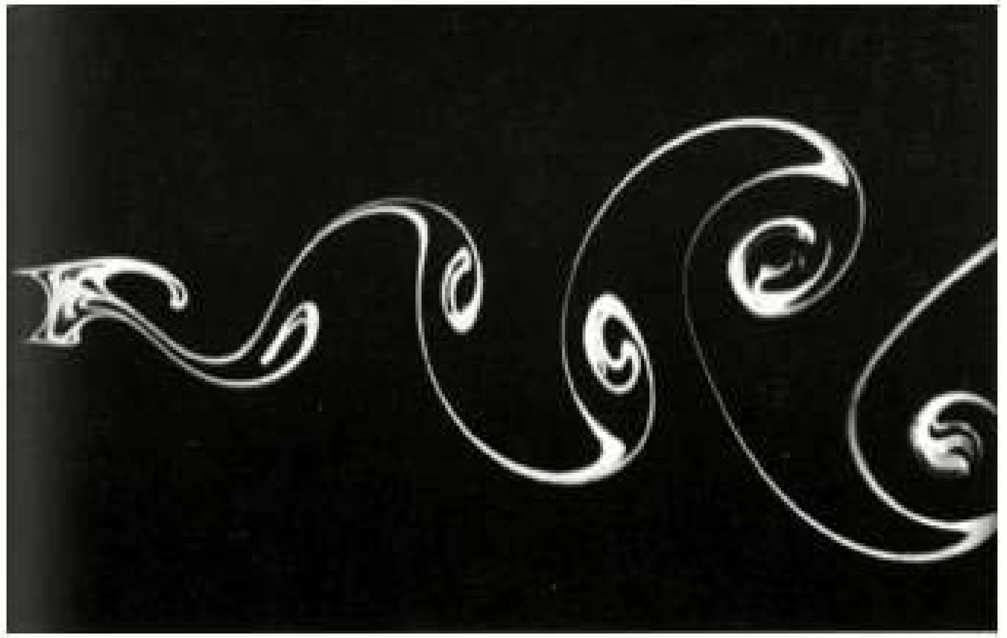
$Re = 9.6$



$Re = 13.1$



$Re = 26$



$Re = 140$

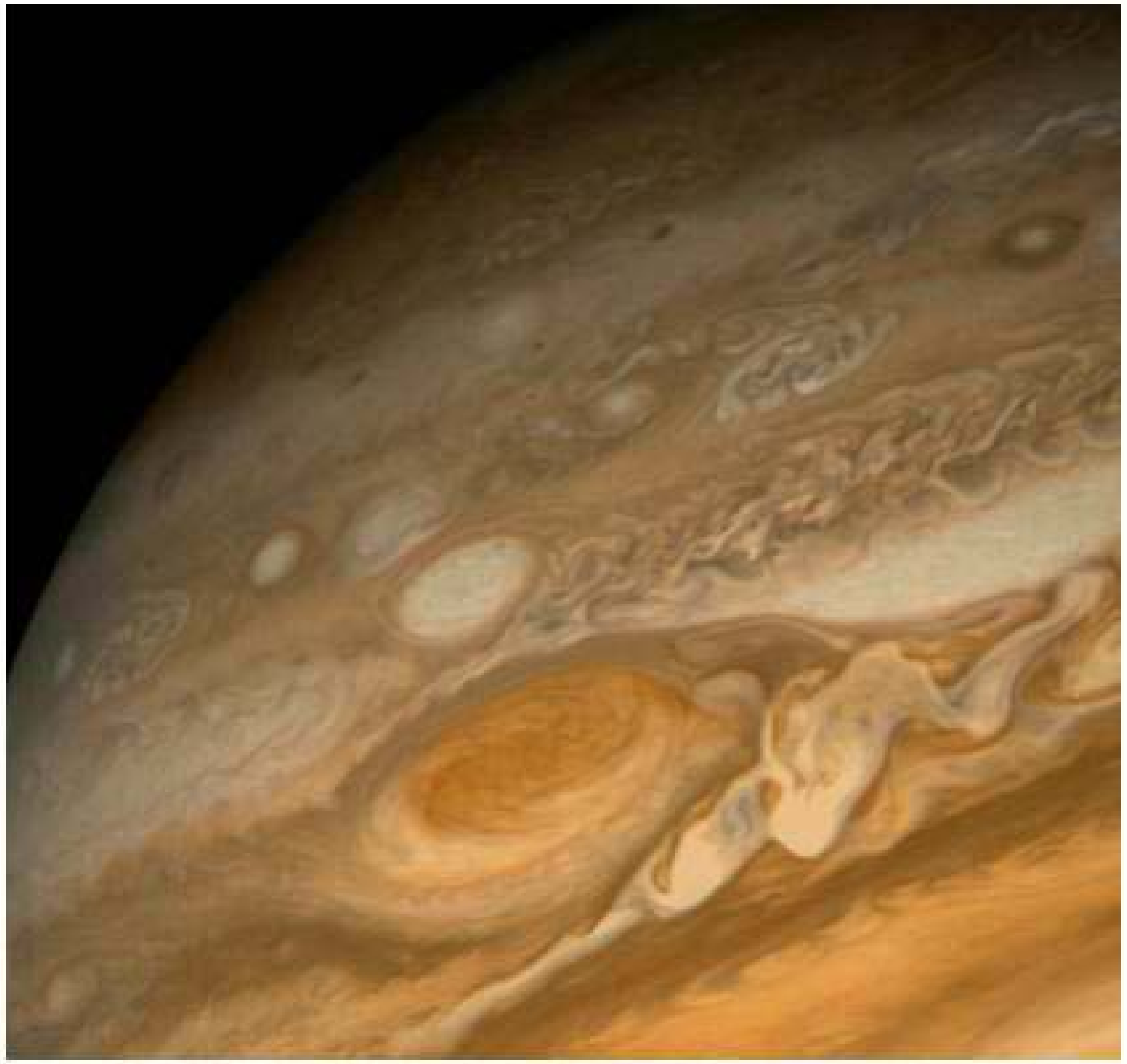
$Re = 1800$

(Photo:Van Dyke M.An Album of Fluid Motion)

Turbulence: exemples (II)

http://www.nasa.gov

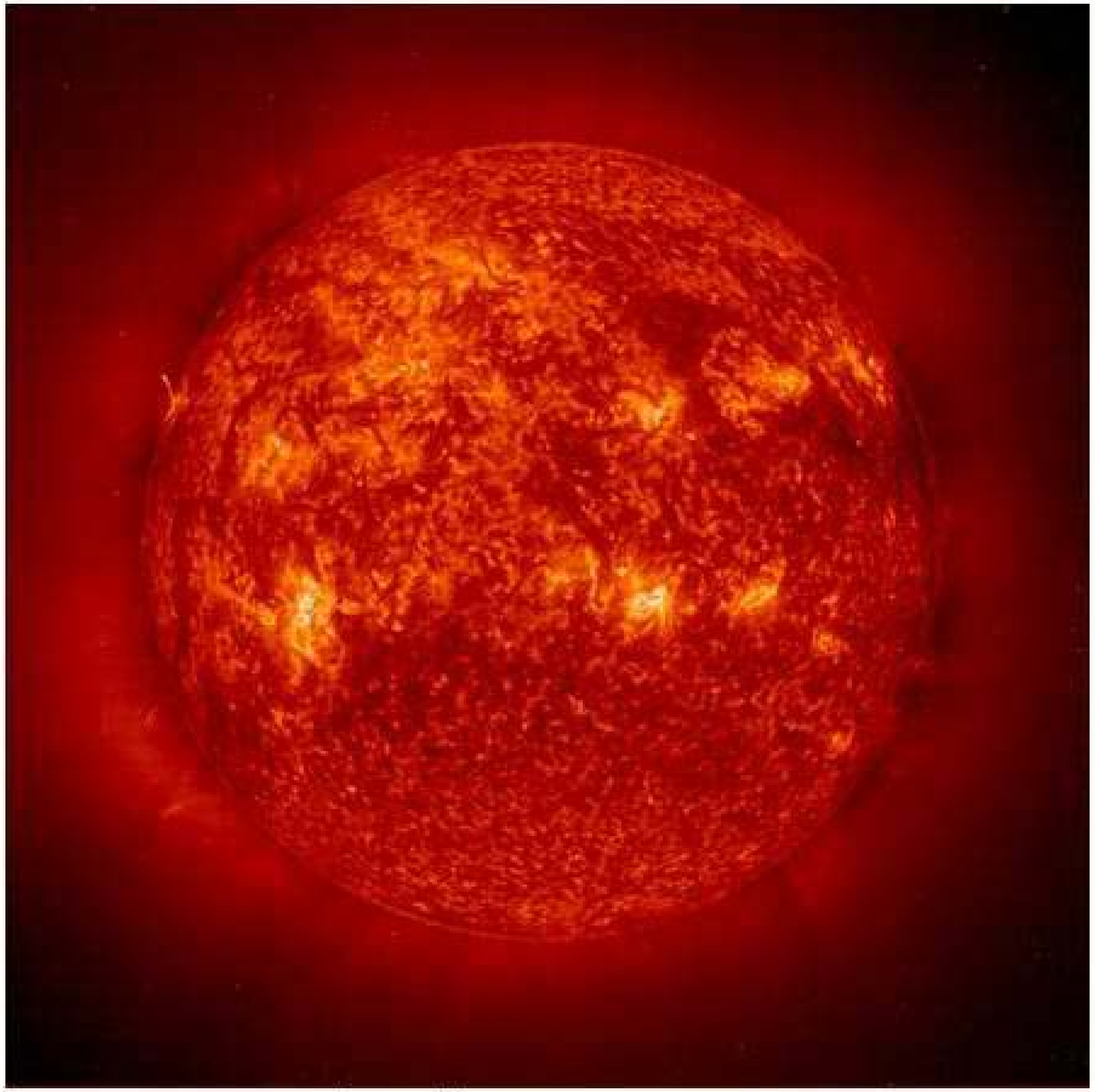
Cyclone (Earth)



http://www.nasa.gov

Red spot (Jupiter)

Turbulence: exemples (III)



http://www.nasa.gov

Convection (Sun)

http://www.nasa.gov

Star formation
(gaz cloud)

Navier–Stokes equations

Incompressible flow (non MHD):

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i (p/\rho) + \nu \Delta u_i \,, \quad \partial_i u_i = 0$$

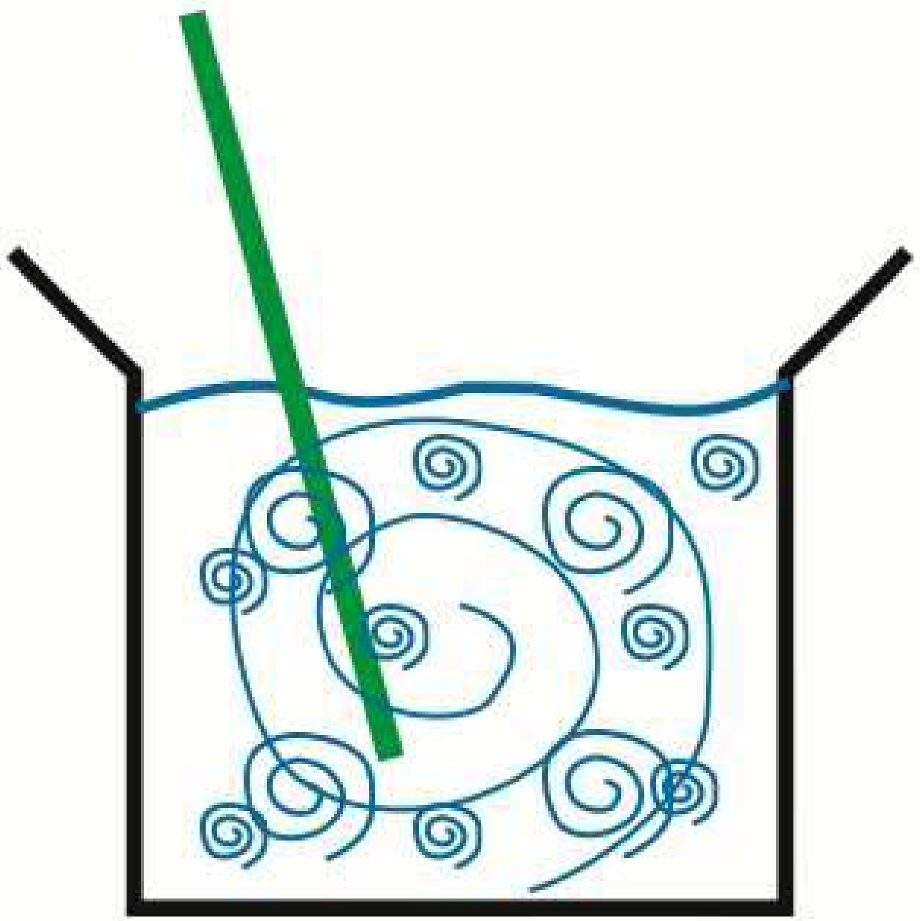
+ initial condition: $u_i(\mathbf{x}, t=0) = u_i^0(\mathbf{x})$

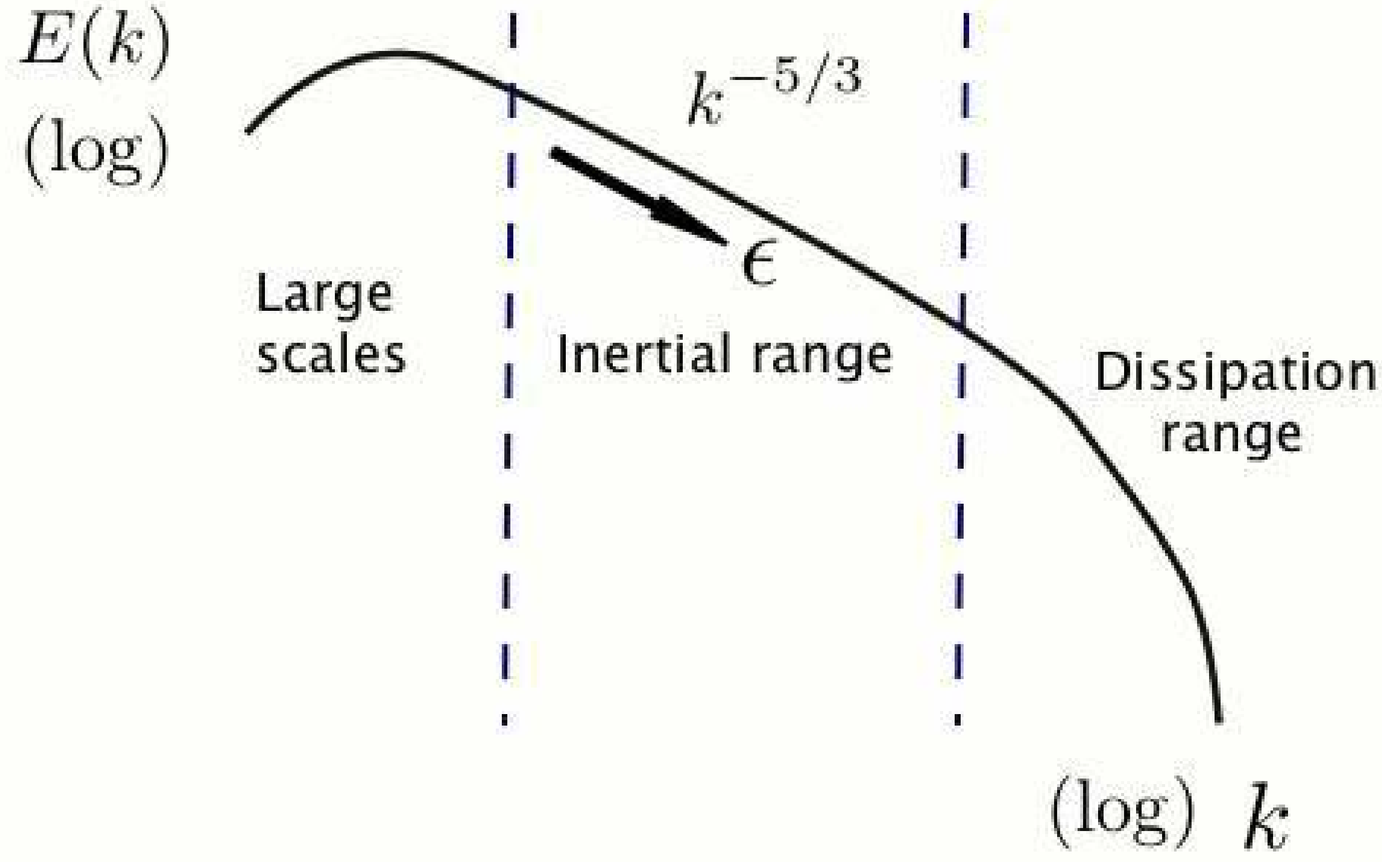
+ boundary conditions

➡ Millenium Prize: http://www.claymath.org

Kolmogorov phenomenology (1941)

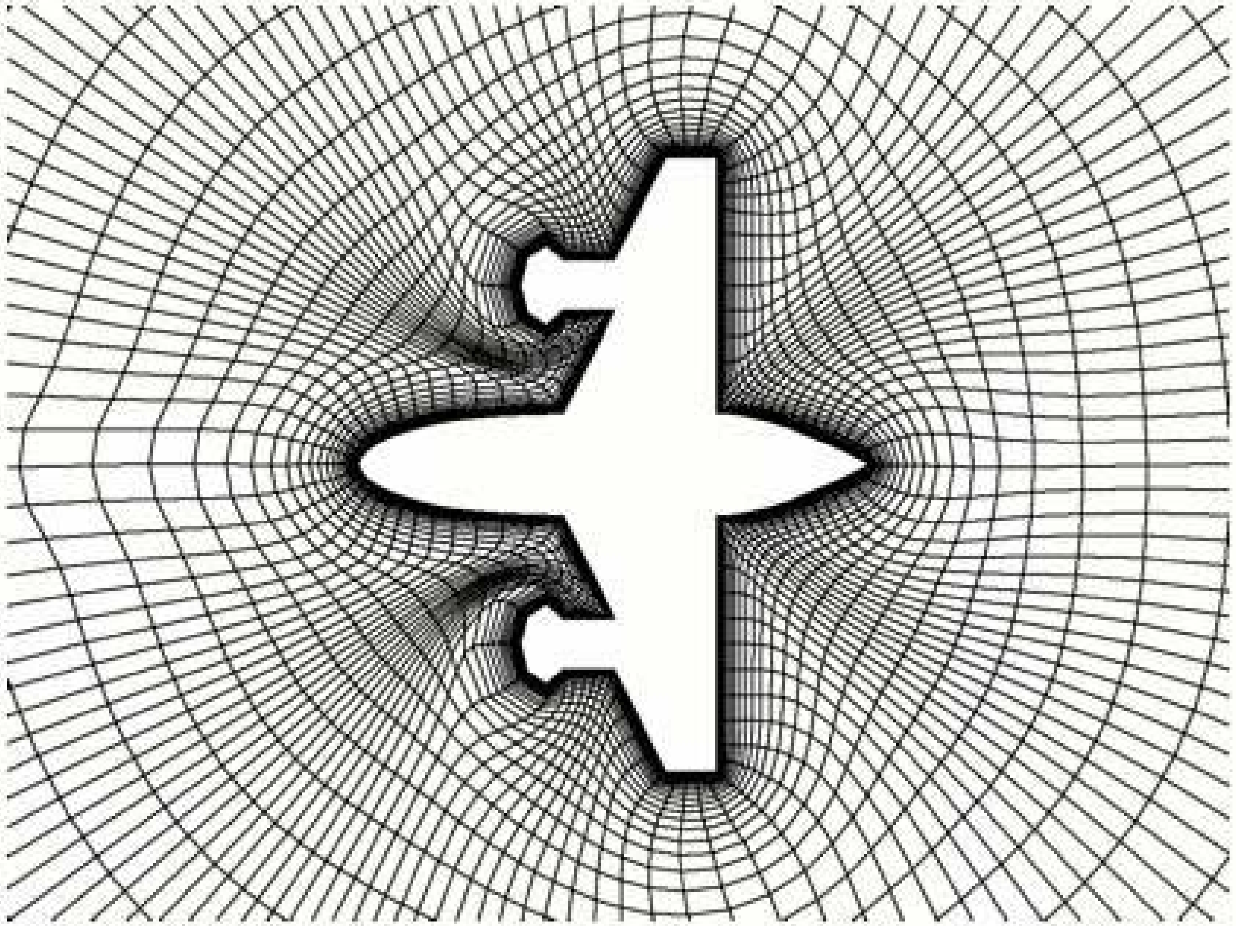
$$\mathbf{u}(\mathbf{k}) = \int \mathbf{u}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathbf{dk}$$





$$E(k) = K\epsilon^{2/3}k^{-5/3}$$

Direct numerical simulations (DNS)



- Step 1: grid generation
- Step 2: Navier–Stokes equations discretization

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i (p/\rho) + \nu \Delta u_i$$

$$\Rightarrow \quad \partial_t u_i = F(x_k) + O(\delta^n)$$

$$(\delta/L \ll 1)$$

Direct numerical simulations (DNS)

Is turbulence a solved problem? ➡ NO

- Not satisfying from the theoretical point of view.
- Inapplicable for most flows encountered in nature or industrial applications.

$$\textit{\# of grid points} \sim Re^{9/4}$$

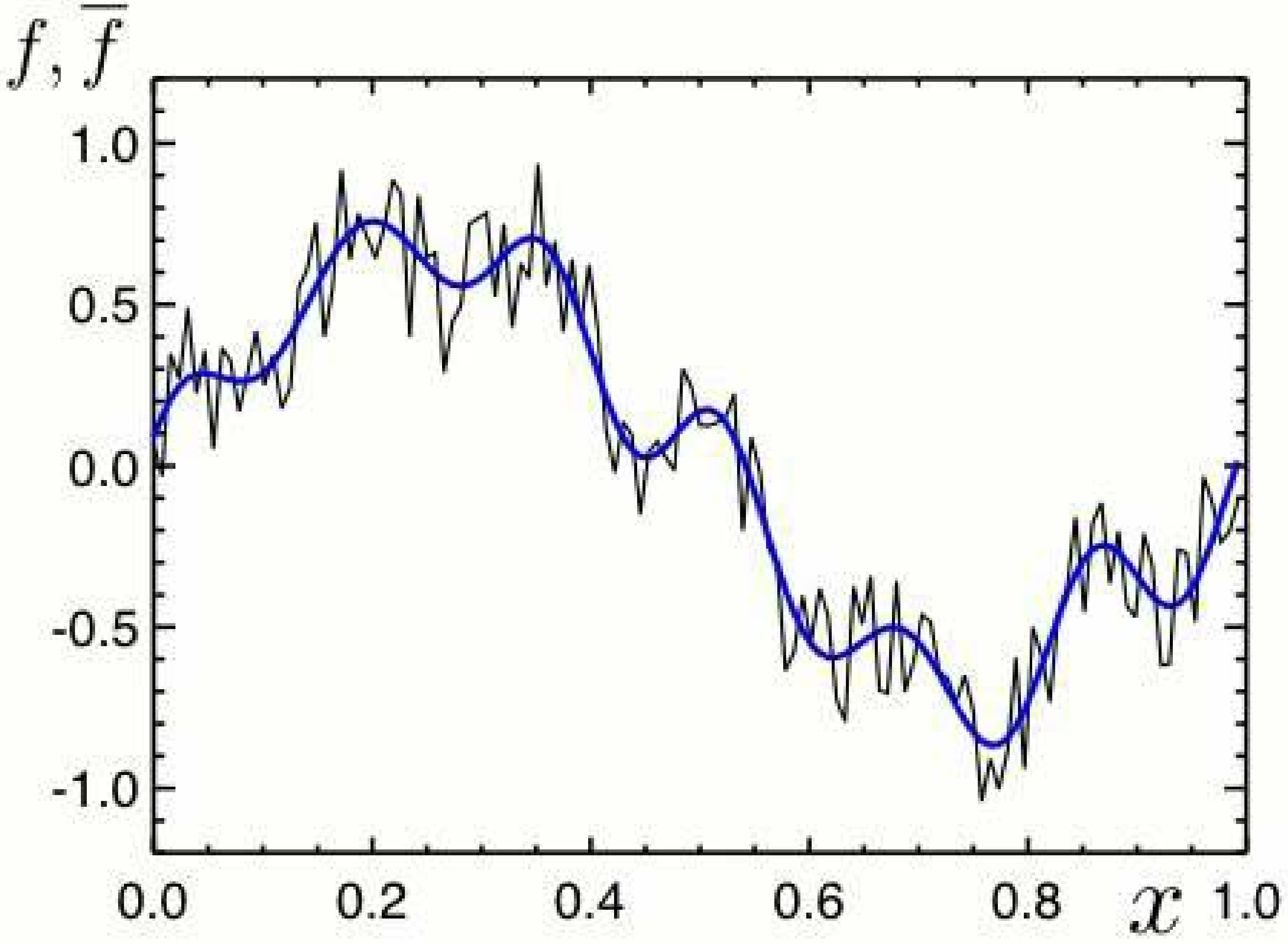
– Commercial aircraft: **Re = $\mathbf{10^8}$** ➡ 10^{18} grid points

"Teraflops" computer $(10^{12}/s)$ ➡ 1s of flight ≈ 1000 years of computing time

But:

DNS provide accurate and non-intrusive measurements for flows at low Reynolds numbers and are efficient to validate and test models and approximate methods.

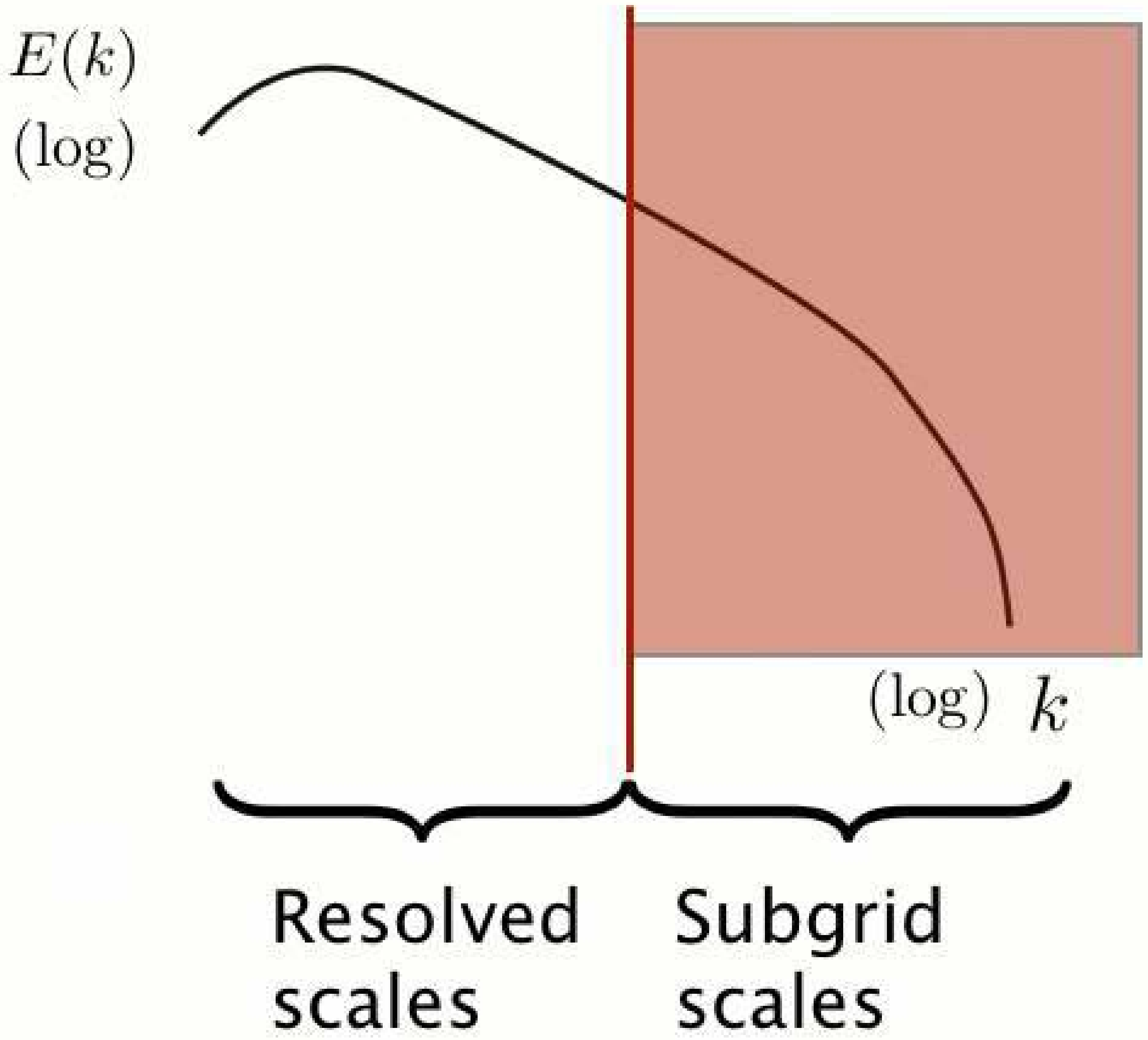
Large-Eddy Simulations (LES)



Complete signal

Resolved signal = LES signal

$$F : f(x) \rightarrow \overline{f}(x)$$



Large-Eddy Simulations (II)

Navier-Stokes equations:

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i (p/\rho) + \nu \Delta u_i$$

$$\overline{u_j \partial_j u_i} \neq \overline{u}_j \partial_j \overline{u}_i \Rightarrow [F, P] \neq 0 \qquad P : \text{product}$$

$$\partial_t \overline{u}_i + \overline{u}_j \partial_j \overline{u}_i = -\partial_i (\overline{p}/\rho) + \nu \Delta \overline{u}_i + (\overline{u}_j \partial_j \overline{u}_i - \overline{u_j \partial_j u_i})$$

Filtered Navier-Stokes equations (LES):

$$\partial_t \overline{u}_i + \overline{u}_j \partial_j \overline{u}_i = -\partial_i (\overline{p}/\rho) + \nu \Delta \overline{u}_i - \partial_j \tau_{ij}$$

$\tau_{ij} = [F, P](u_i, u_j)$: subgrid-scale stress tensor

$$\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$$

Large-Eddy Simulations (III)

➡ τ_{ij} needs to be modelled to close the equations

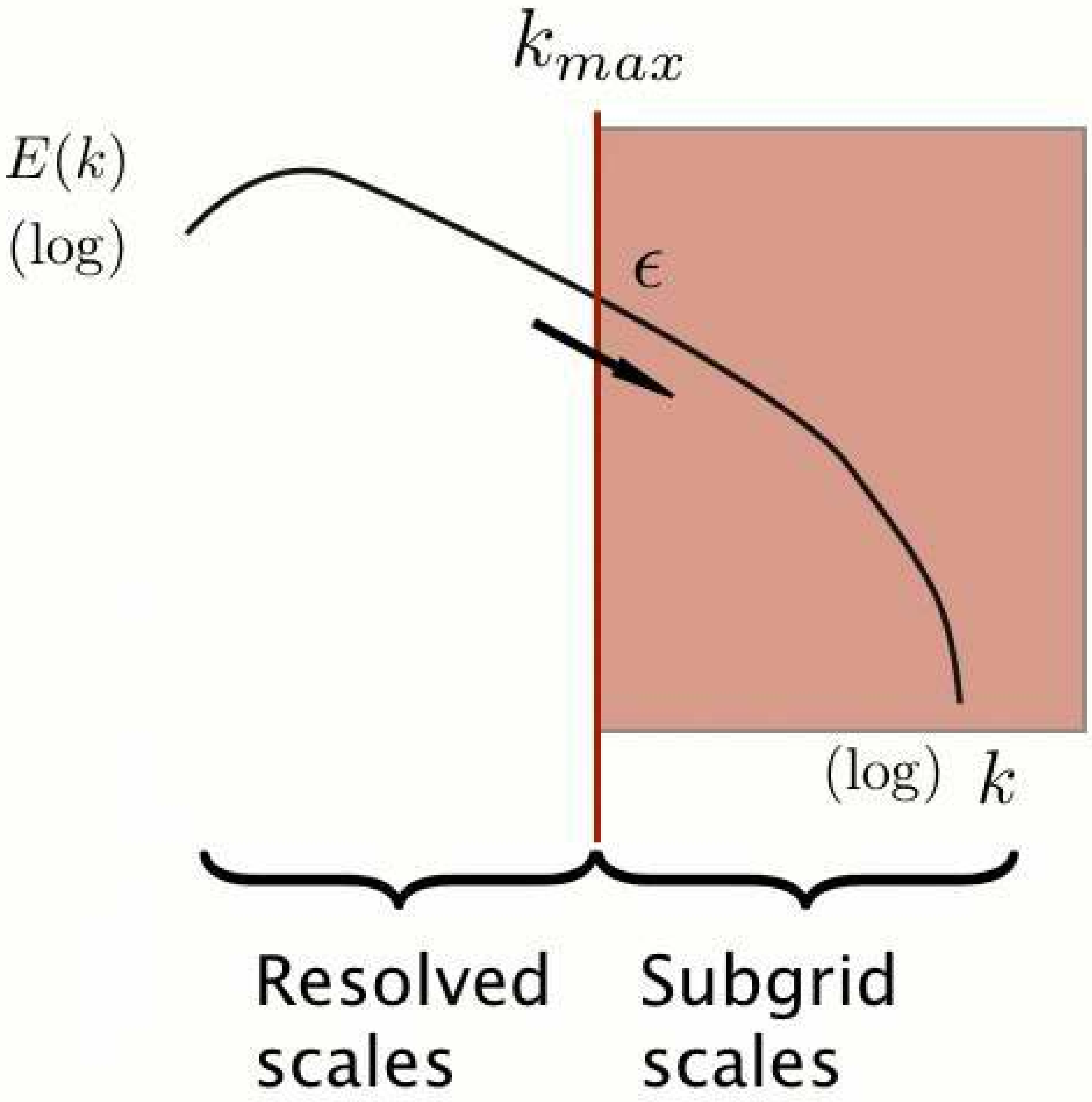
$$\tau_{ij}^{M} = \tau_{ij}^{M}(\overline{u}_i) \approx \tau_{ij}$$

- How to model τ_{ij} ? ➡ eddy viscosity

$$\tau_{ij} = -2\nu_t S_{ij} \quad \text{with} \quad S_{ij} = \frac{1}{2}(\partial_i \overline{u}_j + \partial_j \overline{u}_i)$$

$$\nu \longrightarrow \nu + \nu_t \quad \text{(homogeneous turbulence)}$$

Kolmogorov model



$$[\nu_t] = L^2T^{-1}$$

$$[\epsilon] = L^2T^{-3}$$

$$\overline{\Delta} \sim \frac{1}{k_{max}}$$

$$[\overline{\Delta}] = L$$

$$\nu_t = C_\nu \epsilon^{1/3} \Delta^{4/3}$$

(Kolmogorov model)

Smagorinski model

$$\partial_t \overline{u}_i + \overline{u}_j \partial_j \overline{u}_i = -\partial_i (\overline{p}/\rho) + \nu \Delta \overline{u}_i - \partial_j \tau_{ij}$$

$$\Rightarrow \epsilon_\tau = u_i \partial_j \tau_{ij} = -S_{ij}\tau_{ij}$$

$$= -S_{ij}(-2C_\nu \epsilon^{1/3} \Delta^{4/3} S_{ij}) = 2C_\nu \epsilon^{1/3} \Delta^{4/3} (S_{ij}S_{ij})$$

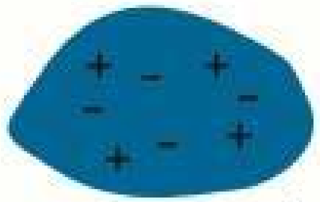
Hypothesis: $\epsilon_\tau = \epsilon$

$$\epsilon = \Delta^2 \left[C_\nu (2S_{ij}S_{ij})\right]^{3/2}$$

$$\boxed{\tau_{ij} = -2C\Delta^2 \sqrt{2S_{ij}S_{ij}} S_{ij}} \qquad (C = C_\nu^{3/2})$$

C is the Smagorinski constant (only parameter)

Magnetohydrodynamics



$\vec{u}$

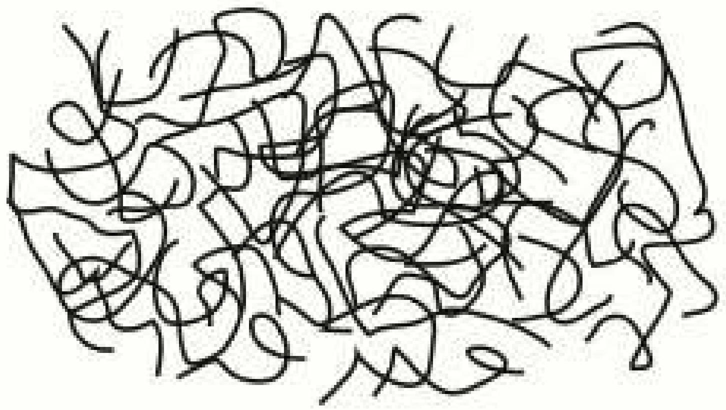
$\vec{b}$

$$\partial_t u_i = -\partial_i(p/\rho) - u_j\partial_j u_i + b_j\partial_j b_i + \nu\Delta u_i$$

$$\partial_t b_i = -u_j\partial_j b_i + b_j\partial_j u_i + \eta\Delta b_i$$

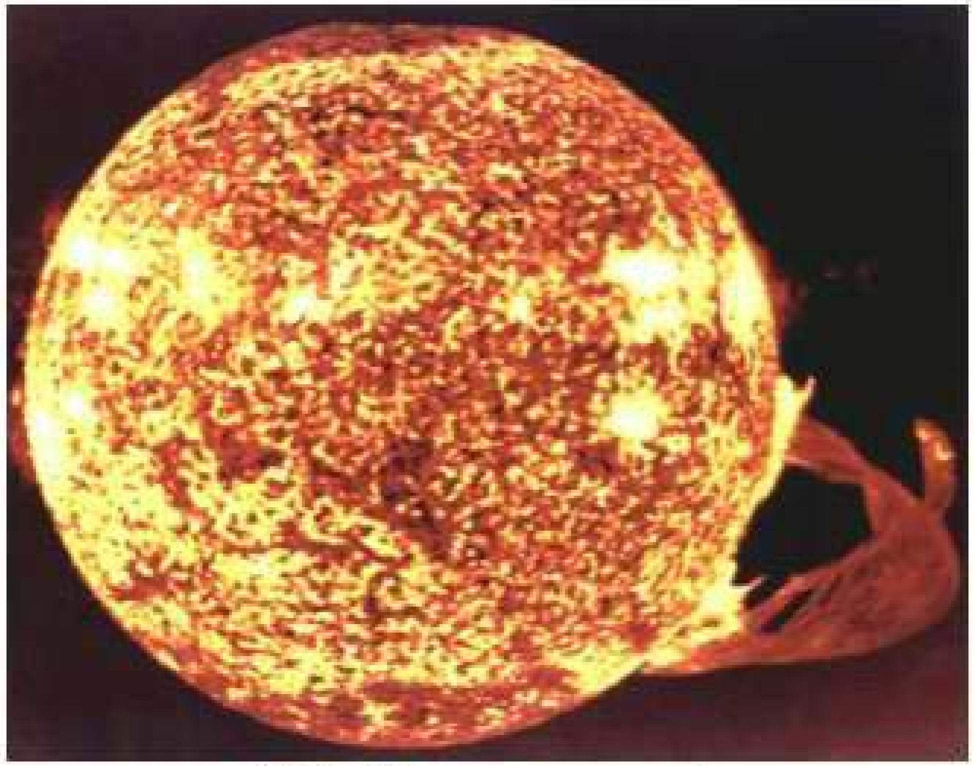
$$\partial_i u_i = 0$$

$$\partial_i b_i = 0$$

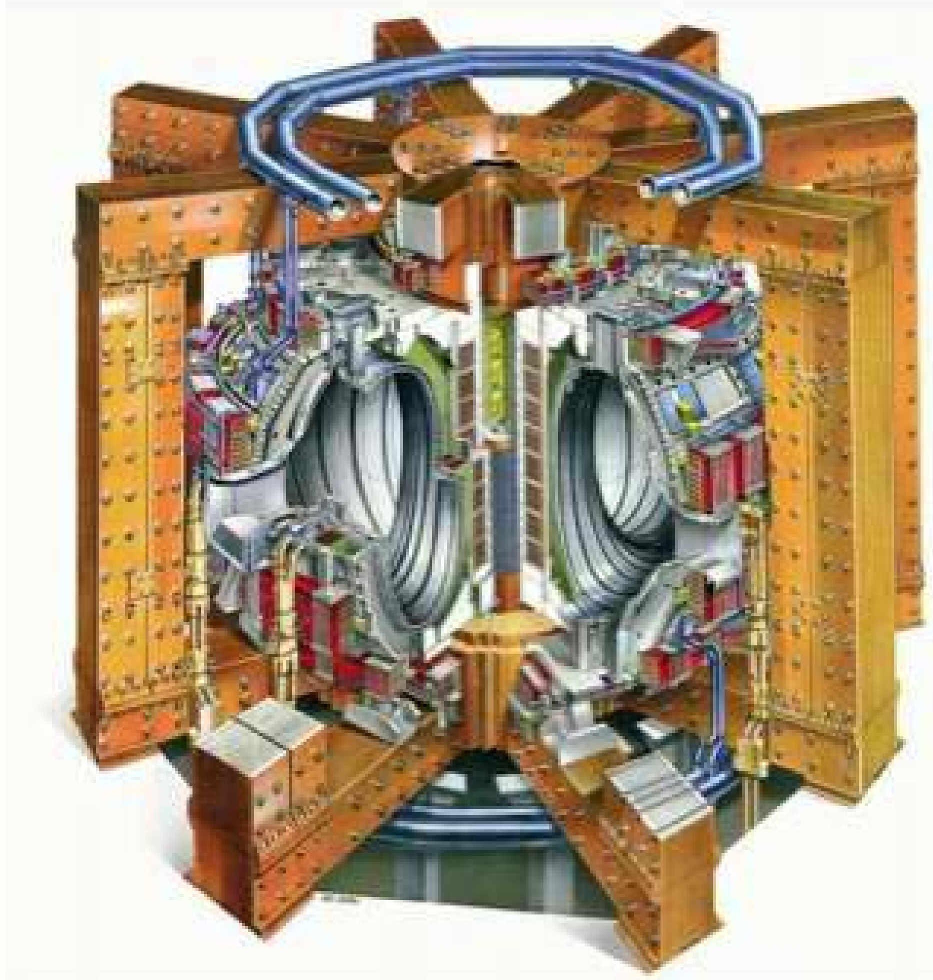


$\vec{B}^{ext}$ (imposed)

Motivations



(http://www.nasa.gov)



http://www.efda.org

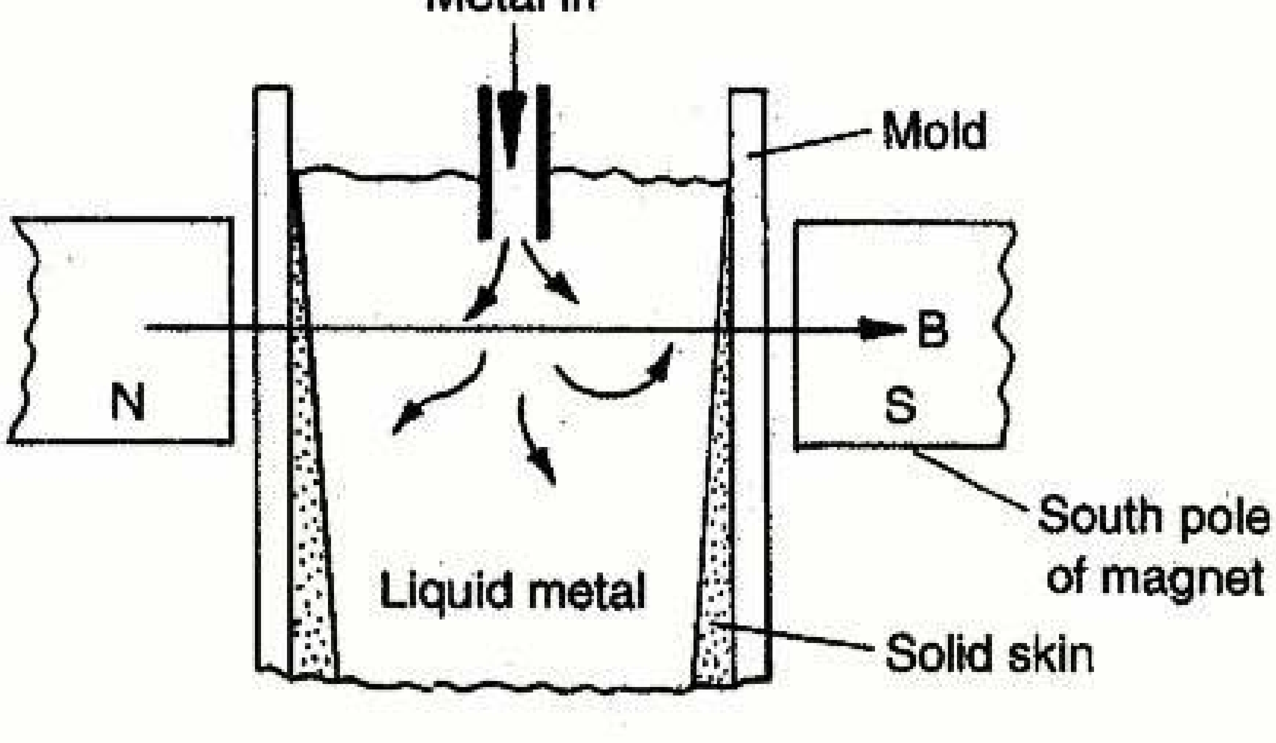
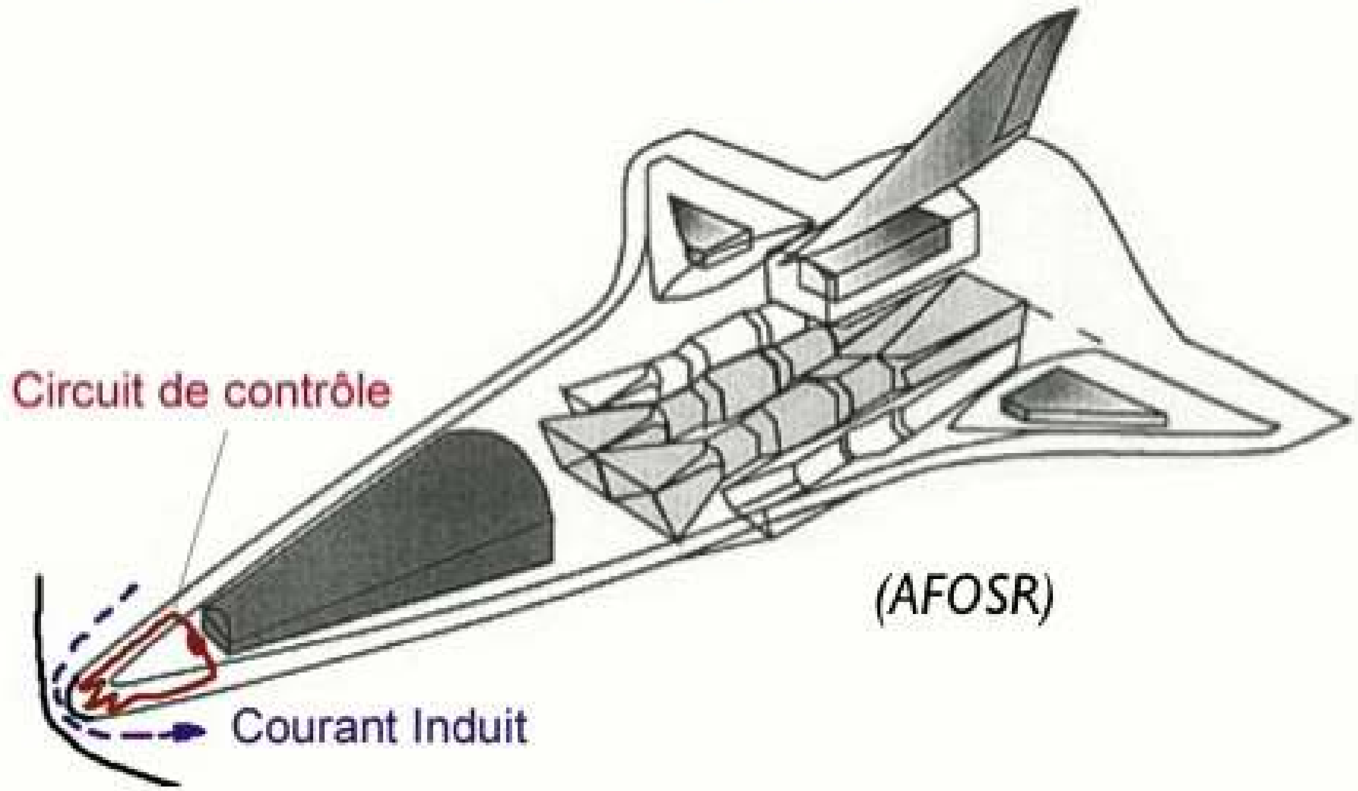


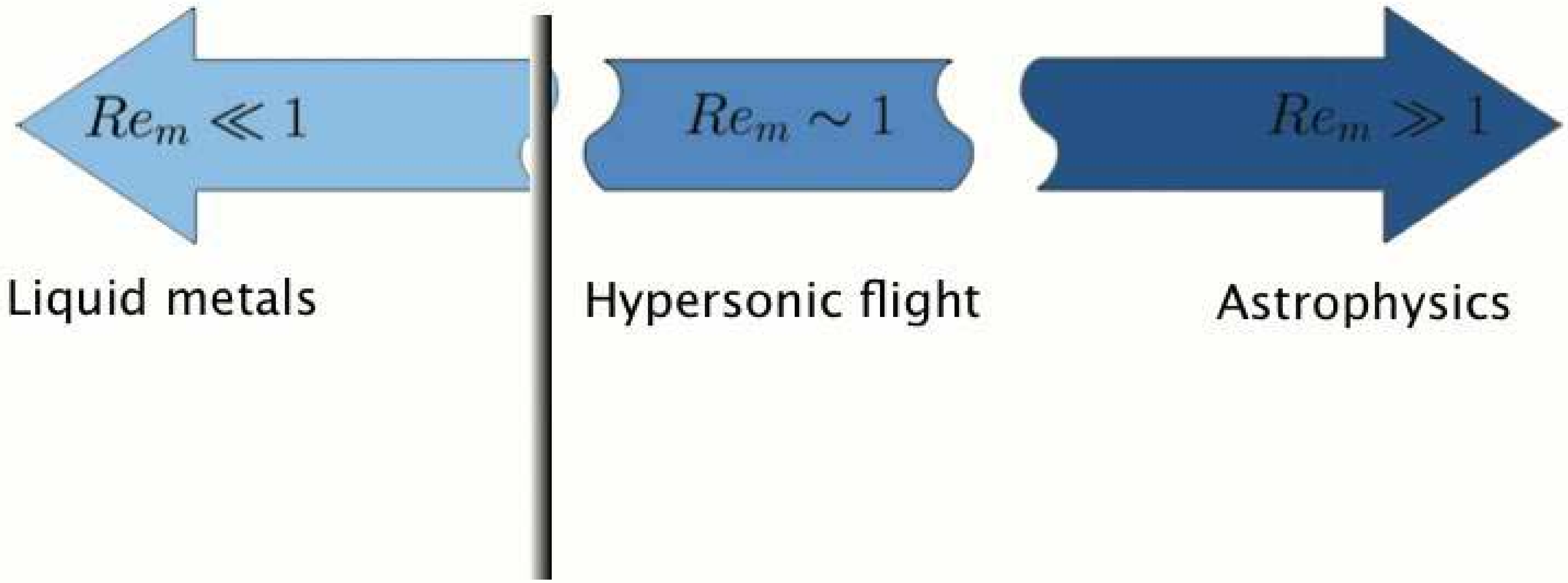
Figure I.5 Magnetic damping.
(Davidson, 2001)



(AFOSR)

Magnetic Reynolds number

$$\partial_t b_i = -u_j \partial_j b_i + b_j \partial_j u_i + \eta \Delta b_i \quad \Rightarrow \quad Re_m = \frac{uL}{\eta}$$



$Re_m \ll 1:$ Quasi-Static approximation

Quasi-Static approximation

$$\partial_t b_i = \cancel{-u_j \partial_j b_i} + \cancel{b_j \partial_j u_i} + B_j^{ext} \partial_j u_i + \eta \Delta b_i$$

➡ 'steady state' $\quad 0 = B_j^{ext} \partial_j u_i + \eta \Delta b_i$

$$\partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + \cancel{\frac{1}{(\mu\rho)} b_j \partial_j b_i} + \frac{1}{(\mu\rho)} B_j^{ext} \partial_j b_i + \nu \Delta u_i$$

$$\partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + \frac{(B_z^{ext})^2}{\mu\eta\rho} \Delta^{-2} \partial_z \partial_z u_i + \nu \Delta u_i$$

Quasi-Static approximation

QS approximation: Large-Eddy Simulation

$$\partial_t u_i = -\partial_i (p/\rho) - u_j \partial_j u_i + \frac{(B_z^{ext})^2}{\mu\eta\rho} \Delta^{-2} \partial_z \partial_z u_i + \nu \Delta u_i$$

Intensity parameter: $N = \frac{(B_z^{ext})^2 L}{\mu\eta\rho v}$

N=1: magnetic effects ~ non-linear effects

N=10: magnetic effects ~ 10 x non-linear effects

LES:

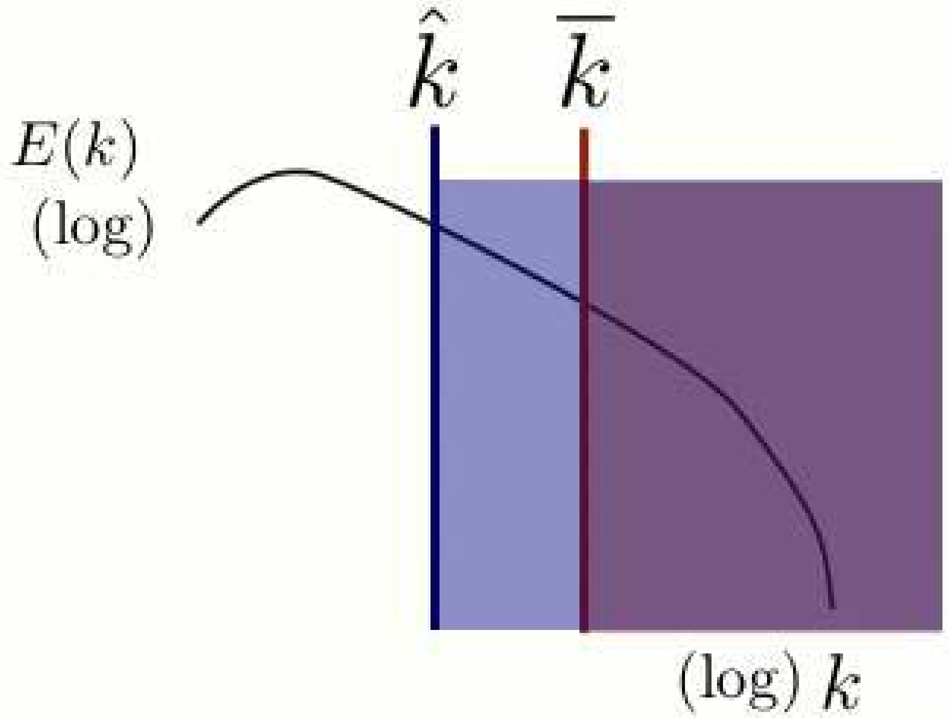
$$\partial_t \overline{u}_i = -\partial_i (\overline{p}/\rho) - \overline{u}_j \partial_j \overline{u}_i + \frac{(B_z^{ext})^2}{\mu\eta\rho} \Delta^{-2} \partial_z \partial_z \overline{u}_i + \nu \Delta \overline{u}_i - \partial_j \tau_{ij}$$

$$\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$$

Dynamic procedure

$$\tau_{ij} = -2C\Delta^2\sqrt{2S_{ij}S_{ij}}S_{ij} \qquad \text{(Smagorinski)}$$

One parameter: Smagorinski constant C



- Self-similarity in the inertial range

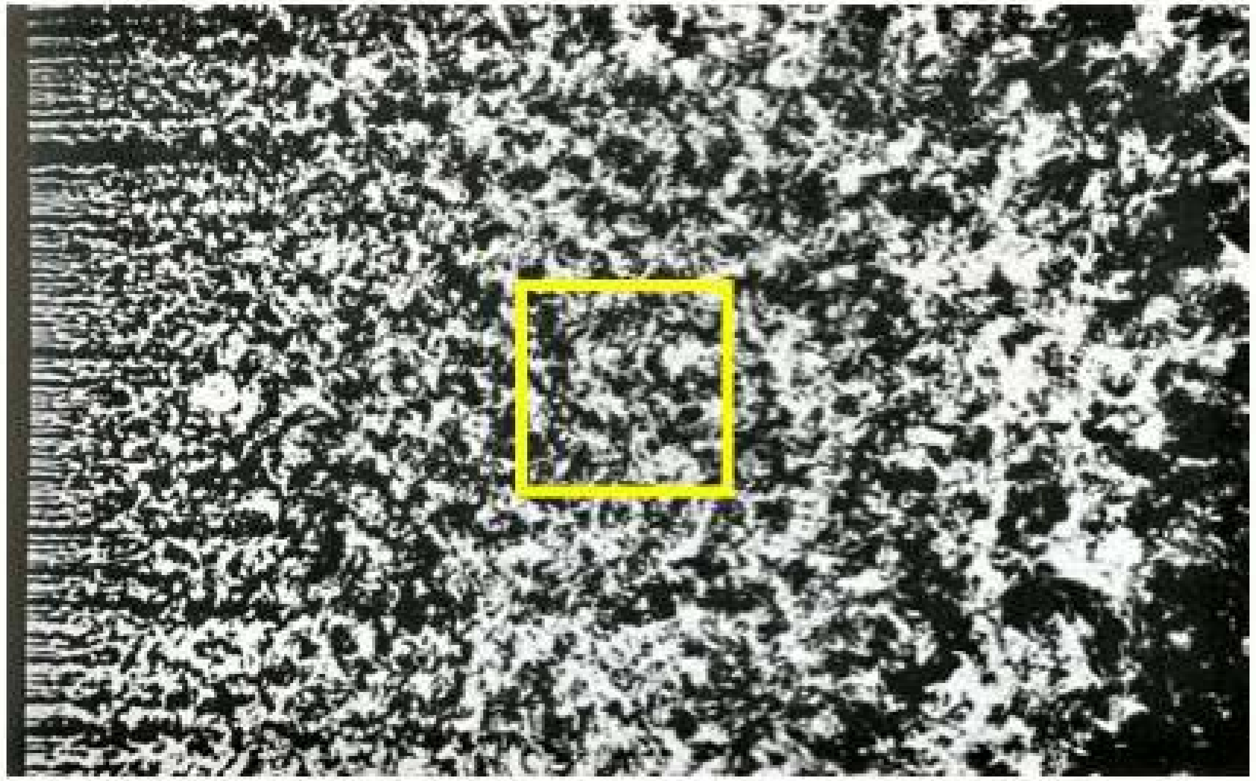
$\hat{\tau_{ij}}$, $\overline{\tau_{ij}}$ → same C

→ $C = C(\overline{u}_i)$

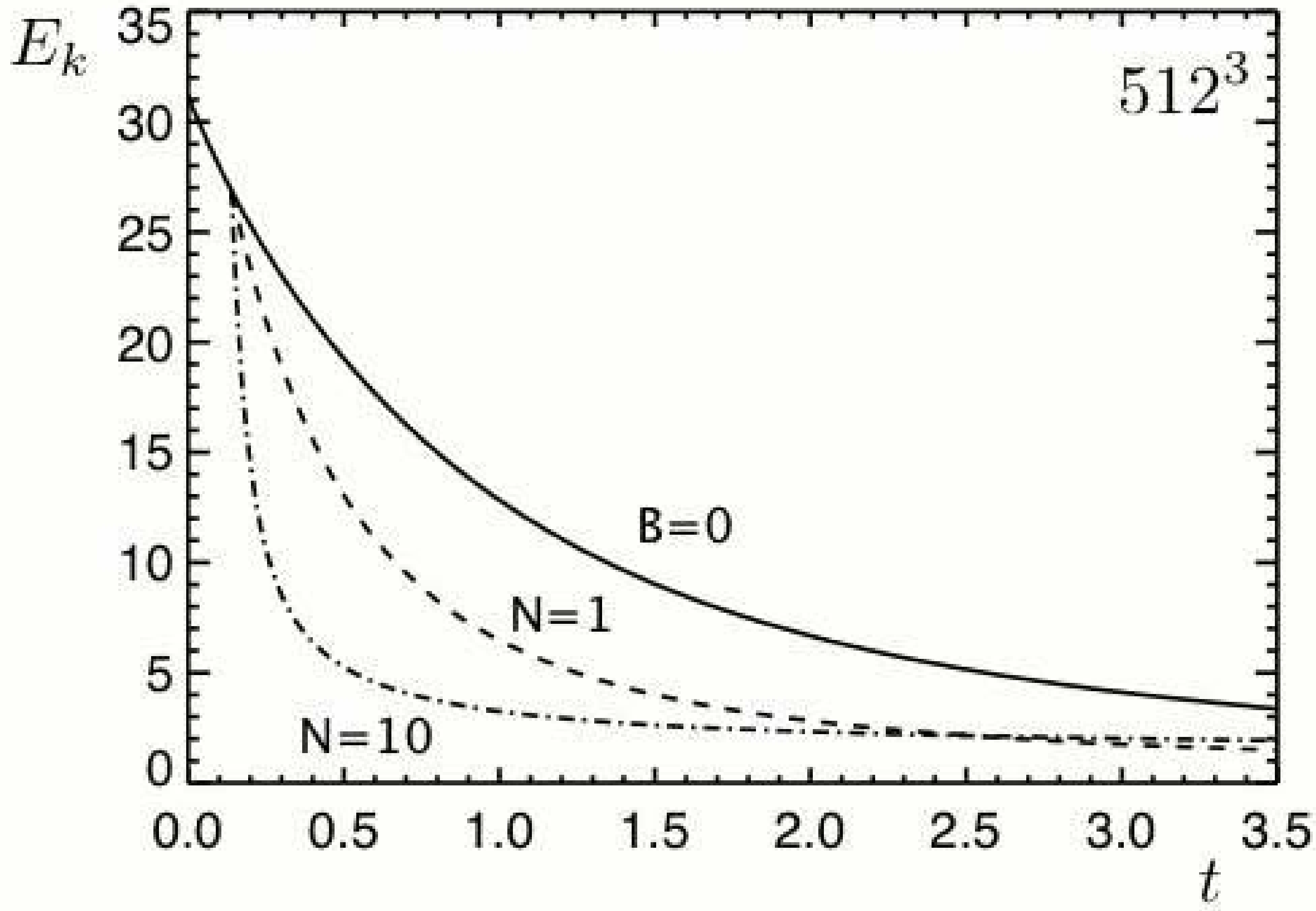
$$\tau_{ij} = -2C(\overline{u}_i)\Delta^2\sqrt{2S_{ij}S_{ij}}S_{ij}$$

Dynamical Model

Decaying turbulence

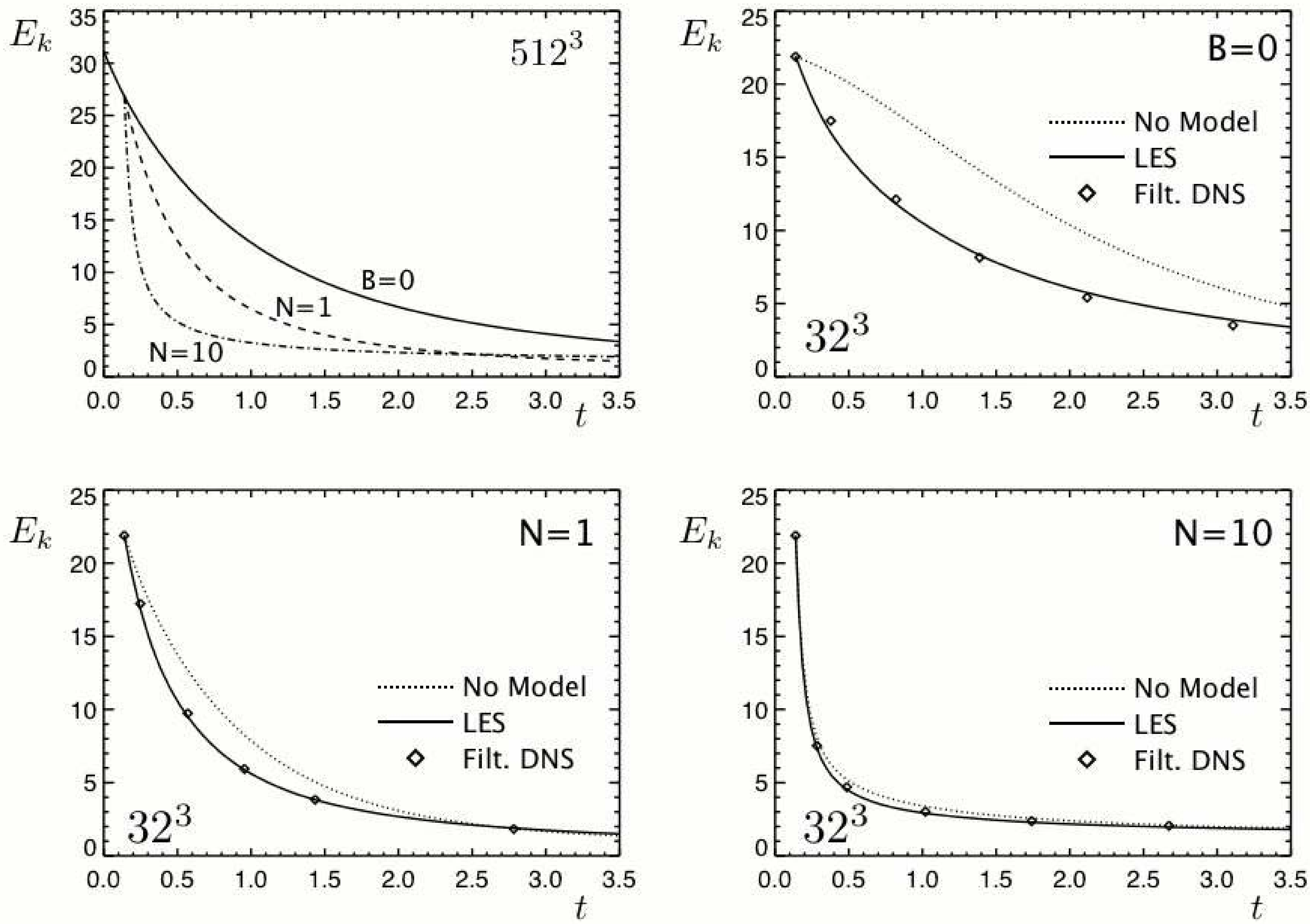


(Photo: Van Dyke M. An Album of Fluid Motion)

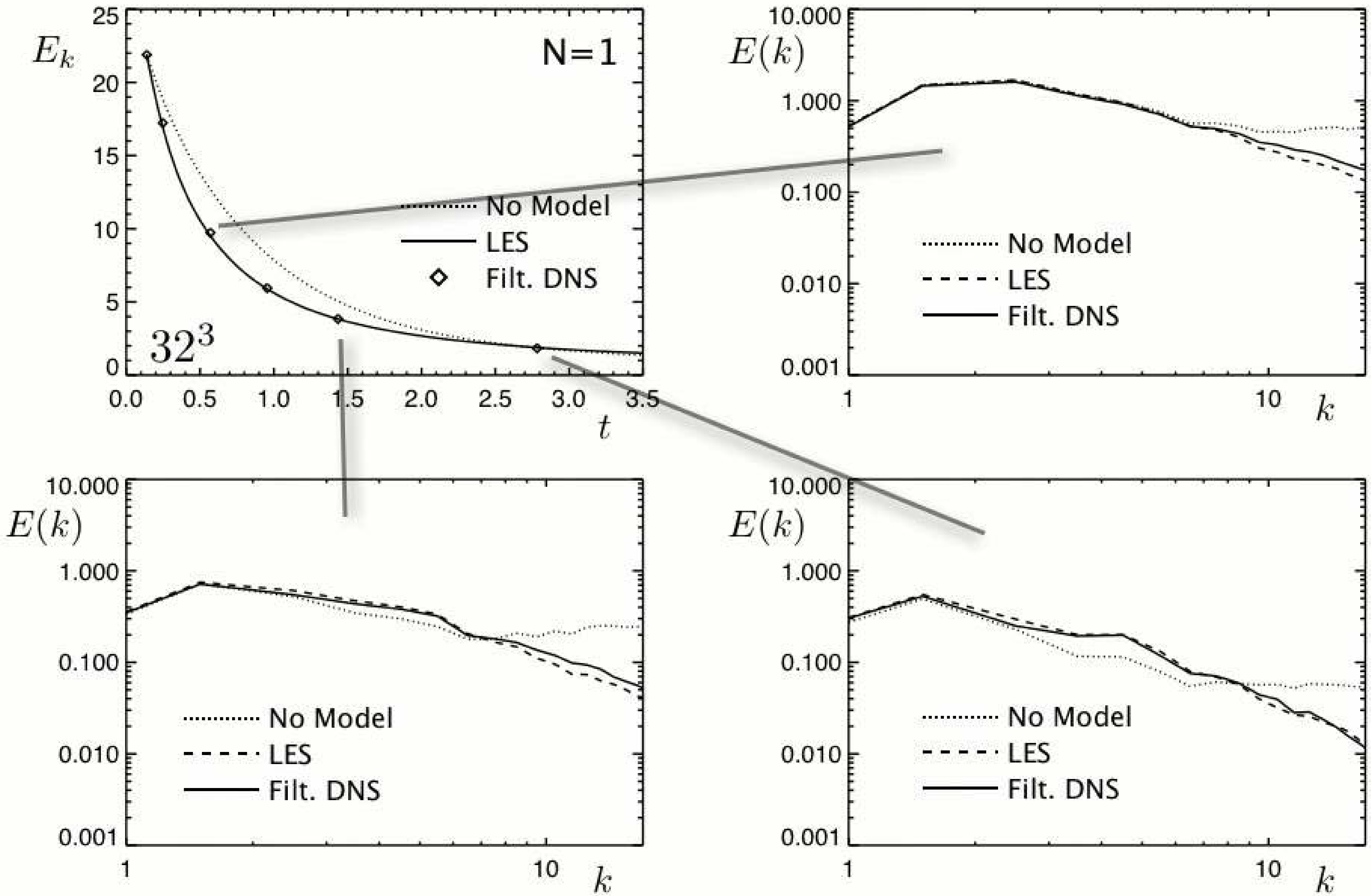


BK, Moin (2003)

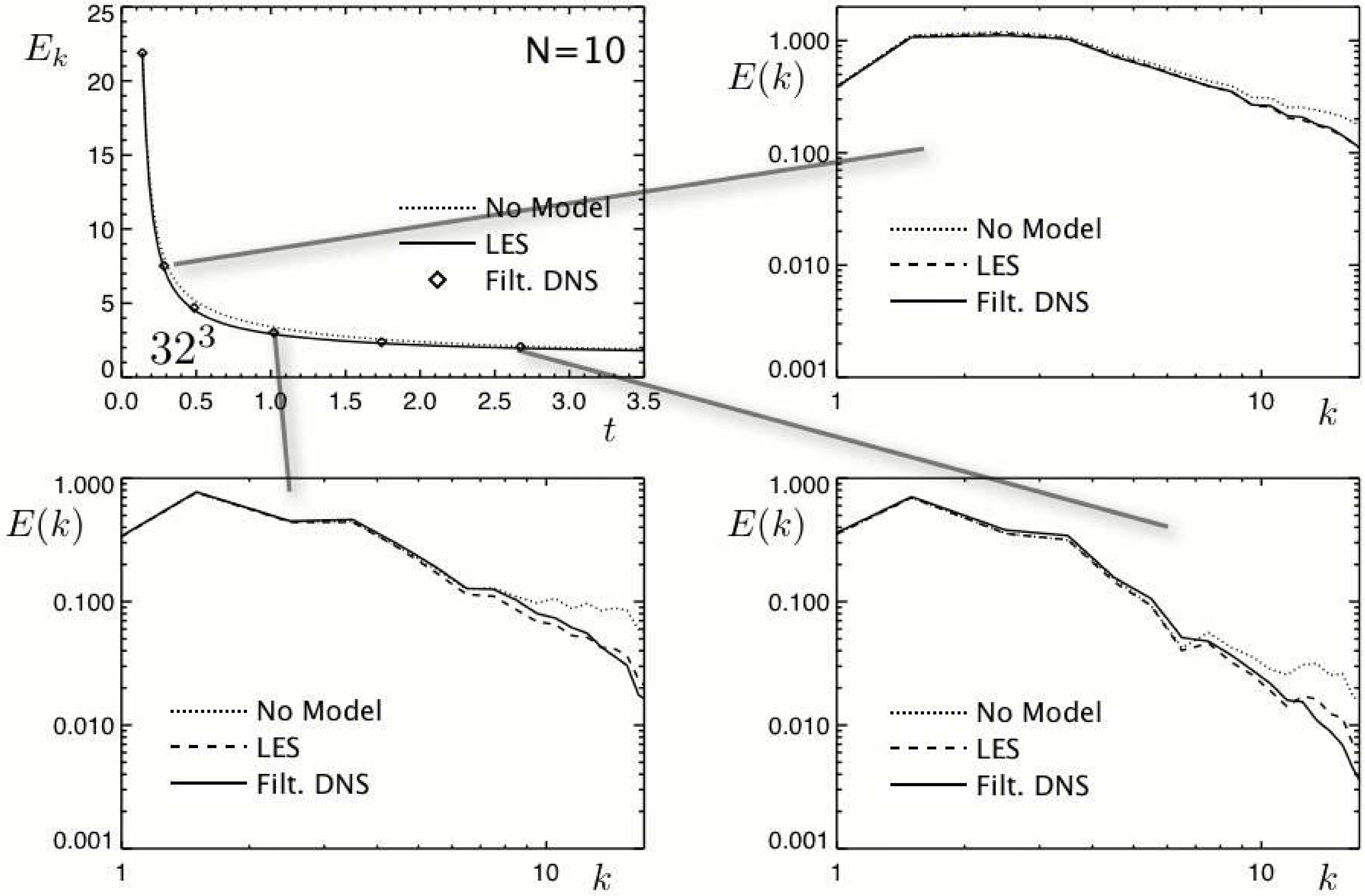
DNS versus LES



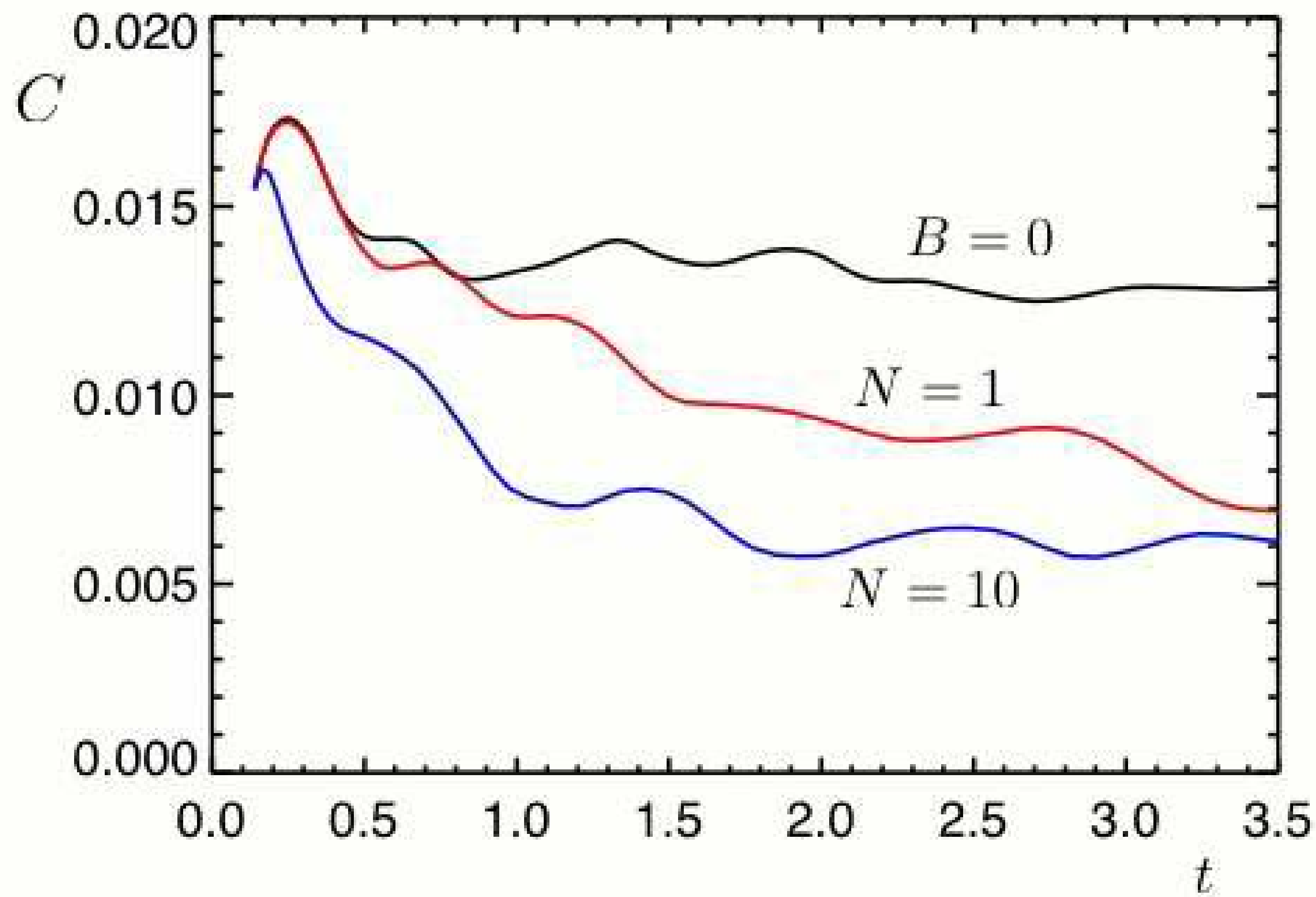
Energy Spectra N=1



Energy Spectra N=10

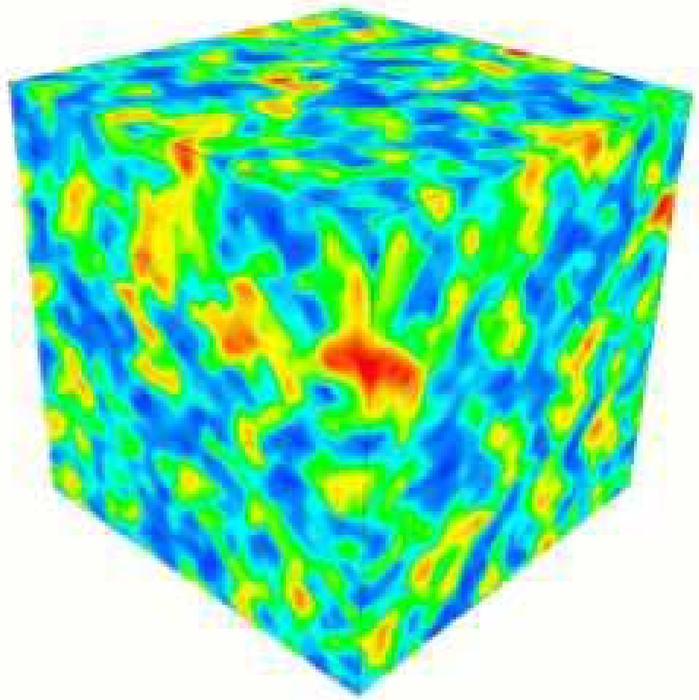


Dynamic constant

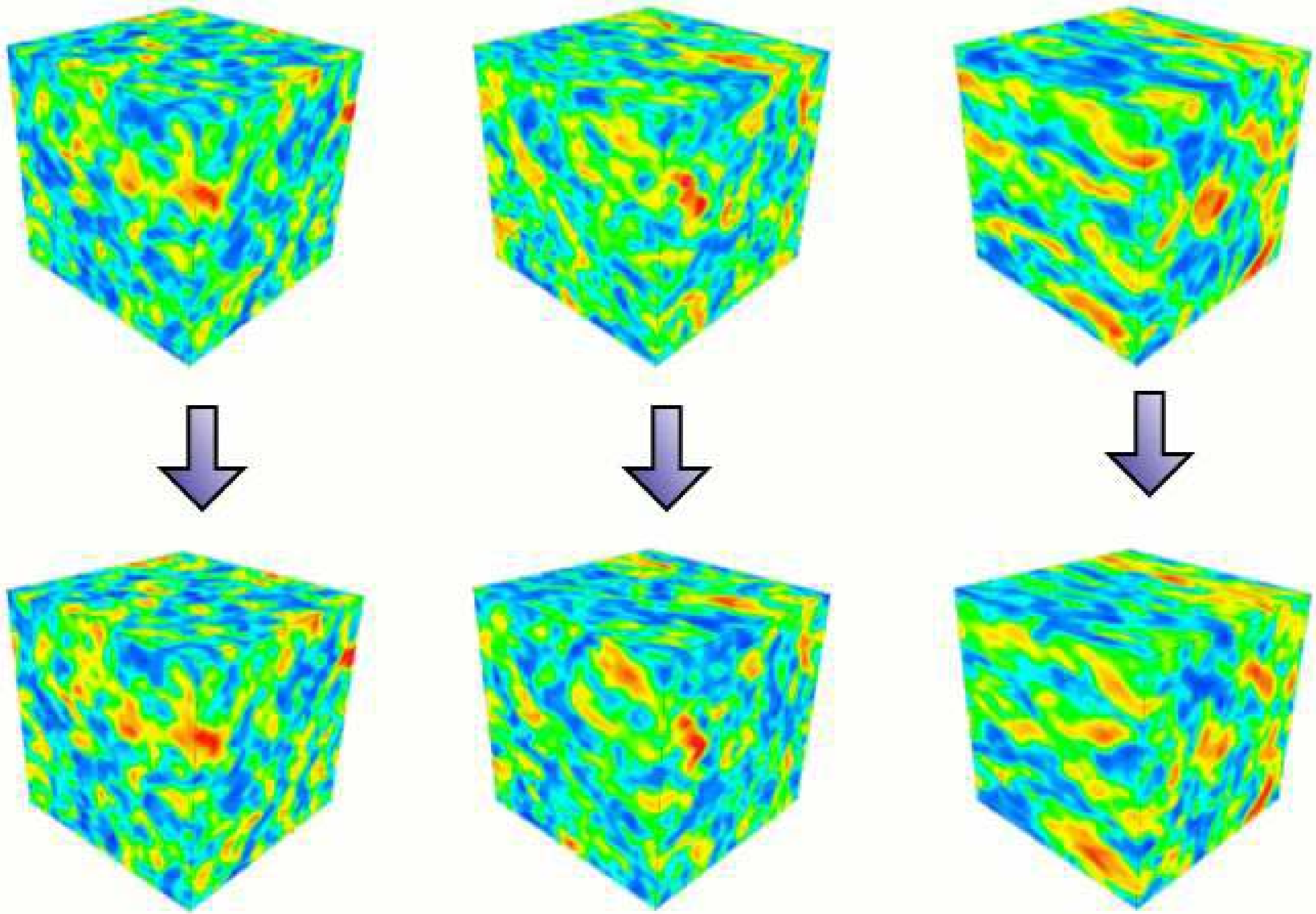


Energy contours N=1

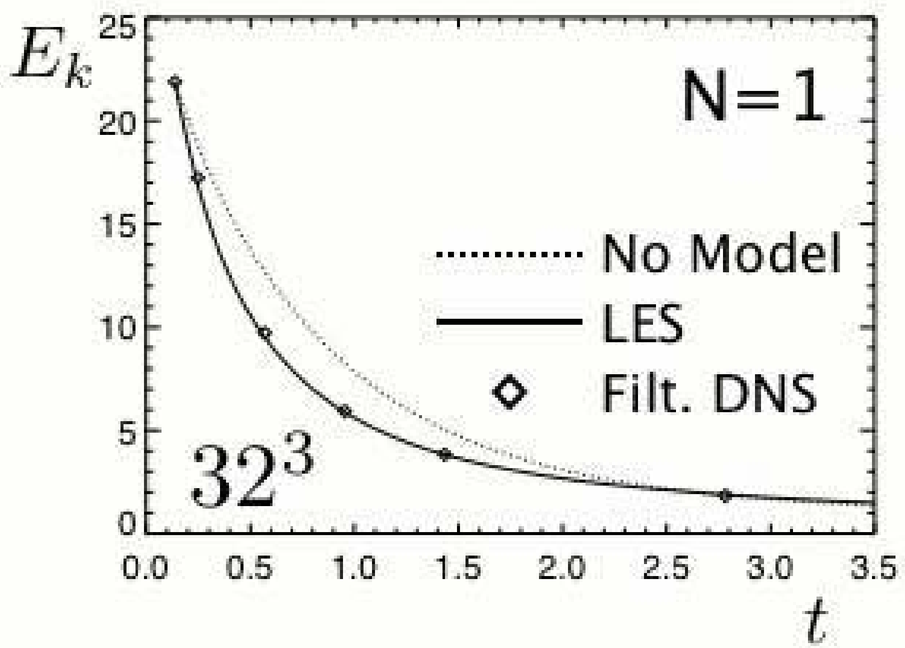
Initial flow



Filtered DNS 32^3

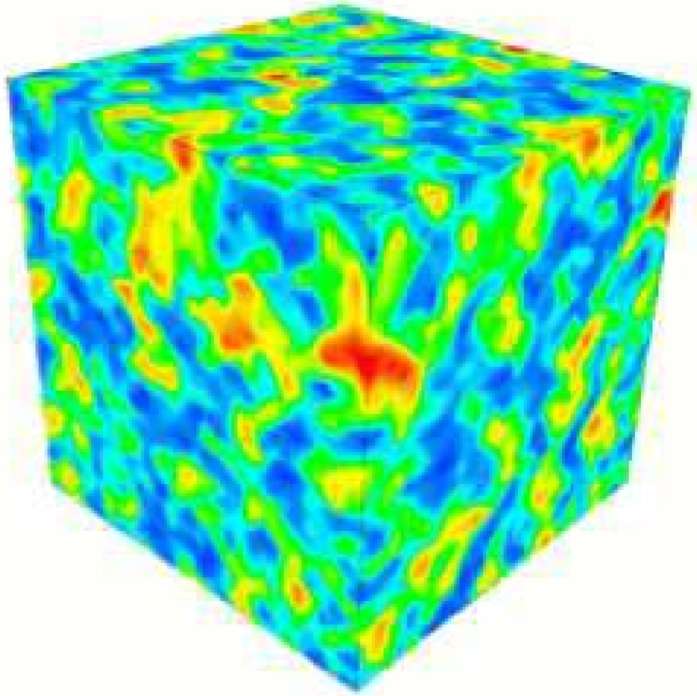


LES 32^3

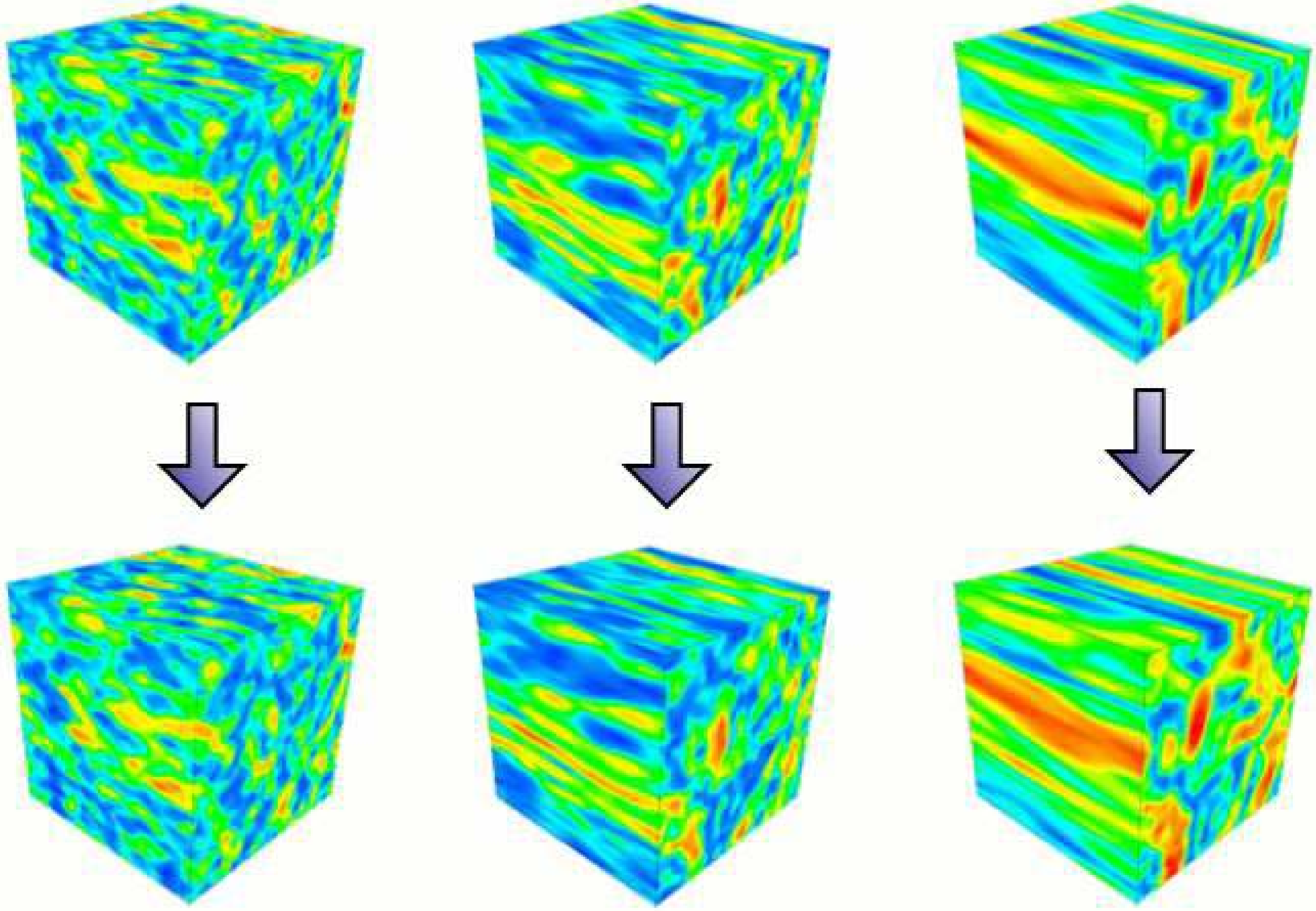


Energy Contours N=10

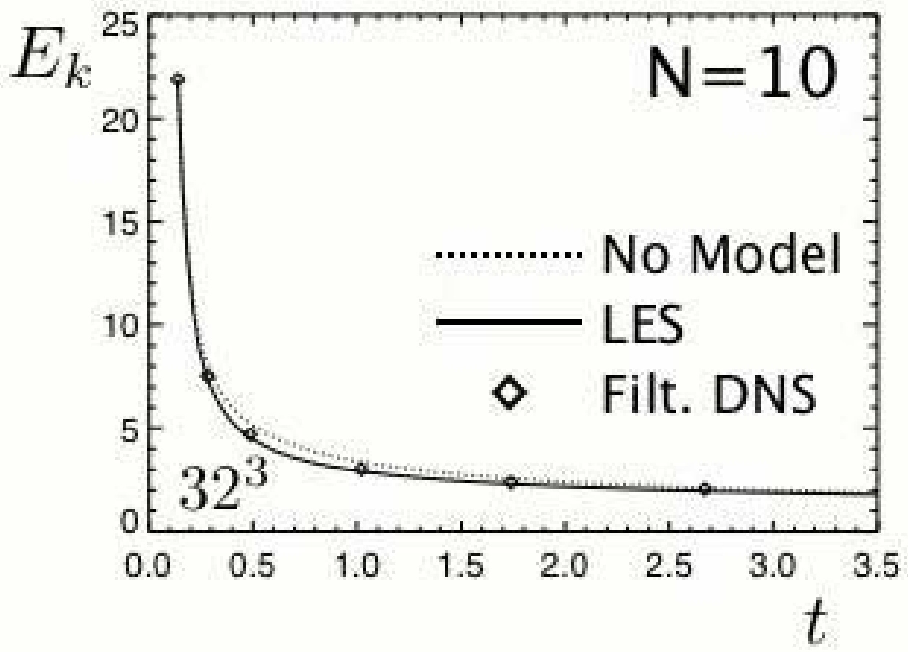
Initial flow



Filtered DNS 32^3



LES 32^3



Conclusions

- Turbulence is a complex process for which very few analytical results are known
- Numerical methods exists but they require theoretical model development in order to explore realistic flows
- Large-Eddy Simulations are a powerful tool for simulating flows at a lower computational cost
- Large-Eddy Simulations are widely used for engineering flows in complex geometries
- Only limited work has been devoted to this method in the case of MHD turbulence