Coupling of Radial and Non-radial Perturbations of Relativistic Stars

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Outline

- Motivation
- Pertubative framework: 2-parameter perturbation theory and GSGM formalism.
- Radial Perturbations
- Non-radial and coupling perturbations: Gauge invariance and perturbative equations.
- Outline of the numerical code and results for the axial case.
- Conclusions and future works

Motivation

$\star$ Gravitational Waves of Neutron stars.

Detectors: i) Direct evidence of the existence of the Gravitational waves.
ii) Development of a new field: Gravitational Wave Astronomy.

$\star$ ***High frequency detectors:*** Ground-based detectors with frequency band range $10 - 10^4\, Hz$

- *Resonant-mass Antennas:* narrow-band detector with a good sensitivity in $\sim 100 Hz$ spectrum region. International Gravitational Event Collaboration (IGEC): ALLEGRO (USA), NIOBE (AUS), EXPLORER (CERN), NAUTILUS and AURIGA (Italy). - Dual bar antenna.

- *Interferometers:* broad-band sensitivity. Ground based laser interferometers: GEO600 (German), VIRGO (Italy), TAMA (Japan), Laser Interferometric Gravitational Wave Observatory (LIGO, USA)

$\star$ ***Low frequency detector:*** Laser Interferometer Space Antenna (LISA) (ESA-NASA collaboration), ($\sim$ 2014−??). Frequency band $\left(10^{-4} - 1\right) Hz$.

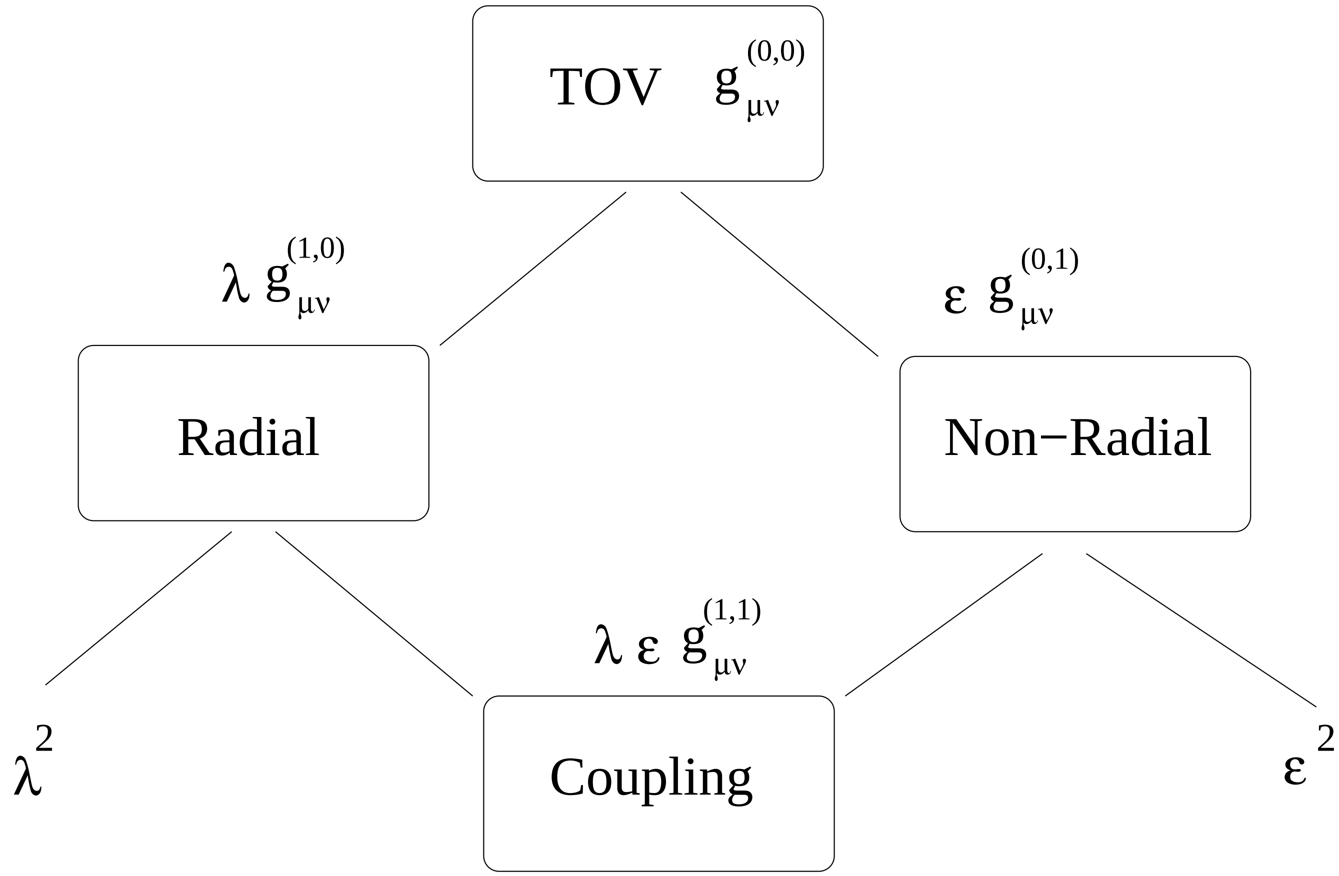
- Templates for the data analysis of current experimental projects and for the design of future detectors.
- Gravitational radiation contains signatures of the internal properties of sources $\longrightarrow$ Gravitational wave Astronomy. - Determination of Masses and Radii from spectra (Andersson & Kokkotas 1996) $\longrightarrow$ constraint the supranuclear EoS.

$\star$ Project aims

- Extension to the second order of the stellar perturbative analysis, in order to investigate the possible changes of the linear description due to a mild non-linearity.
- Coupling radial/non-radial: What are the effects of this non-linear coupling on the gravitational signal? Is there any existence of resonances or non-linear effects which may produce a relevant gravitational radiation? Could the radial perturbations power the non-radial and then lose energy through this channel?
- Time domain analysis: determination of wave forms, allows to investigate the evolution of coupling with the simultaneously excitation of several frequencies. Spectrum through FFT.
- Application of the multi-parameter perturbation theory.

Perturbative Framework

2-parameter second order perturbation theory (C.F.Sopuerta,M.Bruni,L.Gualtieri 2003)



$$\mathrm{g}_{\mu\nu} = \mathrm{g}^{(0,0)}_{\mu\nu} + \lambda\,\mathrm{g}^{(1,0)}_{\mu\nu} + \epsilon\,\mathrm{g}^{(0,1)}_{\mu\nu} + \lambda\,\epsilon\,\mathrm{g}^{(1,1)}_{\mu\nu} + O(\lambda^2,\epsilon^2)\,.$$

Perturbative equations for non-linear coupling

$$\boldsymbol{E}^{(\mathbf{0,1})}\left[\,\mathbf{g}^{(\mathbf{1,1})}\,,\boldsymbol{T}^{(\mathbf{1,1})}\,;\mathbf{g}^{(\mathbf{0,0})}\,\right] = \boldsymbol{\mathcal{S}}\left[\,\boldsymbol{\mathcal{G}}^{(\mathbf{1,0})}\otimes\boldsymbol{\mathcal{J}}^{(\mathbf{0,1})}\,;\boldsymbol{\mathcal{B}}^{(\mathbf{0,0})}\,\right]$$

Gerlach-Sengupta and Gundlach-Martin Garcia formalism (GSGM)

Gerlach & Sengupta (1979), Gundlach & M.Garcia (1999)

- **Gauge invariant description** of first order non-radial perturbations on a *background which is time depending and spherically symmetric.*
- **Background** $\mathcal{M} = \mathcal{M}^2 \times S^2$

  Metric and Energy-momentum tensors:

$$\mathrm{g}_{\alpha\beta} = \begin{pmatrix} \mathrm{g}_{AB} & 0 \\ 0 & r^2 \gamma_{ab} \end{pmatrix} \qquad t_{\alpha\beta} = \begin{pmatrix} t_{AB} & 0 \\ 0 & r^2\, Q(x^C)\, \gamma_{ab} \end{pmatrix}$$

- **Perturbations:** spherically symmetry $\Longrightarrow$ expansion in **tensorial harmonics** of a general perturbations $\Longrightarrow$ ***axial*** (odd) and ***polar*** (even) perturbations.

- ***Perturbative Einstein equations***: system of equations of gauge invariant scalar perturbation fields.

Equilibrium Configuration

- *Spherically-symmetric relativistic star:* $ds^2 = -e^{-\Phi(r)}\,dt^2 + e^{-\Lambda(r)}\,dr^2 + r^2\,d\Omega$

- Perfect Fluid: *polytropic EoS* $p = K\rho^\Gamma$

- TOV equations determine the equilibrium configuration.
- Stellar model:

$\rho_0 = 3\times 10^{15}\,\mathrm{g\ cm^{-3}}$ $K = 100\,\mathrm{km}^2$ $\Gamma = 2 \Longrightarrow M = 1.26\,M_\odot$ $R = 8.86\,\mathrm{km}$

Radial Perturbations

Chandrasekhar (1964)

- Spherically symmetric pulsations $\Longrightarrow$ Not gravitational radiation.
- Frequency domain: Sturm-Liouville eigenvalue problem, complete set of eigenfrequencies and eigenfunctions.
- Time domain: GSGM formalism, gauge invariance breaks down for $l = 0$.

Radial gauge: $\psi^{(1,0)} = 0\,,$ $k^{(1,0)} = 0$

$$\delta g^{(1,0)}_{AB} = \left(r\,S^{(1,0)} - 2\,\eta^{(1,0)}\right)\,e^{2\Phi}\,dt^2 + e^{2\Lambda}\,r\,S^{(1,0)}\,dr^2 \qquad \delta g^{(1,0)}_{ab} = 0$$

$$\delta u_r^{(1,0)} \sim \gamma^{(1,0)} \qquad \delta\rho^{(1,0)} = \omega^{(1,0)}\bar{\rho}\,, \qquad \delta p^{(1,0)} = c_s^2\,\delta\rho^{(1,0)}\,.$$

$$\textit{Enthalpy} \qquad \delta H = \delta p/\left(\bar{\rho}+\bar{p}\right)$$

- **Perturbative equations for the radial pulsations**

$$\begin{aligned} \dot{H}^{(1,0)} &= -\bar{c}_s^2\,\gamma^{(1,0)'} + \dots\ \gamma^{(1,0)}\,, \\ \dot{\gamma}^{(1,0)} &= -H^{(1,0)'} + \dots\ H^{(1,0)} + \dots\ S^{(1,0)}\,, \\ \dot{S}^{(1,0)} &= \dots\ \gamma^{(1,0)}\,, \end{aligned}$$

$$\begin{aligned} S^{(1,0)'} &= \dots\ S^{(1,0)} + \dots\ H^{(1,0)} \qquad \textit{Hamiltonian constraint} \\ \eta^{(1,0)'} &= \dots\ S^{(1,0)} + \dots H^{(1,0)} \end{aligned}$$

- *Wave equation* for the perturbation of the velocity radial component

$$-\gamma_{,\,tt} + \bar{c}_s^2\,\gamma_{,\,rr} + \dots$$

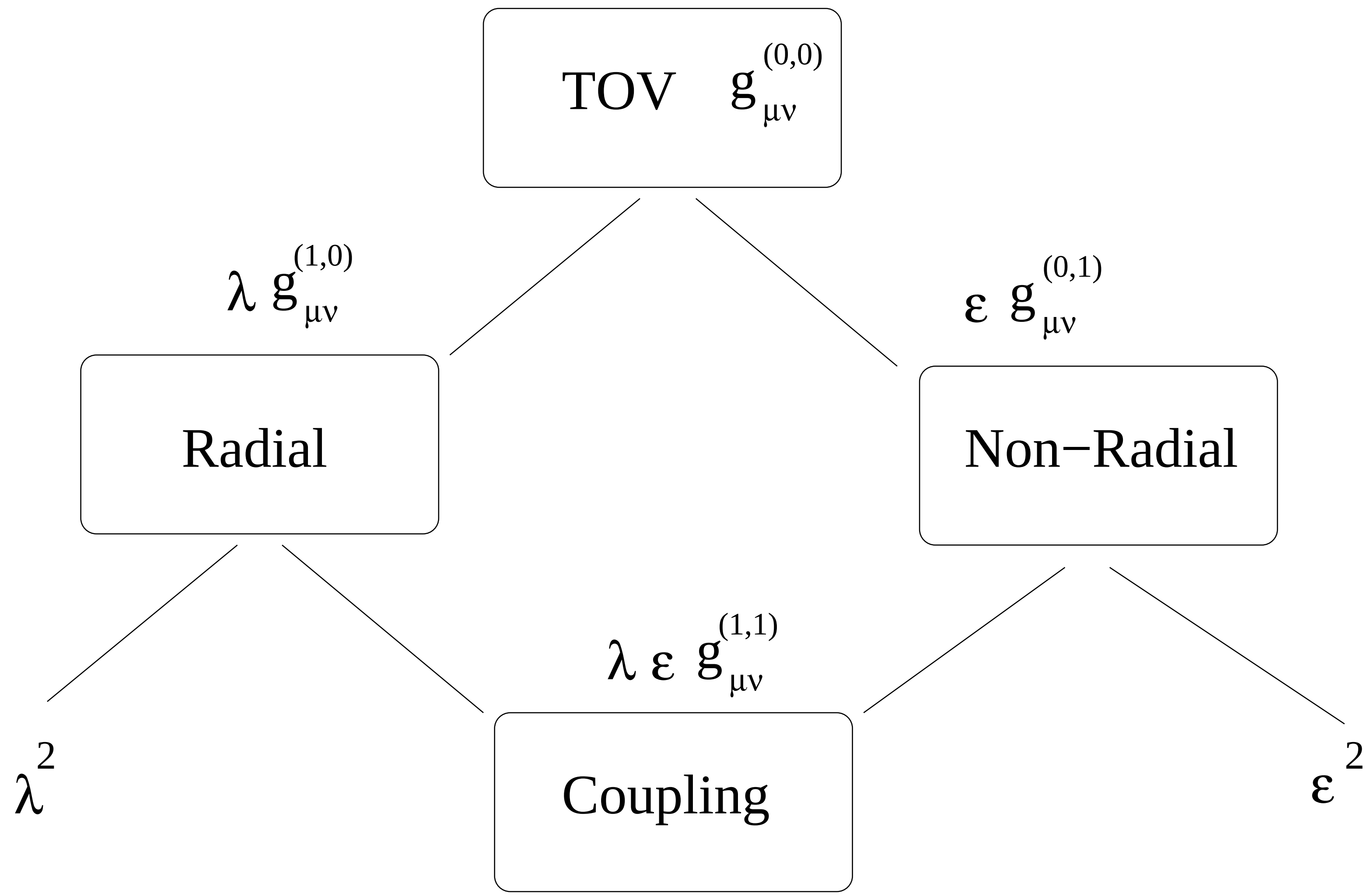
• Boundary Conditions for radial perturbations

Origin: regularity of perturbation fields and perturbative equations.

Surface $\quad \Delta p = 0 \quad \Longrightarrow \quad (\bar{\rho} + \bar{p})\, \bar{c}_s^2\, e^{-\Phi} \left(r^2 e^{-\Lambda} \gamma^{(1,0)} \right)_{,r} = 0$

• Numerical Radial Code:

- Warning: sensibility of radial perturbative equations to the vanishing of c_s on the surface. Convergence problem when the wave packet touches the surface.
- Coordinate transformation: *fluid tortoise coordinate* (Ruoff 2000, Sperhake gr-qc/0201086) $dr = c_s\, dx \quad \Longrightarrow$ refinement of grid close the surface.
- **Numerical algorithms:** *Mac-Cormack*, tridiagonal, trapezoidal formula, RK4.
- Second order convergence with J = 800 in the x-grid, Cf = 0.99 $\longrightarrow$ interpolation to a uniform grid of 400 points for transfer the values to the source terms.
- Initial values: i) Gaussian profile for $\gamma^{(1,0)}$ and zero for the all the remaining variables. ii) Gaussian profile for $H^{(1,0)}$ and the appropriate values derived from the constraints for all the others perturbative variables.



TOV $g_{\mu\nu}^{(0,0)}$
$\lambda\, g_{\mu\nu}^{(1,0)}$
Radial
$\epsilon\, g_{\mu\nu}^{(0,1)}$
Non−Radial
$\lambda\,\epsilon\, g_{\mu\nu}^{(1,1)}$
Coupling
λ^2
ϵ^2

Non-radial Perturbations

Thorne and Campolattaro (1967)

Non-spherically symmetric oscillations of metric and fluid fields $\Longrightarrow$ emission of gravitational waves.

• Frequency domain: Quasi normal mode (QNM) analysis leads to a detailed classification of polar and axial modes of NS:

$\star$ **Fluid modes**: They have a Newtonian counterpart

Mode	**Restoring force**	ν [kHz]	τ [s]
p	**pressure**	~ 5	$\sim$ **few**
g	**gravity-buoyancy**	~ 0.1	$\sim$ **few days**
f	**fundamental**	$\sim 2-3$	$\sim 0.1-0.5$

$\star$ **w-modes**: spacetime modes. Purely relativistic. (Kokkotas & Schutz, 1992)

$\nu \sim 6 - 12KHz, \qquad \tau \sim 0.1ms.$

- *trapped modes* (Chandrasekhar & Ferrari) in ultra-compact stars ($M/R \sim 0.44$).
- w_{II} *modes* (Leins et al. 1993), low ν and extremely fast damping times.

• Time Domain: Initial value formulation Kind et al.(1993). Polar time evolutions using different system of equations in Allen et al. (1998), Ruoff (2000), Nagar et al. (2004). We use the hyperbolic-elliptic system used by Nagar et al. (2004) $\longrightarrow$ stability and hamiltonian constraint used to determine one of the metric variable at every time-slice.

$\star$ Polar case: *Interior perturbative equations*

$$
\begin{aligned}
&-\ddot{\chi} + \chi'' + \ldots. && \text{Gravitational Wave}\\
&-\ddot{H} + c_s^2 H'' + \ldots. && \text{Sound Wave}\\
&k'' + 8\pi\frac{\rho+p}{c_s^2} H'' + \ldots. && \text{Hamiltonian constraint}
\end{aligned}
$$

Exterior: Zerilli formulation.

⋆ Axial case

$$\text{metric} \qquad \delta g^{ax}_{\mu\nu} = \begin{pmatrix} 0 & h^{lm}_A \, S^{lm}_a \\ & \\ h^{lm}_A \, S^{lm}_a & h \, (S^{lm}_{a:b} + S_{b:a}) \end{pmatrix}$$

$$\text{4-velocity} \qquad \delta u_A = 0 \qquad \delta u_\theta = 0 \qquad \delta u_\phi \sim \beta \, S^{lm}_\phi$$

$$\{h_0, h_1, h\} \longrightarrow \Pi \longrightarrow \Psi \equiv r^{-3} \, \Pi$$

Interior perturbative equations

$$- \Psi^{(0,1)}_{,tt} + \Psi^{(0,1)}_{,r^* r^*} + V(r) \, \Psi^{(0,1)} + a_1 \, \beta^{(0,1)} + a_2 \, \beta^{(0,1)}_{,r} = 0 \qquad \text{Gravitational Wave}$$

$$\beta^{(0,1)}_{,t} = 0 \qquad \text{Stationary differential horizontal fluid motion}$$

Exterior: Regge-Wheeler formulation.

- Numerical Non-Radial Axial Code:
  - **Numerical algorithms:** *Leapfrog*.
  - Boundary conditions: regularity at origin, outgoing boundary conditions at infinity, Junction conditions on the surface.

- Initial values: we can impose the initial values of the following quantities: $\Psi^{(0,1)}$, $\Psi^{(0,1)}_{,t}$, $\beta^{(0,1)}$.

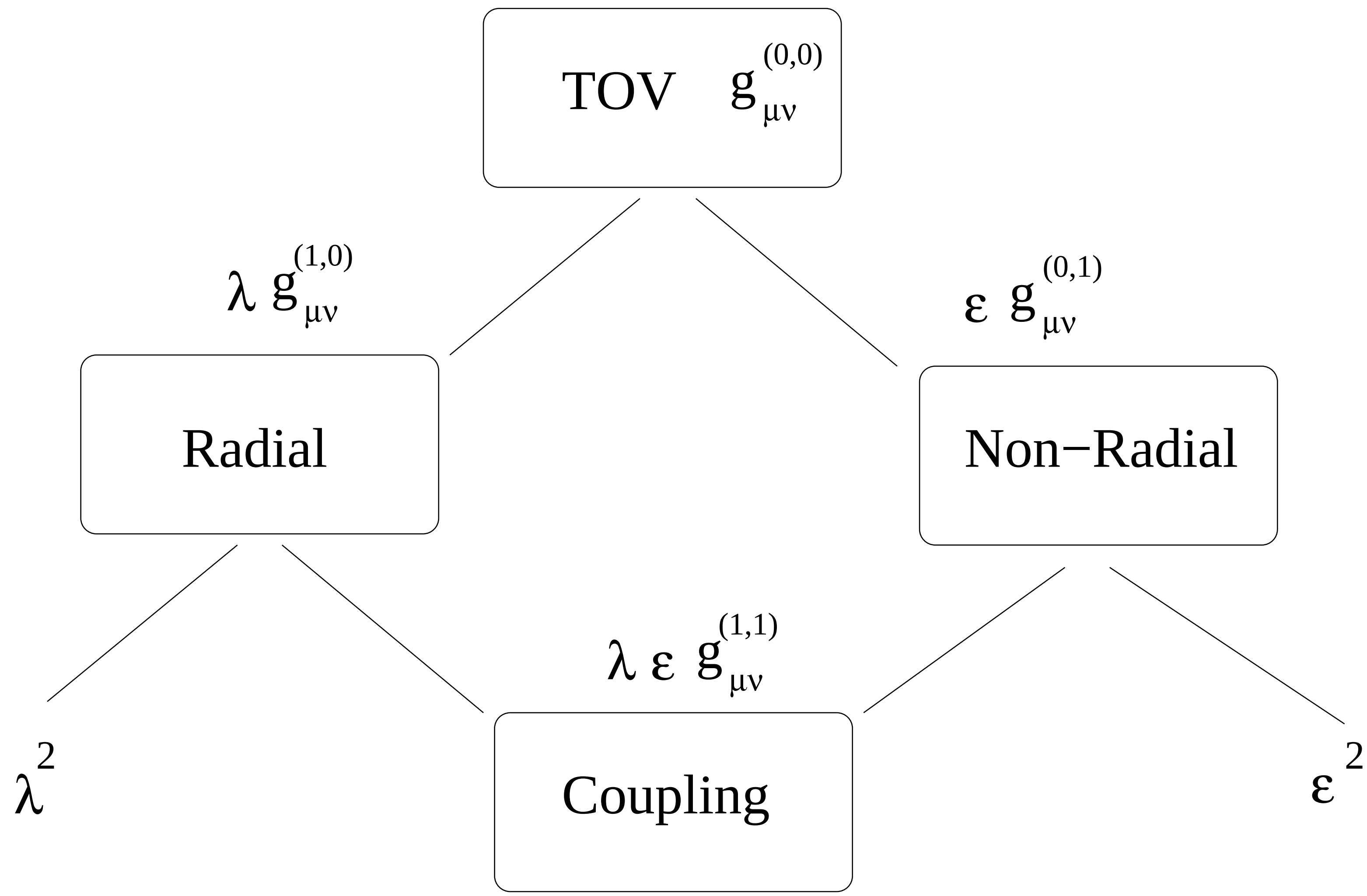
  a) Scattering of a gravitational wave on a static non-perturbed stars.

  b) Scattering of a gravitational wave on a static star but with the presence of the velocity perturbation $\beta^{(0,1)} = \Omega\, r$ that induces an stationary differential rotation in the fluid.

  c) Presence of the only stationary differential rotation perturbation.

- The particular solution describes the frame dragging:

$$\omega(t,\theta) = -g_{0\phi}/g_{\phi\phi} = \left(h_0/r^2 \sin\theta\right) \partial_\theta P_l\left(\cos\theta\right)$$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Radial
Non−Radial
$\lambda\, \varepsilon\, g_{\mu\nu}^{(1,1)}$
λ^2
Coupling
ε^2

Non-linear Coupling Perturbations

Derivation of perturbative equations trough the application of 2-parameter perturbations to the GSGM formalism

GSGM: $$\mathrm{g}_{\mu\nu} = \mathrm{g}_{\mu\nu}^{(0)\,GS}(t,r) + \epsilon\delta\mathrm{g}_{\mu\nu}^{(1)\,GS}(t,r)$$

$$\Downarrow$$

$$\mathrm{g}_{\mu\nu} = \mathrm{g}_{\mu\nu}^{TOV}(r) + \lambda\,\delta\mathrm{g}_{\mu\nu}^{(1,0)}(t,r) + \epsilon\left(\delta\mathrm{g}_{\mu\nu}^{(0,1)}(t,r) + \lambda\,\mathrm{g}_{\mu\nu}^{(1,1)}(t,r)\right)$$

We have proved the gauge invariant character of the non-linear coupling perturbations provided that we fix a gauge for the radial perturbations.

• *Gauge invariance of the coupling perturbations*

First order gauge transformation:

$$\mathcal{T}_Y^{(0,1)} = \mathcal{T}_X^{(0,1)} + \pounds_{\xi_{(0,1)}} \mathcal{T}_0$$

Second order gauge transformation for Coupling: Gauge transformation of coupling perturbations:

$$\mathcal{T}_Y^{(1,1)} = \mathcal{T}_X^{(1,1)} + \pounds_{\xi_{(0,1)}} \mathcal{T}^{(1,0)} + \pounds_{\xi_{(1,0)}} \mathcal{T}^{(0,1)} + \left(\pounds_{\xi_{(1,1)}} + 2\left\{\pounds_{\xi_{(1,0)}},\, \pounds_{\xi_{(0,1)}}\right\}\right) \mathcal{T}^{(0,0)}\,,$$

(C.Sopuerta,M.Bruni and L.Gualtieri 2003).

Fixing the gauge for radial perturbations:

$$\mathcal{T}^{(1,1)} = \pounds_{\xi_{(0,1)}} \mathcal{T}^{(1,0)} + \pounds_{\xi_{(1,1)}} \mathcal{T}^{(0,0)}$$

Expansion of GSGM gauge invariant quantities using the 2-parameter perturbation theory:

$$\mathcal{G}^{(1)} = \mathcal{G}^{(0,1)} + \lambda \mathcal{G}^{(1,1)} \qquad \mathcal{G}^{(1,1)} = \mathcal{H}^{(1,1)} + \sum_\sigma \mathcal{I}_\sigma^{(1,0)} \mathcal{J}_\sigma^{(0,1)}\,,$$

Gauge invariant condition for the coupling perturbations

$$\mathcal{G}_Y^{(1,1)} - \mathcal{G}_X^{(1,1)} = \mathcal{H}_Y^{(1,1)} - \mathcal{H}_X^{(1,1)} + \sum_\sigma \mathcal{I}_\sigma^{(1,0)} \left(\mathcal{J}_{\sigma\,Y}^{(0,1)} - \mathcal{J}_{\sigma\,X}^{(0,1)}\right) \equiv 0$$

Coupling: Polar Perturbative equations

- ***Stellar interior***: hyperbolic-elliptic system

$$
\begin{array}{ll}
- S^{(1,1)}_{,tt} + e^{2(\Phi-\Lambda)} S^{(1,1)}_{,rr} + \ldots\ldots = \mathcal{S}_S \left[\mathcal{J}^{(1,0)} \otimes \mathcal{G}^{(1,0)} \right] & \textit{Gravitational wave} \\
- H^{(1,1)}_{,tt} + c_s^2\, e^{2(\Phi-\Lambda)} H^{(1,1)}_{,rr} + .. = \mathcal{S}_H \left[\mathcal{J}^{(1,0)} \otimes \mathcal{G}^{(1,0)} \right] & \textit{Sound Wave} \\
k^{(1,1)}_{,rr} + 8\,\pi\, \dfrac{\rho + p}{c_s^2} H^{(1,1)} + \ldots = \mathcal{S}_{ham} \left[\mathcal{J}^{(1,0)} \otimes \mathcal{G}^{(1,0)} \right] & \textit{Hamilt. constr.}
\end{array}
$$

- ***Stellar exterior:*** Schwarzschild spacetime and ripples of gravitational waves.

Birkhoff's Theorem $\Longrightarrow$ No radial oscillations $\Longrightarrow$ Vanishing of Source terms

$$
\begin{array}{ll}
-S^{(1,1)}_{,tt} + e^{2(\Phi-\Lambda)} S^{(1,1)}_{,rr} + \ldots\ldots = 0 & \textit{Gravitational Wave} \\
k^{(1,1)}_{,rr} + \dfrac{2}{r\bar{c}_s^2} \left(\Lambda_{,r} + \Phi_{,r}\right) H^{(1,1)} + \ldots = 0 & \textit{Hamilton. constraint}
\end{array}
$$

- ***Zerilli equation*** (Moncrief 1974, M.Garcia & Gundlach 2000)

$$-Z^{(1,1)}_{,tt} + e^{2(\Phi-\Lambda)}\, Z^{(1,1)}_{,rr} + \frac{M}{r^2}\, e^{2\,\Phi}\, Z^{(1,1)}_{,r} - V\left(r\right)\; Z^{(1,1)} = 0$$

$$Z^{(1,1)} = \frac{2\,r^2\,e^{-2\,\Phi}}{\left(l+2\right)\,\left(l-1\right)\,r + 6\,M}\,\left[r\,S^{(1,1)} + \left(\frac{l}{2}\,\left(l+1\right) + \frac{M}{r}\right)\,e^{2\,\Phi}\,k^{(1,1)} - r\,k^{(1,1)}_{,r}\right]$$

Gravitational wave energy at infinity (Cunnigham, Price and Moncrief 1978)

$$\frac{d\,E}{d\,t} = \frac{1}{64\,\pi}\sum_{l\,,m}\frac{\left(l+2\right)!}{\left(l-2\right)!}\,\left|\dot{Z}_{lm}\right|^2$$

Coupling: Axial Perturbative equations

Interior

Gravitational Wave

$$-\Psi^{(1,1)}_{,tt} + \Psi^{(1,1)}_{,r^* r^*} + V(r)\,\Psi^{(1,1)} + a_1\,\beta^{(1,1)} + a_2\,\beta^{(1,1)}_{,r} = \mathcal{S}_\Psi\left[\mathcal{J}^{(1,0)} \otimes \mathcal{G}^{(1,0)}\right]$$

Evolution equation

$$\beta^{(1,1)}_{,t} = \mathcal{S}_\beta\left[\mathcal{J}^{(1,0)} \otimes \mathcal{G}^{(1,0)}\right]$$

Exterior: Regge-Wheeler formulation.

$$-\Psi^{(1,1)}_{,tt} + \Psi^{(1,1)}_{,r^* r^*} + V_{RW}(r)\,\Psi^{(1,1)} = 0$$

Gravitational wave energy at infinity

$$\frac{d\,E}{d\,t} = \frac{1}{16\,\pi}\sum_{l\,,m}\frac{l\,(l+1)}{(l-1)\,(l+1)}\,|\dot{\Psi}_{lm}|^2$$

Coupling: Boundary Conditions

- Infinity: outgoing boundary condition.
- Origin: regularity of perturbation fields and equations.
- Surface: a) Junction conditions: continuity of first and second fundamental forms and their perturbations on the surface. Continuity conditions and Zerilli extraction formula (M.Garcia-Gundlach 2000). b) Dynamical Boundary condition for H from the Vanishing of Lagrangian perturbations on the surface,

$$\Delta p^{(1,1)} = \delta p^{(1,1)} + \left(\pounds_{\xi_{(1,1)}} + \left\{\pounds_{\xi_{(1,0)}}, \pounds_{\xi_{(0,1)}}\right\}\right) p_0$$

Warning:

Problem: first order pulsations induce the movement of stellar surface $\longrightarrow$ where do we fix the matching hypersurface?

\- Possible solutions (Sperhake gr-qc/0201086): i) addition of a stellar atmosphere (low density problem). ii) Implementation of dynamical boundary conditions following the first order surface movement. iii) Lagrangian formulation. iv) Consider a stellar model without the outer layers, in order to impose the J.C. always inside the star.

The interesting non-linear effect is given by the following term in the source,

$$\sim \delta\rho^{(1,0)}_{,r}\,\beta^{(0,1)}$$

Conclusions

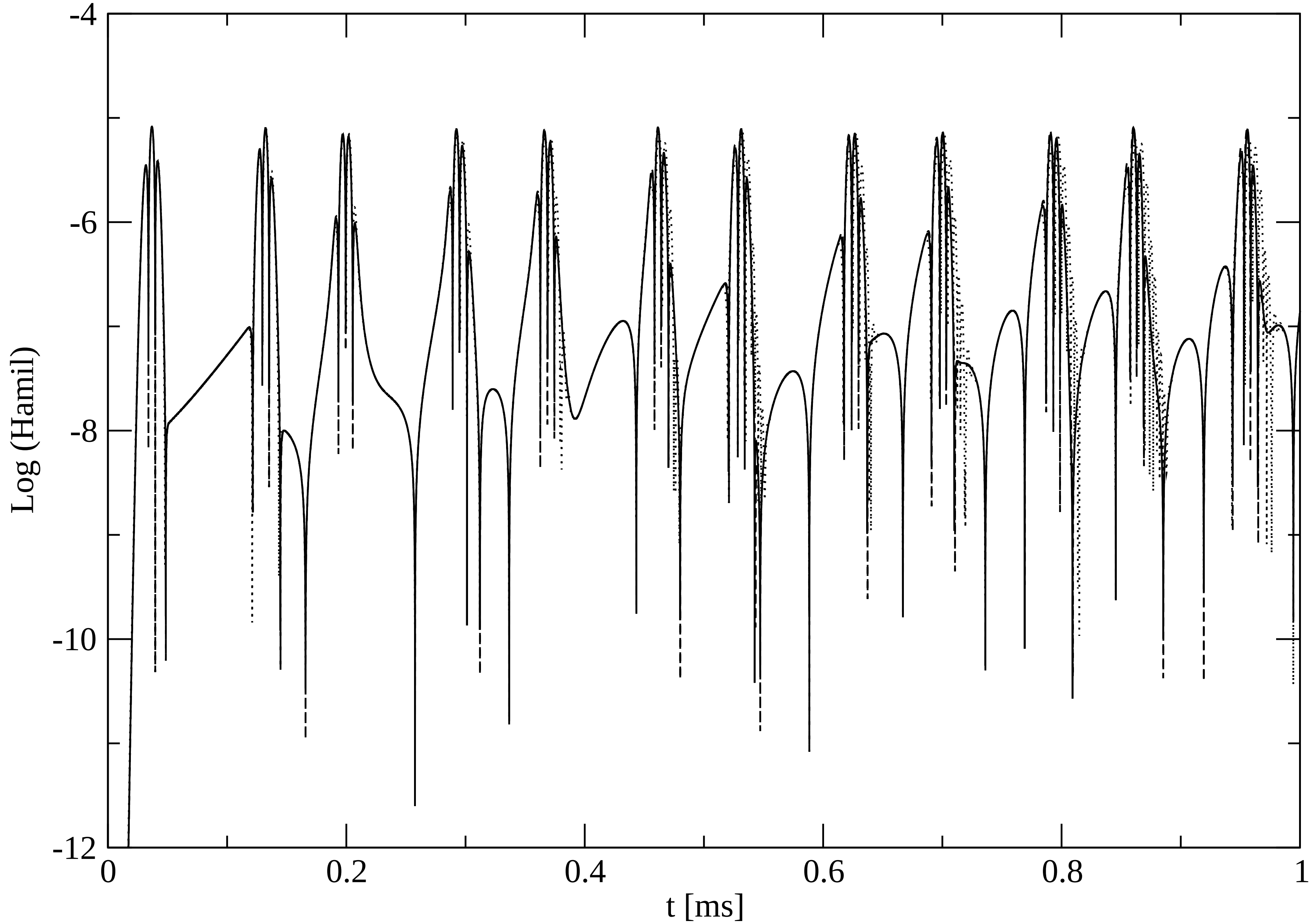
- In a general relativistic approach, we have set up a system of equations to study in the time domain the coupling of radial and polar and axial non-radial perturbations of relativistic stars, based on a 2-parameters perturbations theory and on the GSGM formalism.
- We have proved the gauge invariance of coupling variables by fixing the radial gauge.
- We have implemented a numerical code to study the axial coupling oscillations.
- We have found an interesting effects in the axial coupling gravitational signal, due to the coupling between the first order velocity perturbations (first order differential rotation) in the case $l = 2$ and the radial perturbations $l = 0$.

Future works:

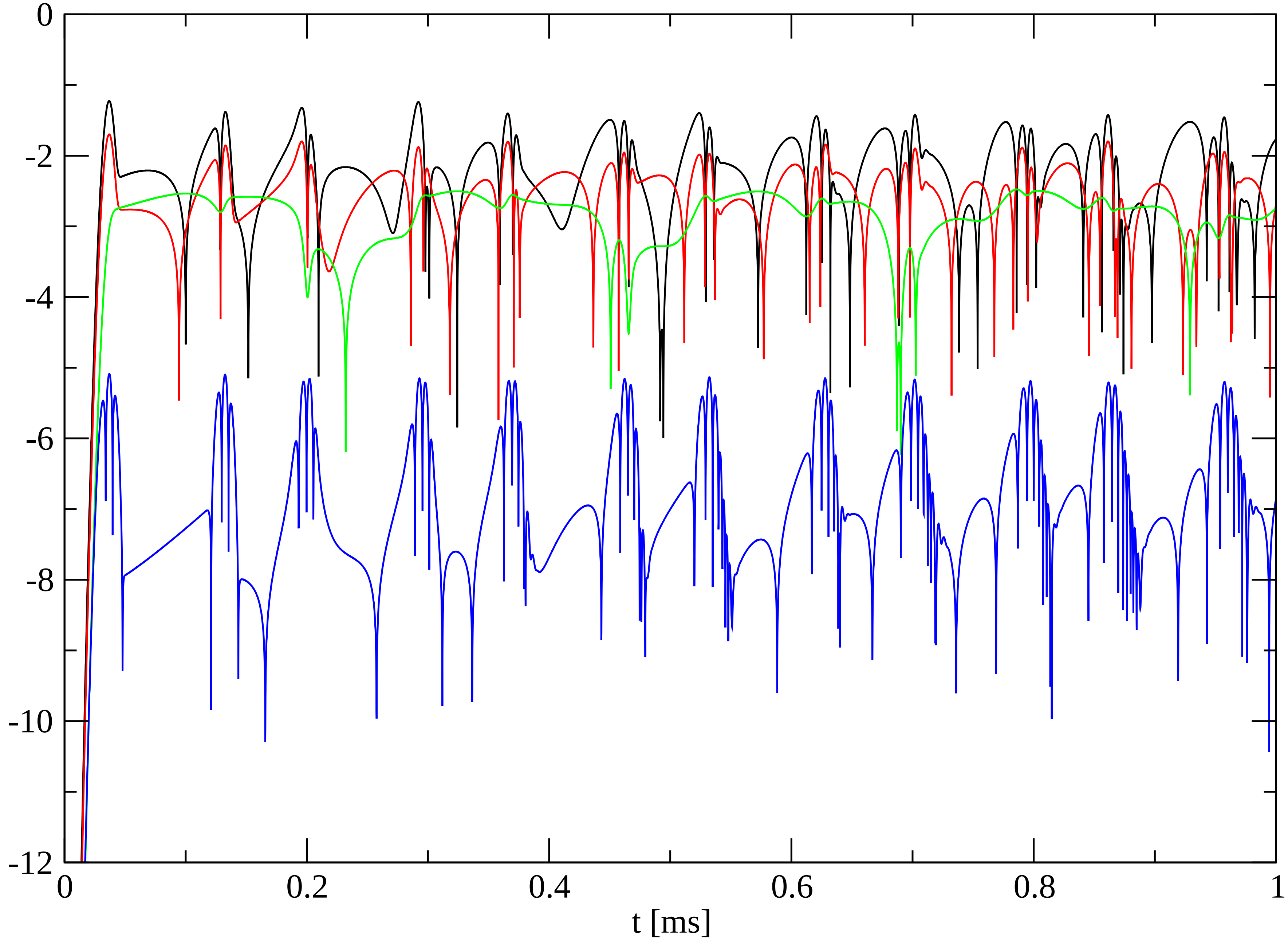
- Reach a better understanding of the matching conditions at second order.
- Analysis of the axial perturbation results and extraction of the physical quantities.
- Explore the evolution with different choices of initial data.

- Investigate the Back-reaction problem.
- Complete the implementation of the numerical code for the polar case.
- Consider the ϵ^2 perturbations, corresponding to the pure second order non-radial perturbations.

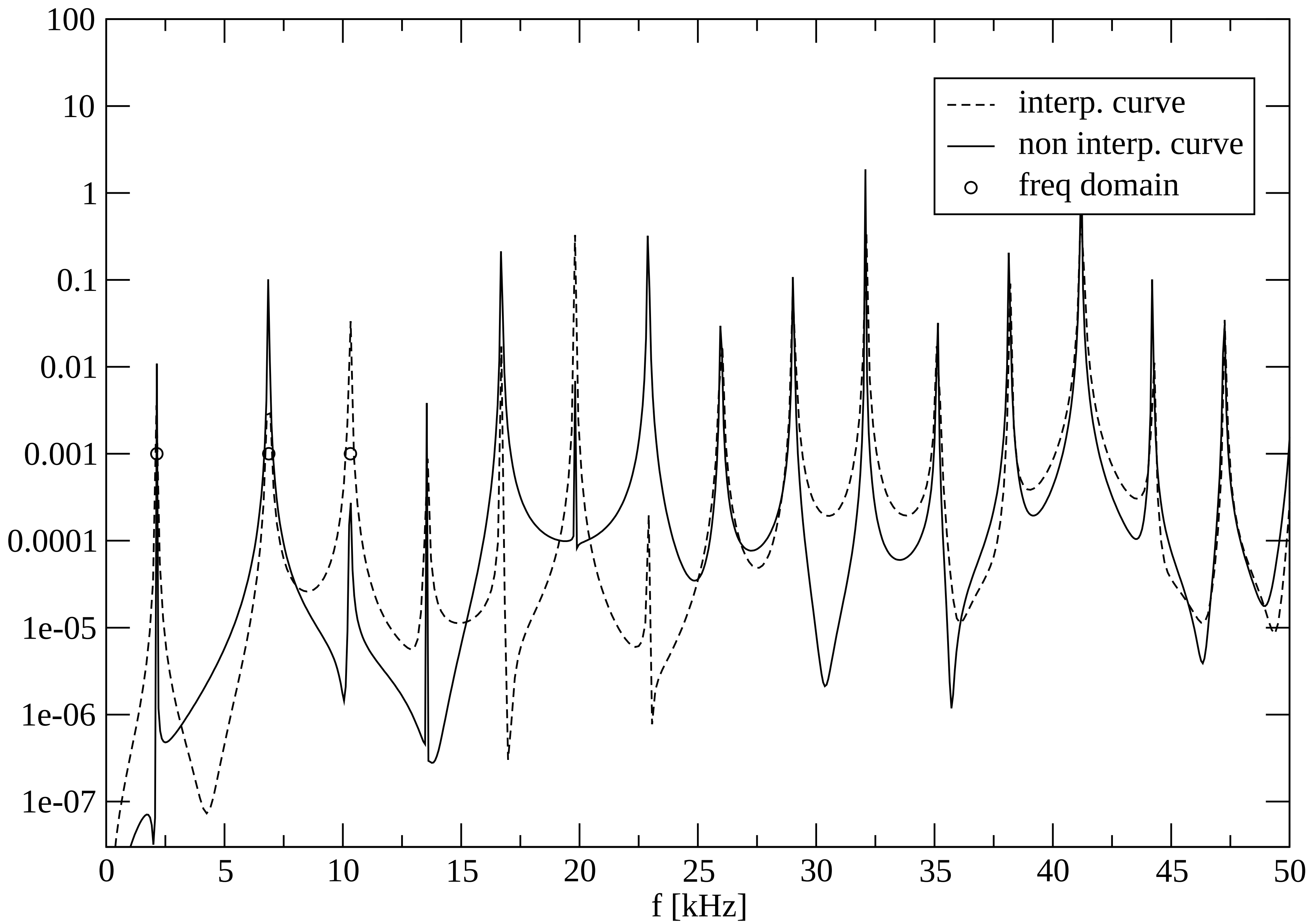
Coupling of Radial and Non-radial Perturbations of Relativistic Stars



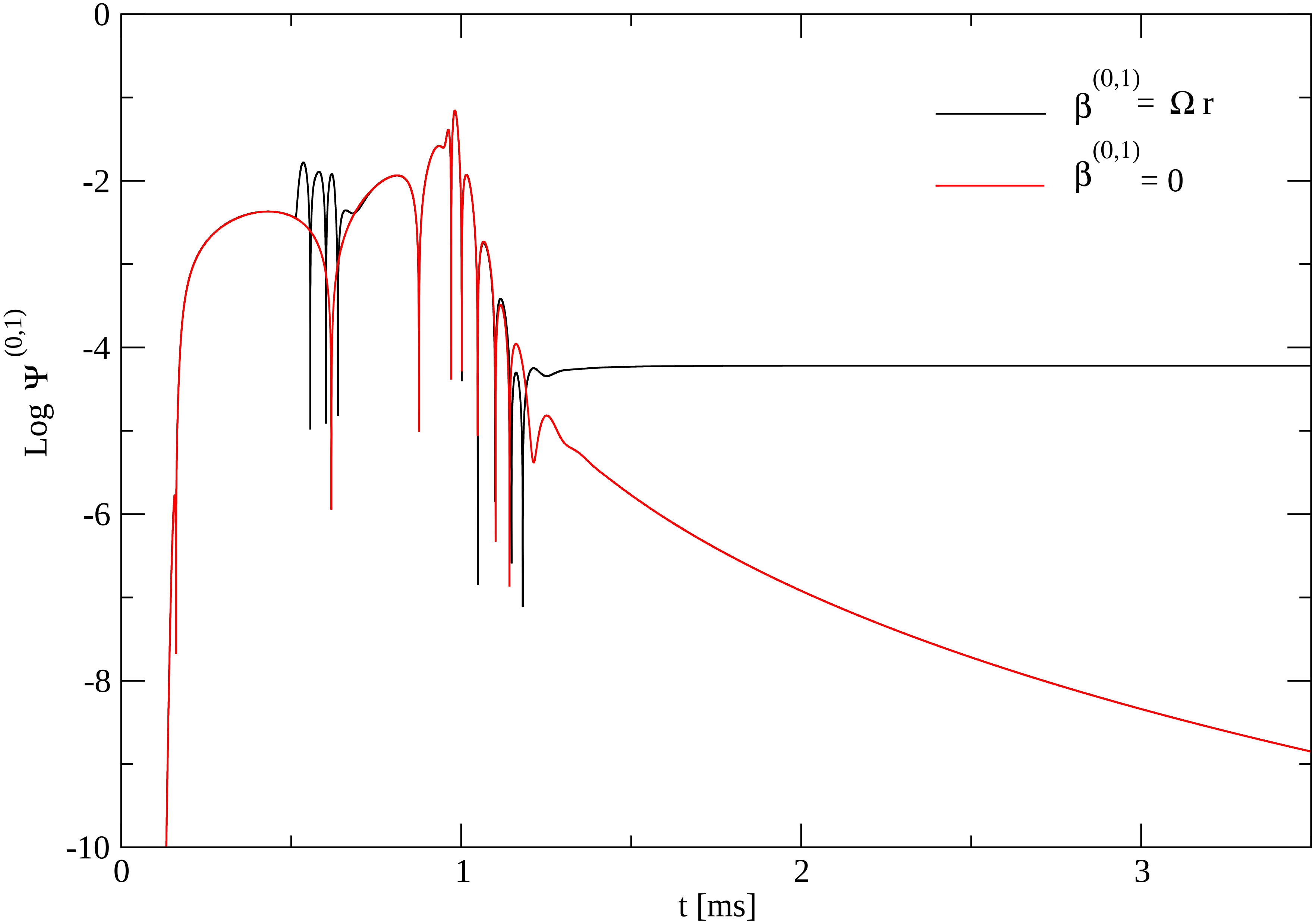
-4
-6
-8
-10
-12
Log (Hamil)
0
0.2
0.4
0.6
0.8
1
t [ms]



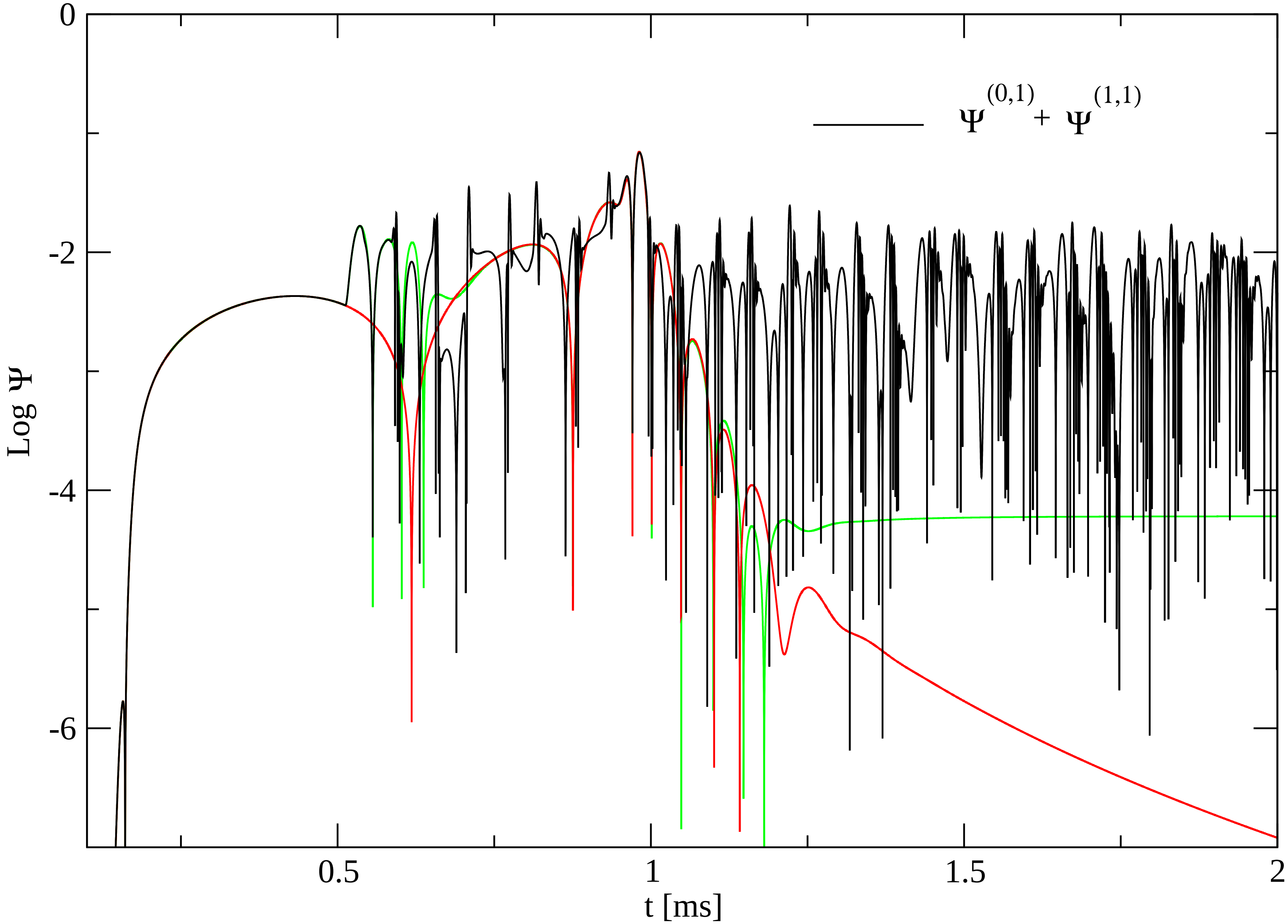
0
-2
-4
-6
-8
-10
-12
0
0.2
0.4
0.6
0.8
1
t [ms]



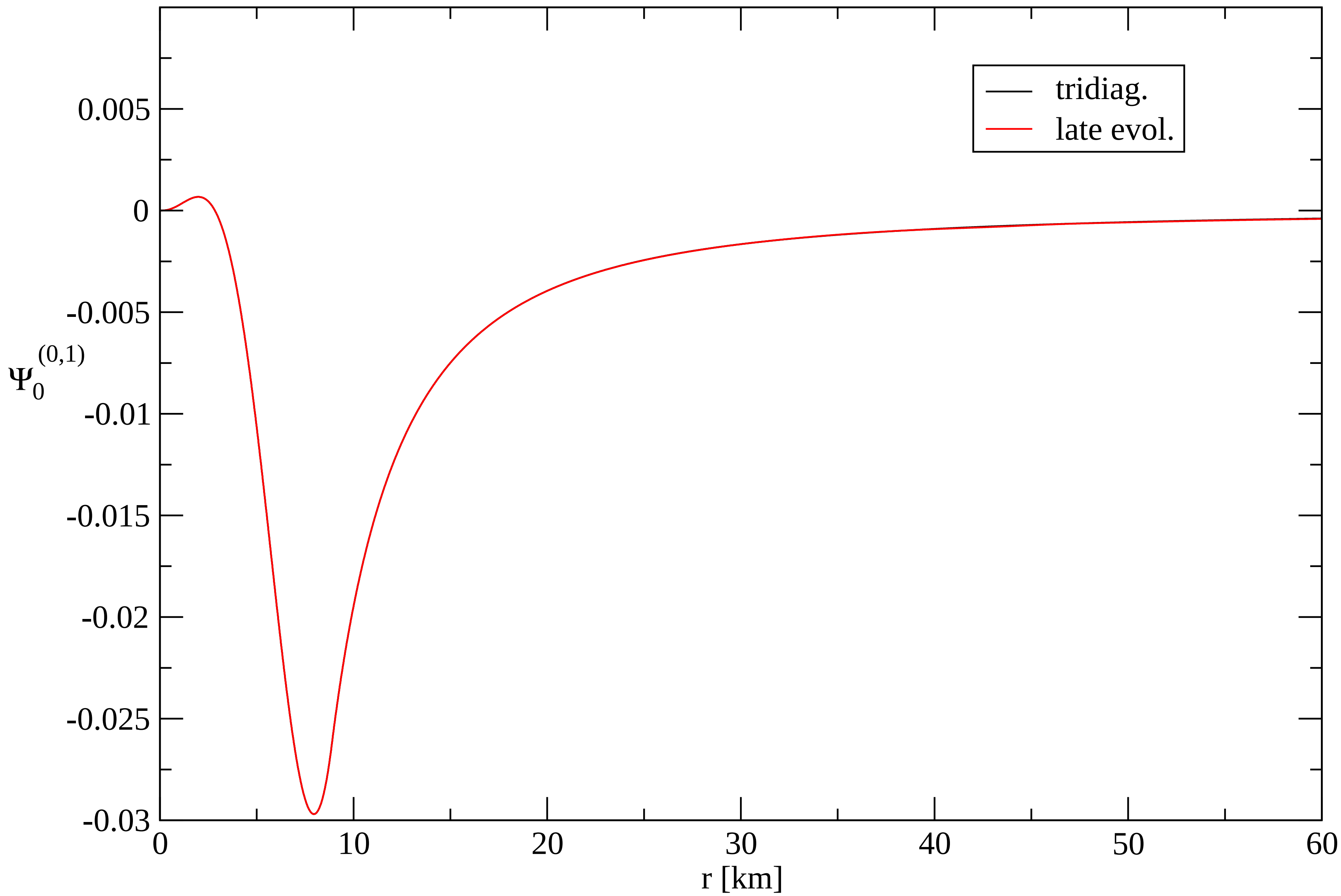
100
10
1
0.1
0.01
0.001
0.0001
1e-05
1e-06
1e-07
0
5
10
15
20
25
30
35
40
45
50
f [kHz]
interp. curve
non interp. curve
freq domain



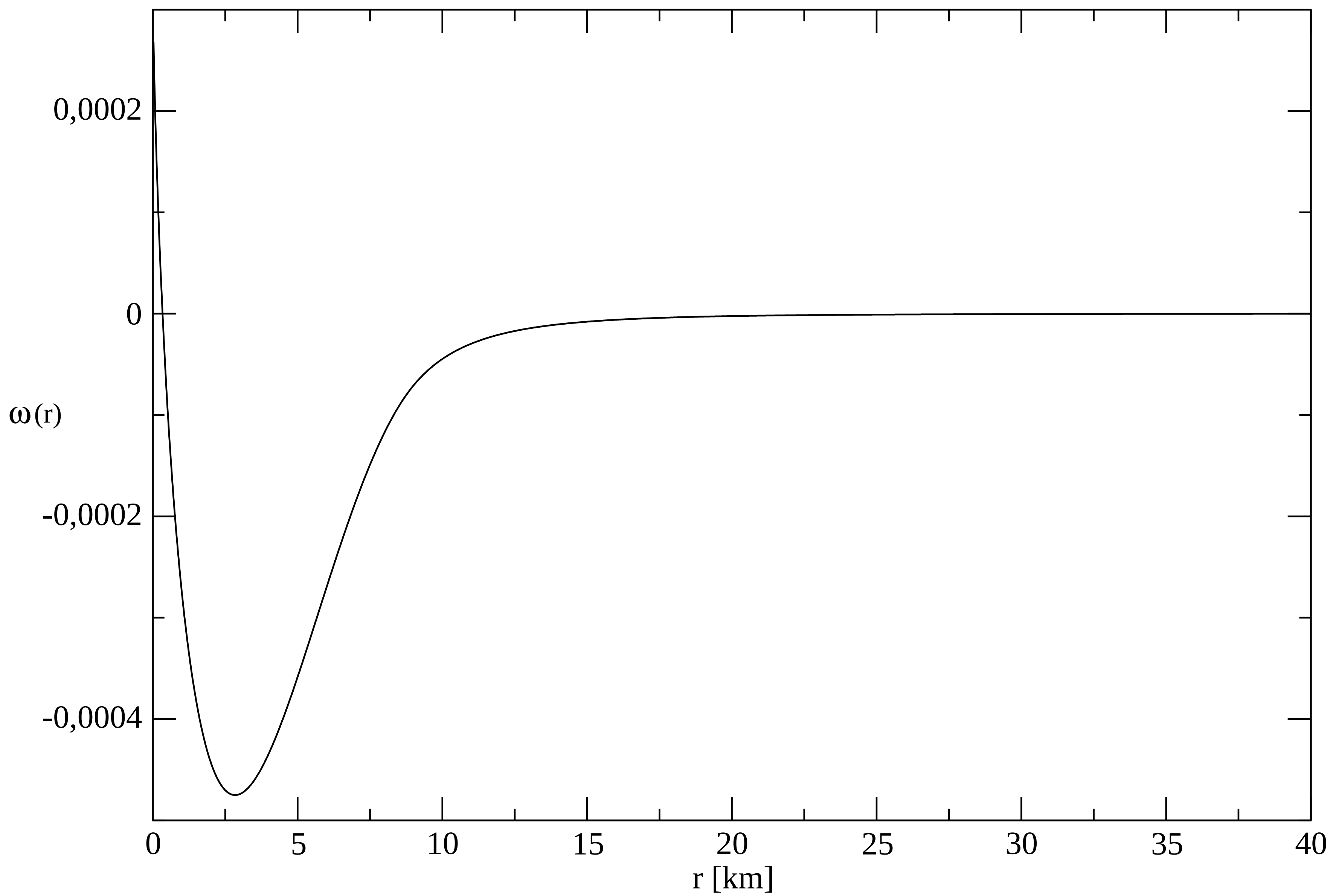
0
-2
-4
-6
-8
-10
0
1
2
3
t [ms]
Log Ψ^(0,1)
β^(0,1) = Ω r
β^(0,1) = 0

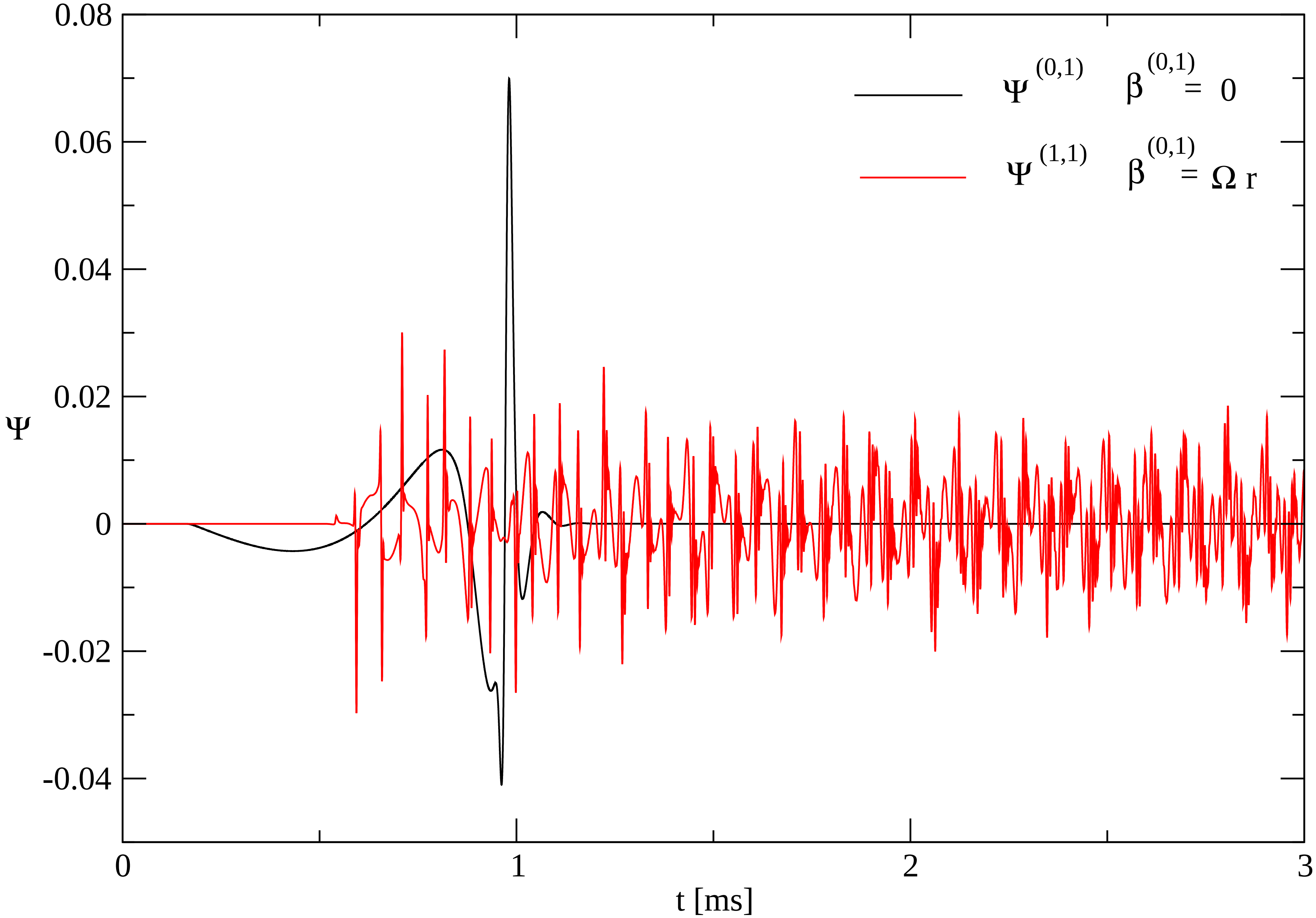


$\Psi^{(0,1)} + \Psi^{(1,1)}$
Log Ψ
t [ms]
0
-2
-4
-6
0.5
1
1.5
2

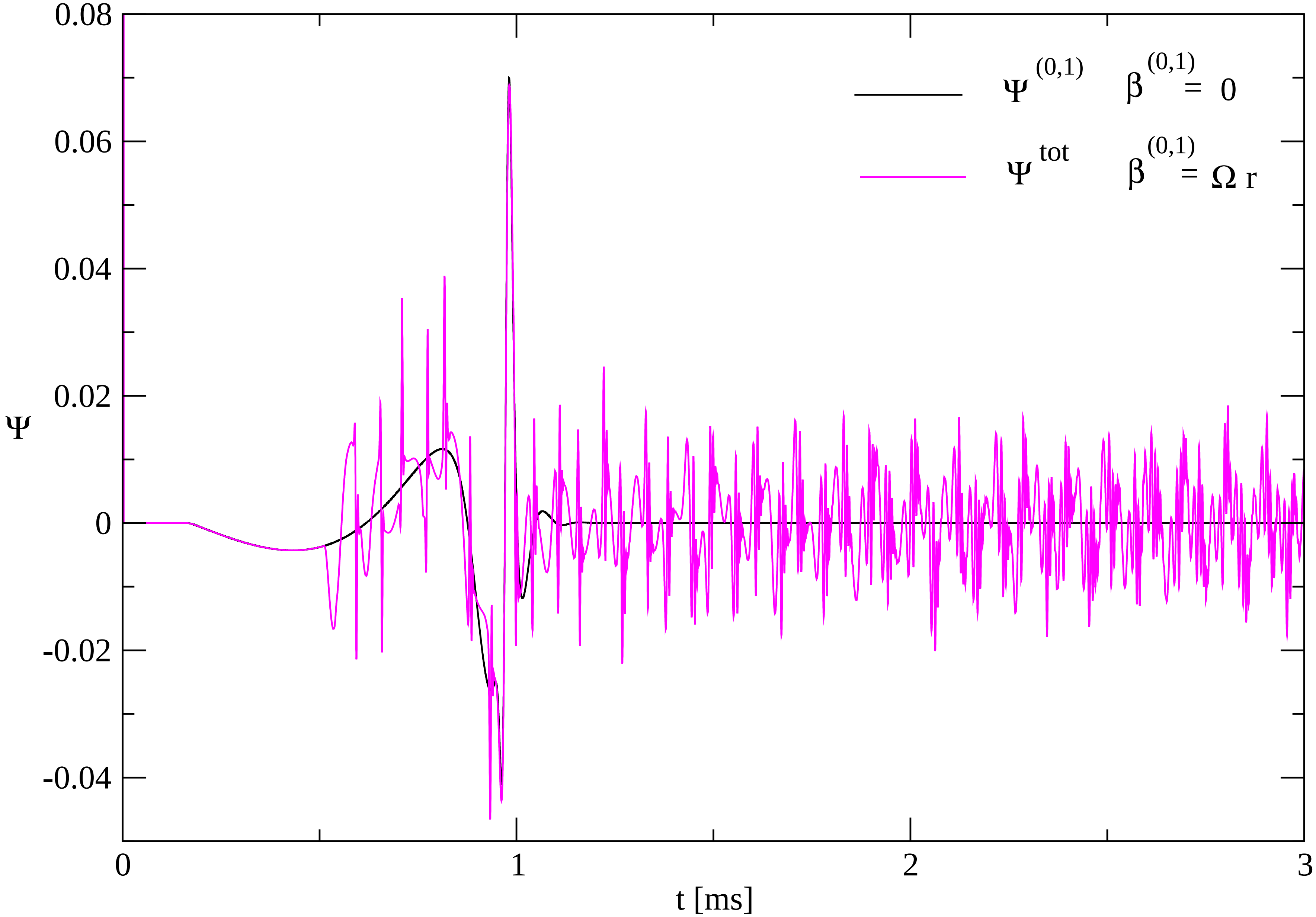


tridiag.
late evol.
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
$\Psi_0^{(0,1)}$
0
10
20
30
40
50
60
r [km]


0,0002
0
-0,0002
-0,0004
ω(r)
0
5
10
15
20
25
30
35
40
r [km]



0.08
0.06
0.04
0.02
0
-0.02
-0.04
Ψ
0
1
2
3
t [ms]
$\Psi^{(0,1)}$ $\beta^{(0,1)} = 0$
$\Psi^{(1,1)}$ $\beta^{(0,1)} = \Omega r$



0.08
0.06
0.04
0.02
0
-0.02
-0.04
Ψ
0
1
2
3
t [ms]
Ψ (0,1) β (0,1) = 0
Ψ tot β (0,1) = Ω r