Smaller Alignment Index (SALI): An efficient method of Chaos detection

Charalampos (Haris) Skokos

Research Center for Astronomy and Applied Mathematics,
Academy of Athens, Athens, Greece
and
Center for Research and Applications of Nonlinear Systems (CRANS),
University of Patras, Patras, Greece

**Work in collaboration with
Chris Antonopoulos, Thanos Manos,
Tassos Bountis, Michael Vrahatis**

E-mail: hskokos@cc.uoa.gr

URL: http://www.math.upatras.gr/~skokos/

Outline

- Definition of the SALI
- Applications
  - ✓ 2D map
  - ✓ 4D map
  - ✓ 4D map – A case of weak Chaos
  - ✓ 2D Hamiltonian (Hénon-Heiles system)
  - ✓ 3D Hamiltonian
- Behavior of the SALI
  - ✓ Ordered orbits
  - ✓ Chaotic orbits

Definition of the Smaller Alignment Index (SALI)

Consider the n-dimensional phase space of a conservative dynamical system (a symplectic map or a Hamiltonian flow).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a deviation vector

$$v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of a deviation vector is defined by:

- the variational equations (for Hamiltonian flows) and
- the equations of the tangent map (for mappings)

Definition of the SALI

We follow the evolution in time of two different initial deviation vectors (e.g. $v_1(0)$, $v_2(0)$), and define SALI (Skokos Ch., 2001, J. Phys. A, 34, 10029) as:

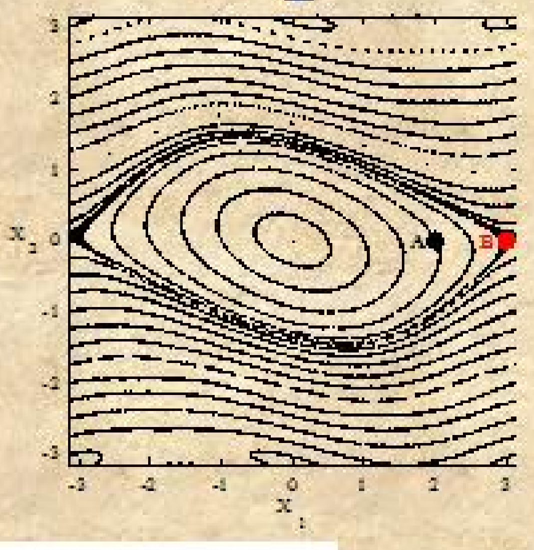
$$\mathrm{SALI}(t) = \min\left\{ \left\| \frac{v_1(t)}{\|v_1(t)\|} + \frac{v_2(t)}{\|v_2(t)\|} \right\|, \left\| \frac{v_1(t)}{\|v_1(t)\|} - \frac{v_2(t)}{\|v_2(t)\|} \right\| \right\}$$

When the two vectors tend to coincide or become opposite

$$\mathrm{SALI}(t) \rightarrow 0$$

Applications – 2D map

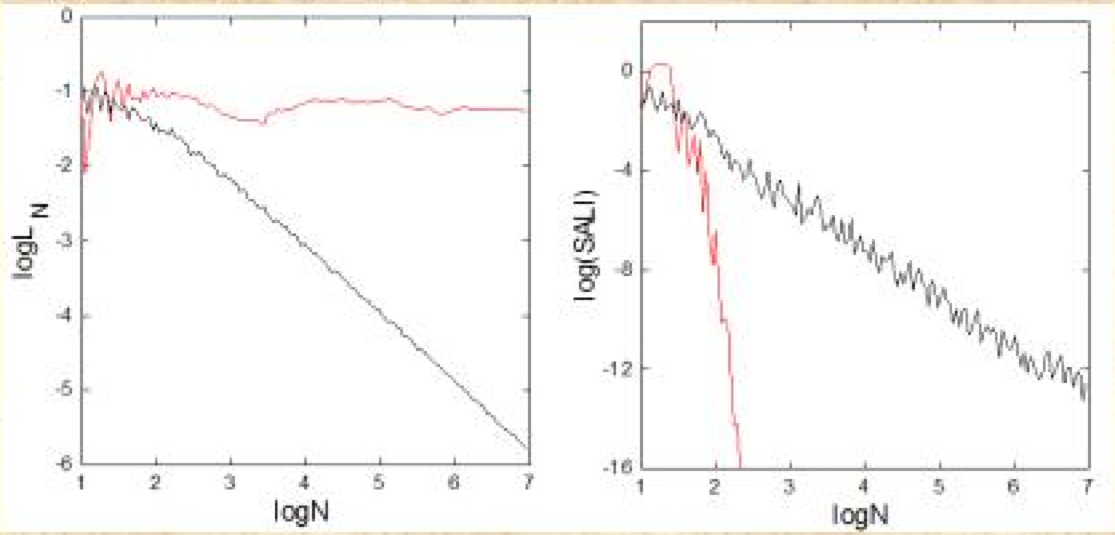
$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \sin(x_1 + x_2) \end{aligned} \qquad (\text{mod } 2\pi)$$



For ν=0.5 we consider the orbits:

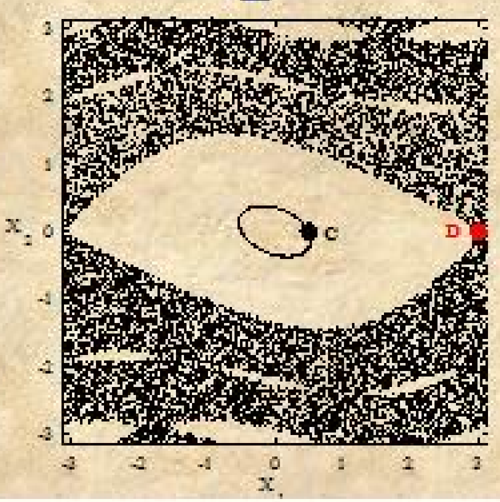
ordered orbit A *with initial conditions $x_1=2$, $x_2=0$.*

chaotic orbit B *with initial conditions $x_1=3$, $x_2=0$.*



Applications – 4D map

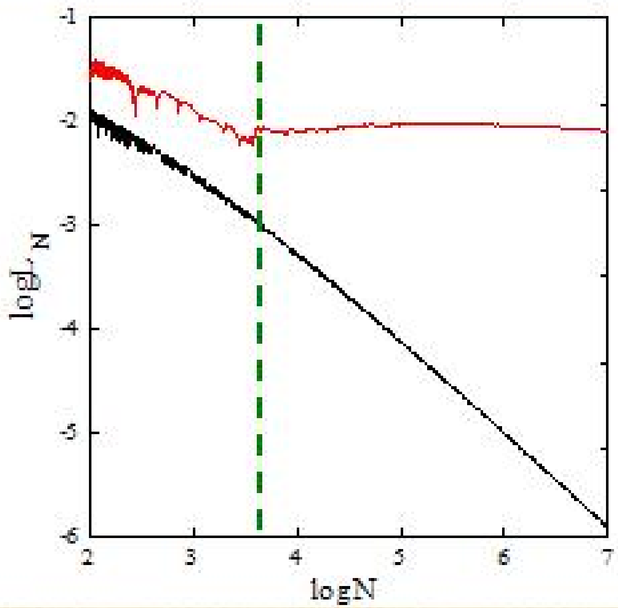
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\qquad (\text{mod } 2\pi)
$$

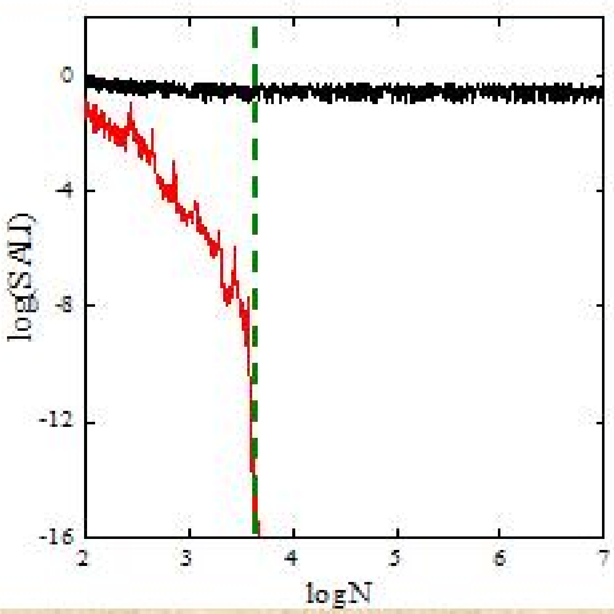


For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

ordered orbit C *with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.*

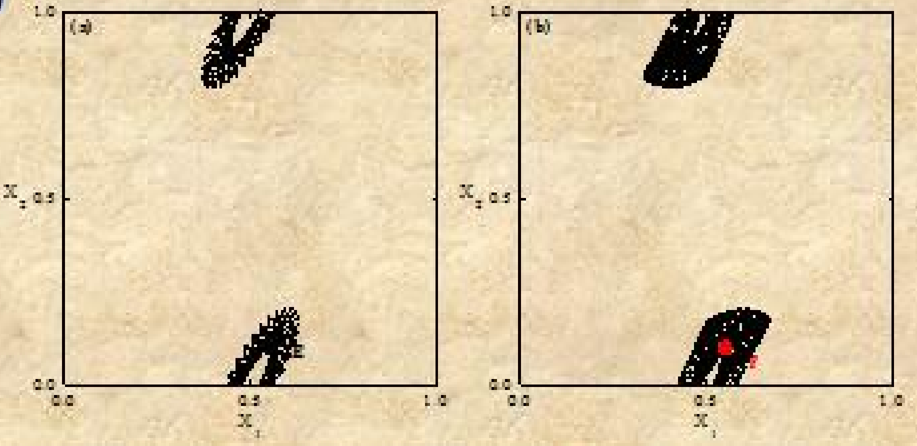
chaotic orbit D *with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.*





Applications – 4D map (weak chaos)

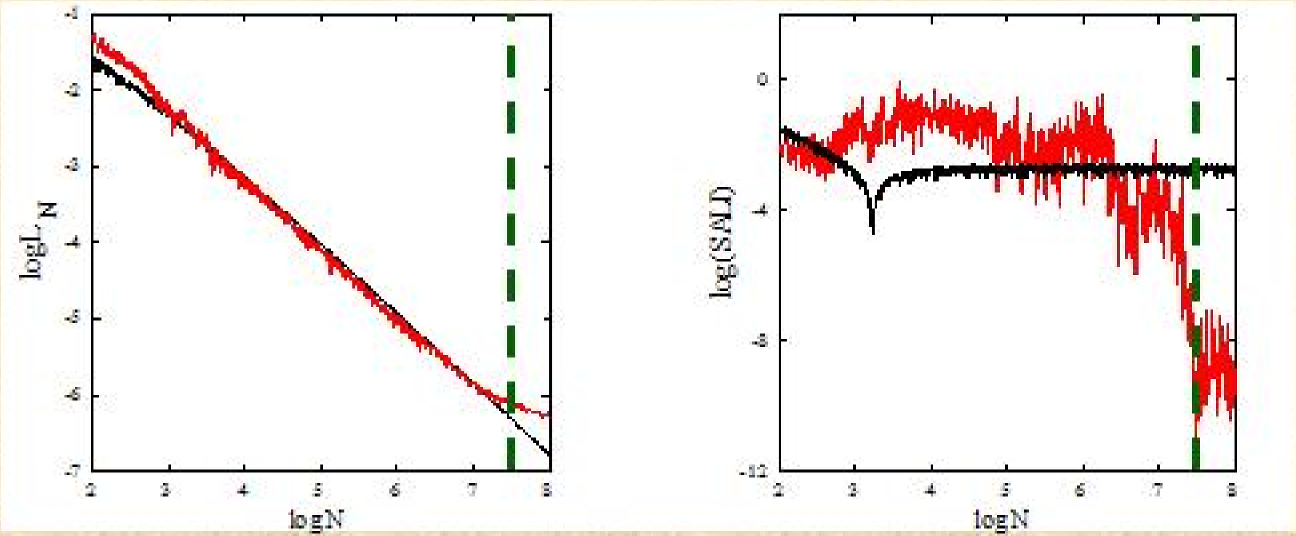
$$
\begin{aligned}
x_1' &= x_1 + x_2' \\
x_2' &= x_2 + \tfrac{K}{2\pi}\sin(2\pi x_1) - \tfrac{\beta}{\pi}\sin[2\pi(x_3 - x_1)] \\
x_3' &= x_3 + x_4' \\
x_4' &= x_4 + \tfrac{K}{2\pi}\sin(2\pi x_3) - \tfrac{\beta}{\pi}\sin[2\pi(x_1 - x_3)]
\end{aligned}
\qquad (\text{mod } 1)
$$



For K=3 we consider two orbits with the same initial conditions

ordered orbit E with initial conditions x1=0.55, x2=0.1, x3=0.62, x4=0.2 for β=0.1.

chaotic orbit F with initial conditions x1=0.55, x2=0.1, x3=0.62, x4=0.2 for β=0.3051.



Applications – Hénon-Heiles system

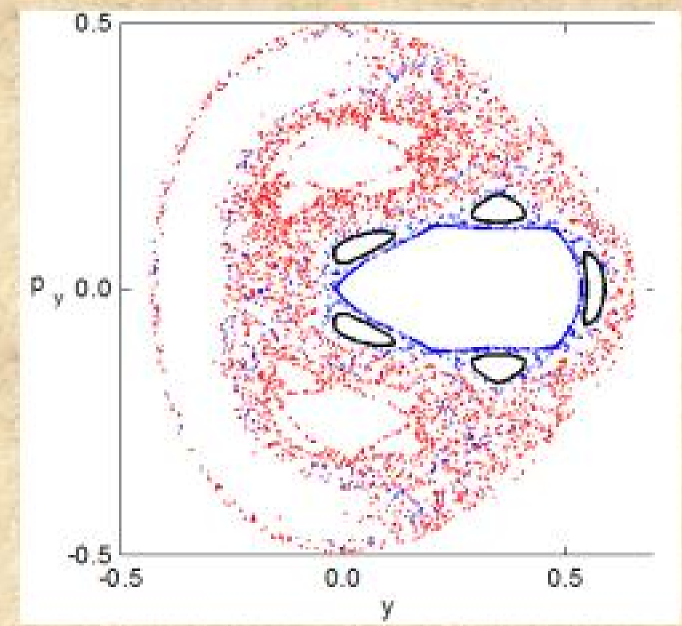
$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

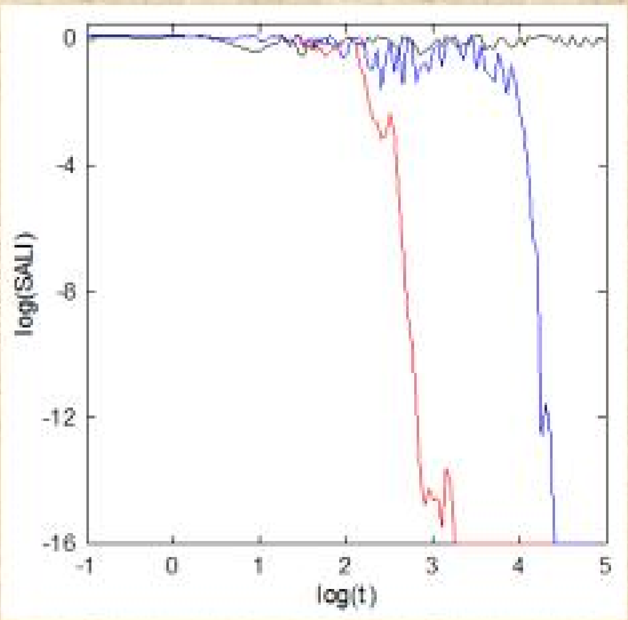
For E=1/8 we consider the orbits with initial conditions:

Ordered orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

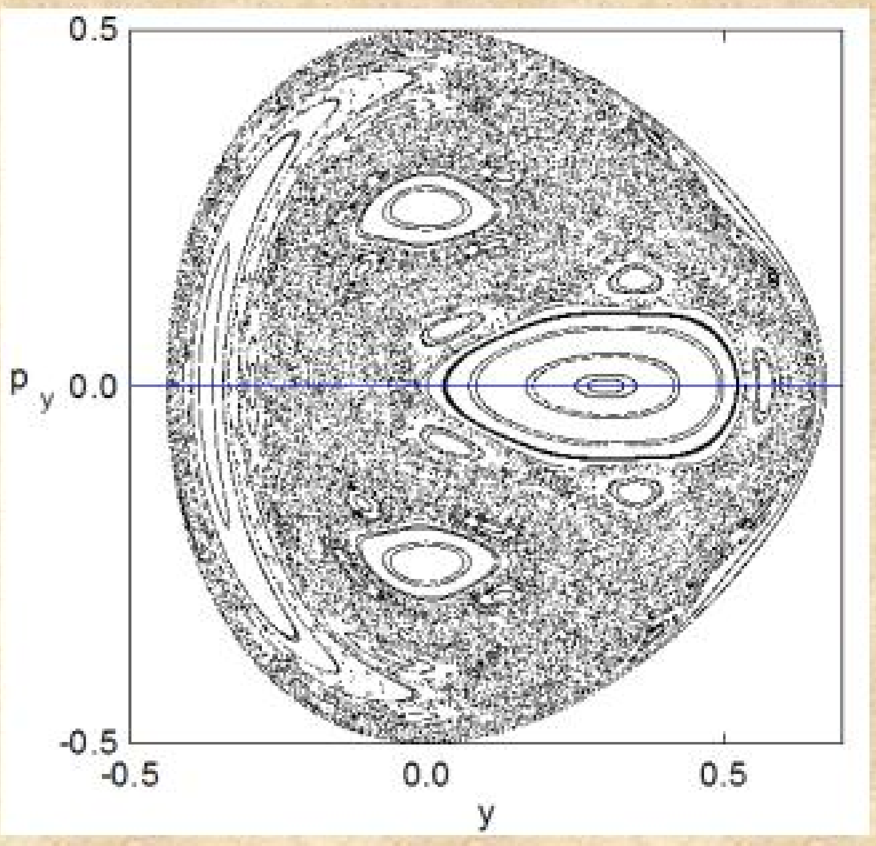
Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

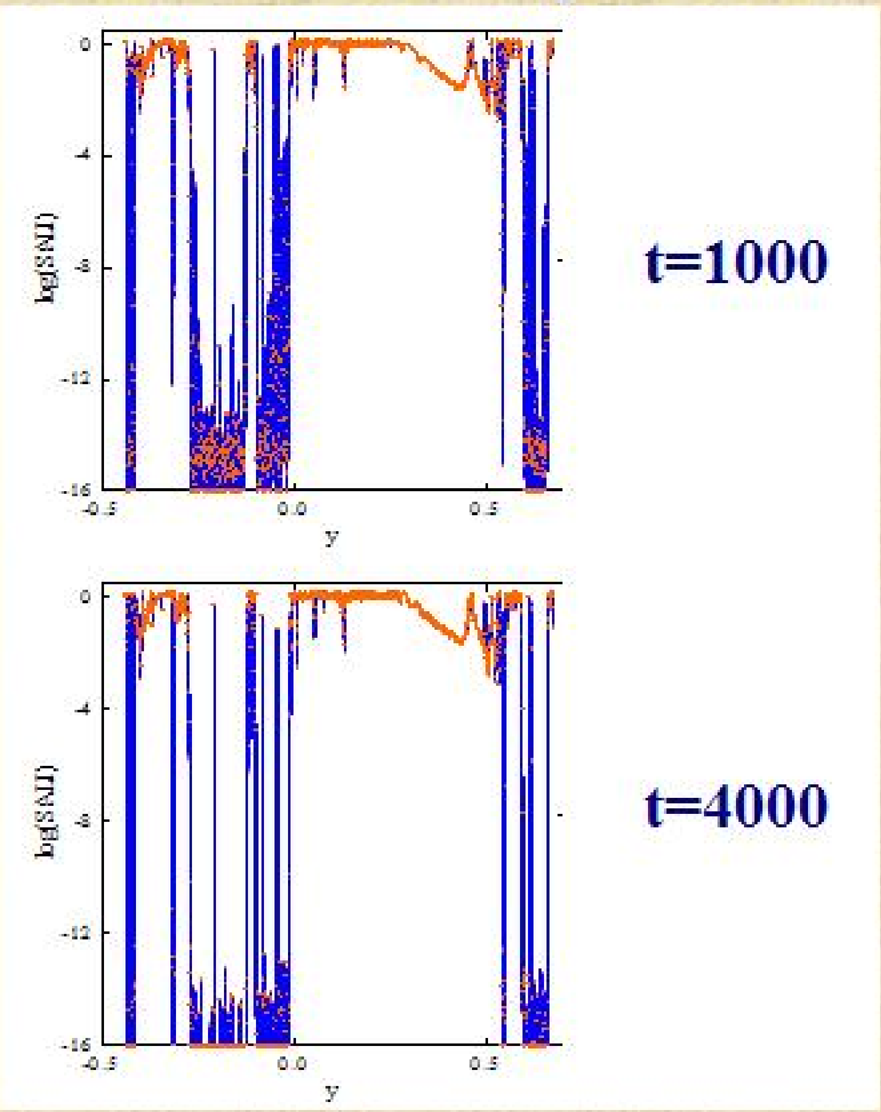
Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



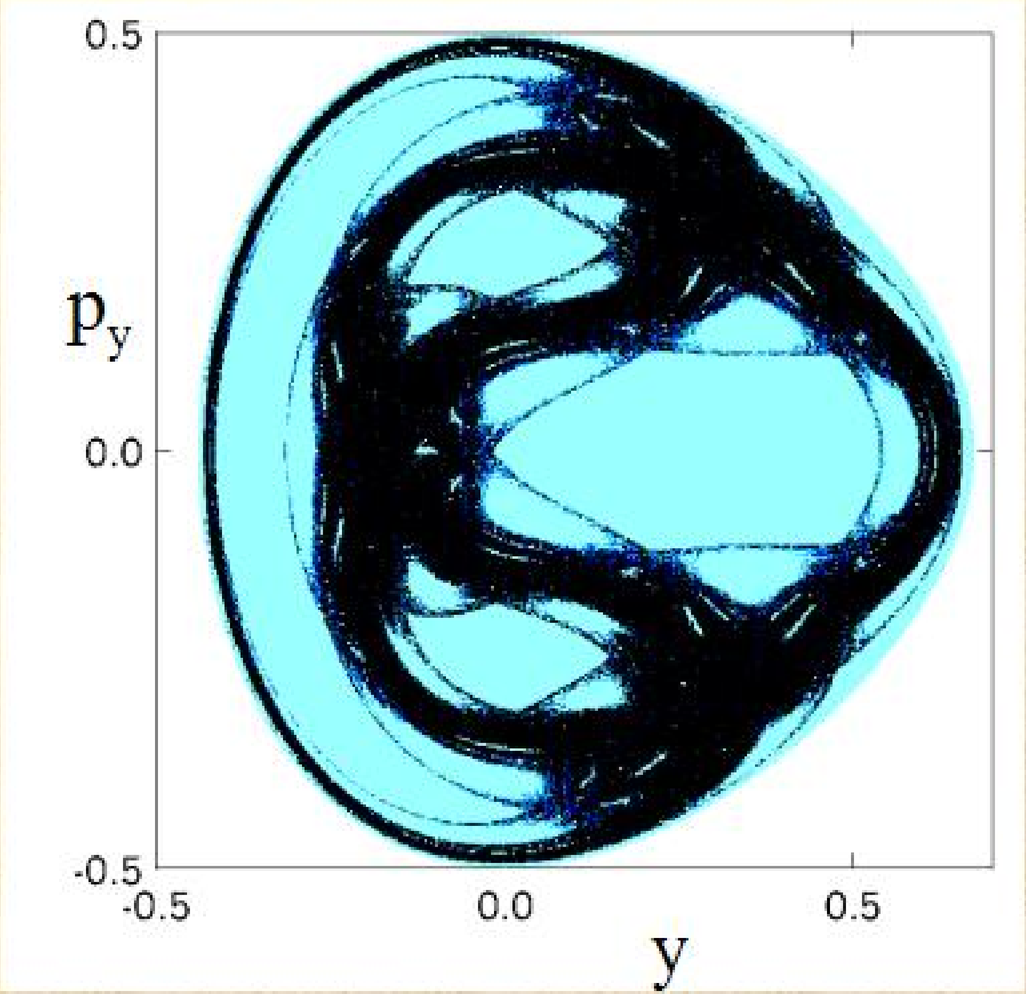


Applications – Hénon-Heiles system



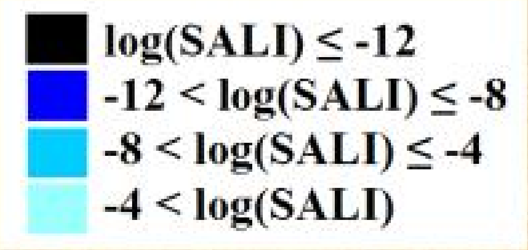


Applications – Hénon-Heiles system



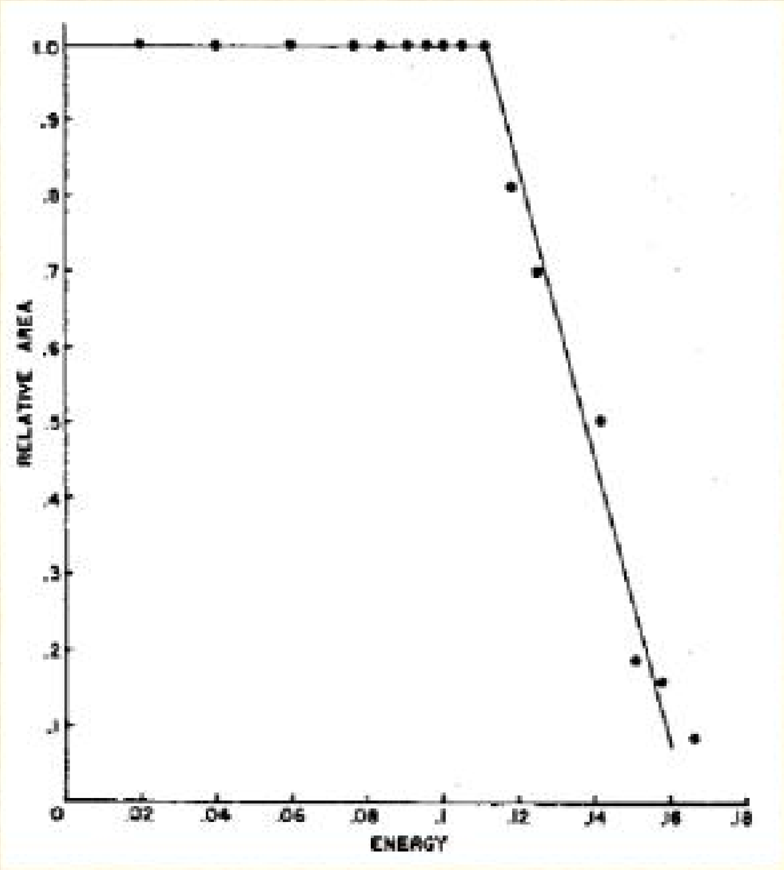
E=1/8

t=1000

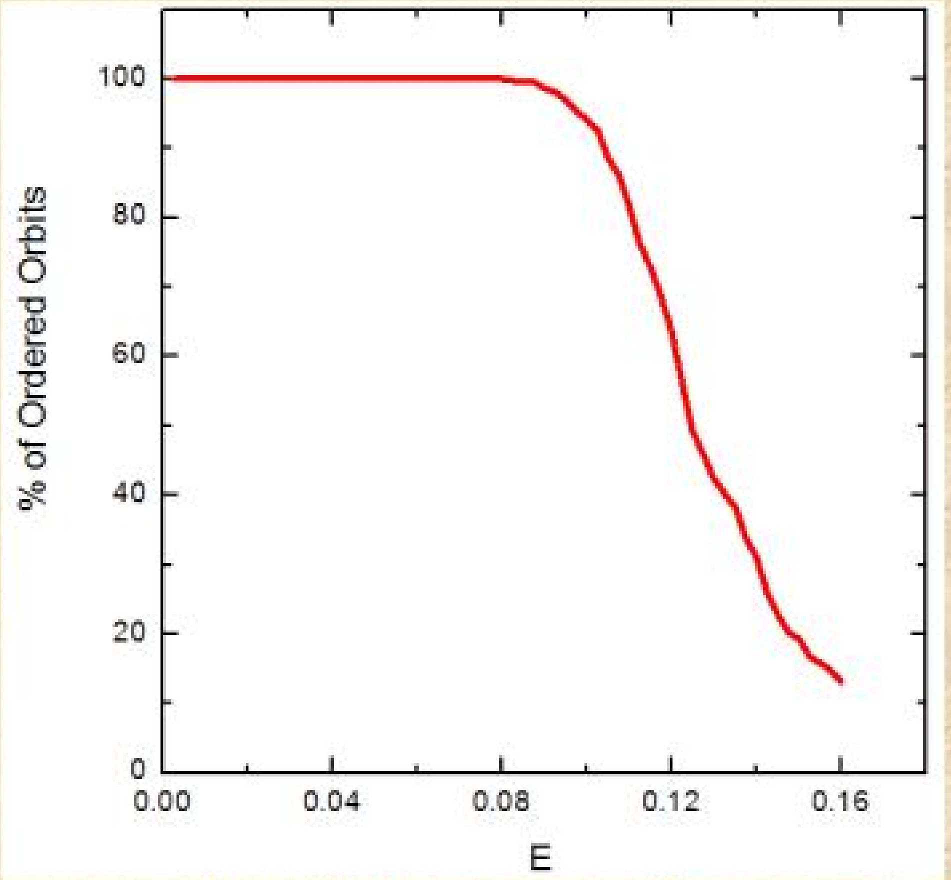


Applications – Hénon-Heiles system

The percentage of non chaotic orbits (SALI > 10^{-8} for t=1000)



Hénon-Heiles (1964) Astron. J. 69, 73.



A. Manos (2004) Master Thesis, Univ. of Patras

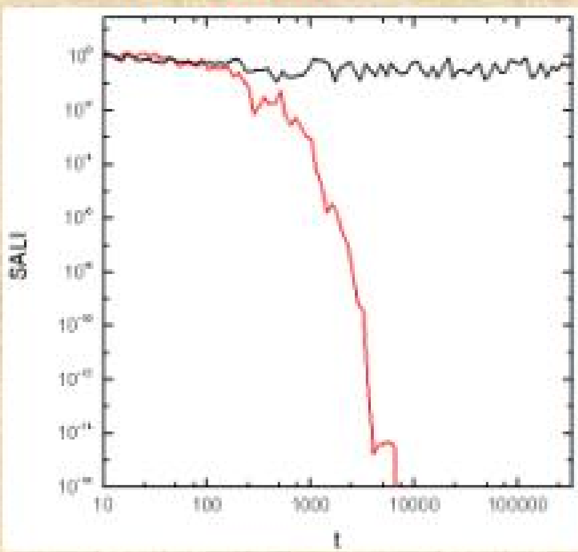
Applications – 3D Hamiltonian

We consider the 3D Hamiltonian

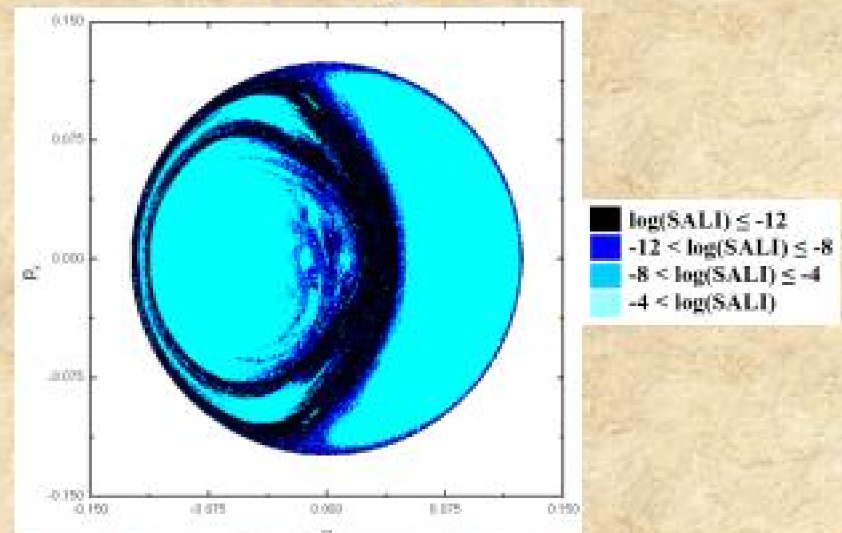
$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}(Ax^2 + By^2 + Cz^2) - \varepsilon xz^2 - \eta yz^2$$

with A=0.9, B=0.4, C=0.225, ε=0.56, η=0.2, H=0.00765.

Behavior of the SALI for ordered and chaotic orbits



Color plot of the subspace $y=z=p_y=0$, $p_z>0$



Behavior of the SALI

2D maps

SALI→0 both for ordered and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

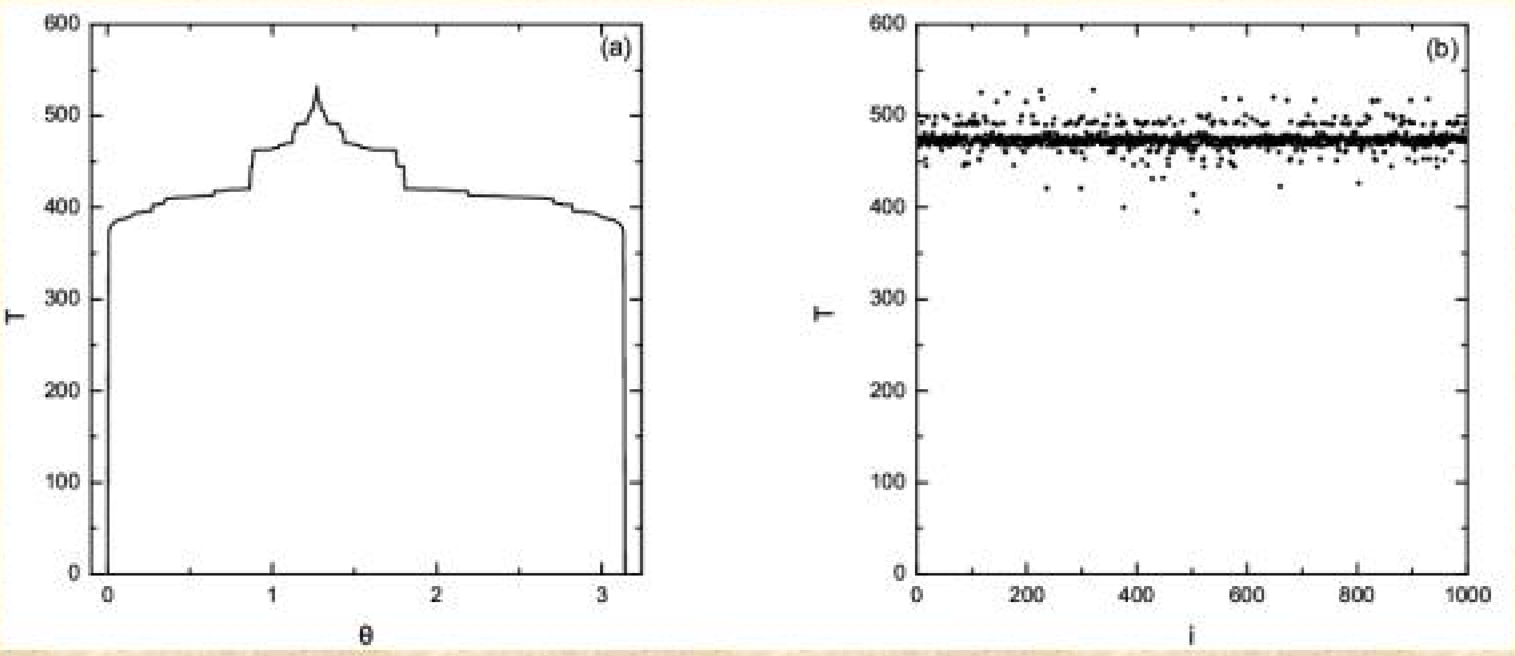
SALI→0 for chaotic orbits

SALI≠ 0 for ordered orbits

Dependence on the choice of the initial deviation vectors

For the chaotic orbit $x=0$, $y=0.1$, $p_x=0.49058$, $p_y=0$ of the Hénon-Heiles system with $E=1/8$ we compute the time needed for SALI to become less that 10^{-12} using $v_1=(0,1,0,0)$ and

$v_2=(0,\cos\theta,0,\sin\theta)$ $\theta\in[0,2\pi]$ **v_2=random**



Behavior of the SALI (Ordered motion)

An integrable 2D Hamiltonian system (Skokos et al., 2003, Prog. Th. Phys. Supp., 150, 439)

$$H = \frac{1}{2}(p_x^2 + p_y^2) - E(x^2 + y^2) + A(x^6 + y^6) + B(x^4y^2 + x^2y^4)$$

For $B=3A$ and $E \in \mathbb{R}$ there is a <u>second integral</u> (Ganesan & Lakshmanan, 1990): $F=(x\,p_y - y\,p_x)^2$

<u>**Vector base of the 4D phase space**</u>

vectors along the direction of the flow:

$$f_H=(H_{px}, H_{py}, -H_x, -H_y), \quad f_F=(F_{px}, F_{py}, -F_x, -F_y)$$

and perpendicular to the flow:

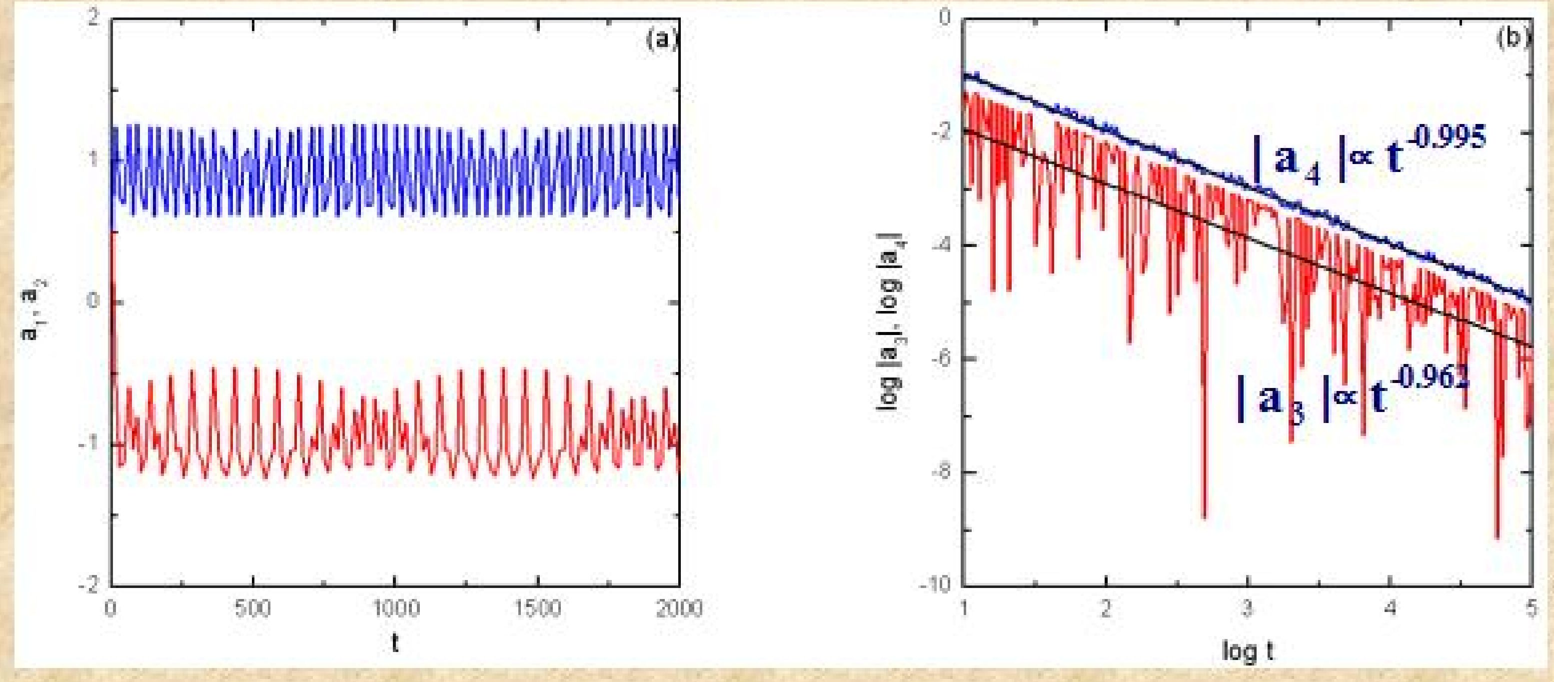
$$\nabla H=(H_x, H_y, H_{px}, H_{py}), \quad \nabla F=(F_x, F_y, F_{px}, F_{py})$$

Behavior of the SALI (Ordered motion)

The deviation vector can be written as

$$v = a_1 \overline{f_H} + a_2 \overline{f_F} + a_3 \overline{\nabla H} + a_4 \overline{\nabla F}$$

The deviation vector tends to the **tangent space** of the torus as $|a_3|, |a_4| \propto t^{-1}$.



Behavior of the SALI (Chaotic motion)

The evolution of a deviation vector can be approximated by:

$$\mathbf{v}_1(t)=\sum_{i=1}^{2n} c_i^{(1)} e^{\sigma_i t}\hat{e}_i \approx c_1^{(1)} e^{\sigma_1 t}\hat{e}_1 + c_2^{(1)} e^{\sigma_2 t}\hat{e}_2$$

where $\sigma_1>\sigma_2>\ldots>\sigma_{2n}$ are the Lyapunov exponents.

In this approximation, we derive a leading order estimate of the ratio

$$\frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|} \approx \frac{c_1^{(1)} e^{\sigma_1 t}\hat{e}_1 + c_2^{(1)} e^{\sigma_2 t}\hat{e}_2}{\left|c_1^{(1)}\right| e^{\sigma_1 t}} = \pm\hat{e}_1 + \frac{c_2^{(1)}}{\left|c_1^{(1)}\right|} e^{-(\sigma_1-\sigma_2)t}\hat{e}_2$$

and an analogous expression for $\mathbf{v}_2$

$$\frac{\mathbf{v}_2(t)}{\|\mathbf{v}_2(t)\|} \approx \frac{c_1^{(2)} e^{\sigma_1 t}\hat{e}_1 + c_2^{(2)} e^{\sigma_2 t}\hat{e}_2}{\left|c_1^{(2)}\right| e^{\sigma_1 t}} = \pm\hat{e}_1 + \frac{c_2^{(2)}}{\left|c_1^{(2)}\right|} e^{-(\sigma_1-\sigma_2)t}\hat{e}_2$$

So we get:

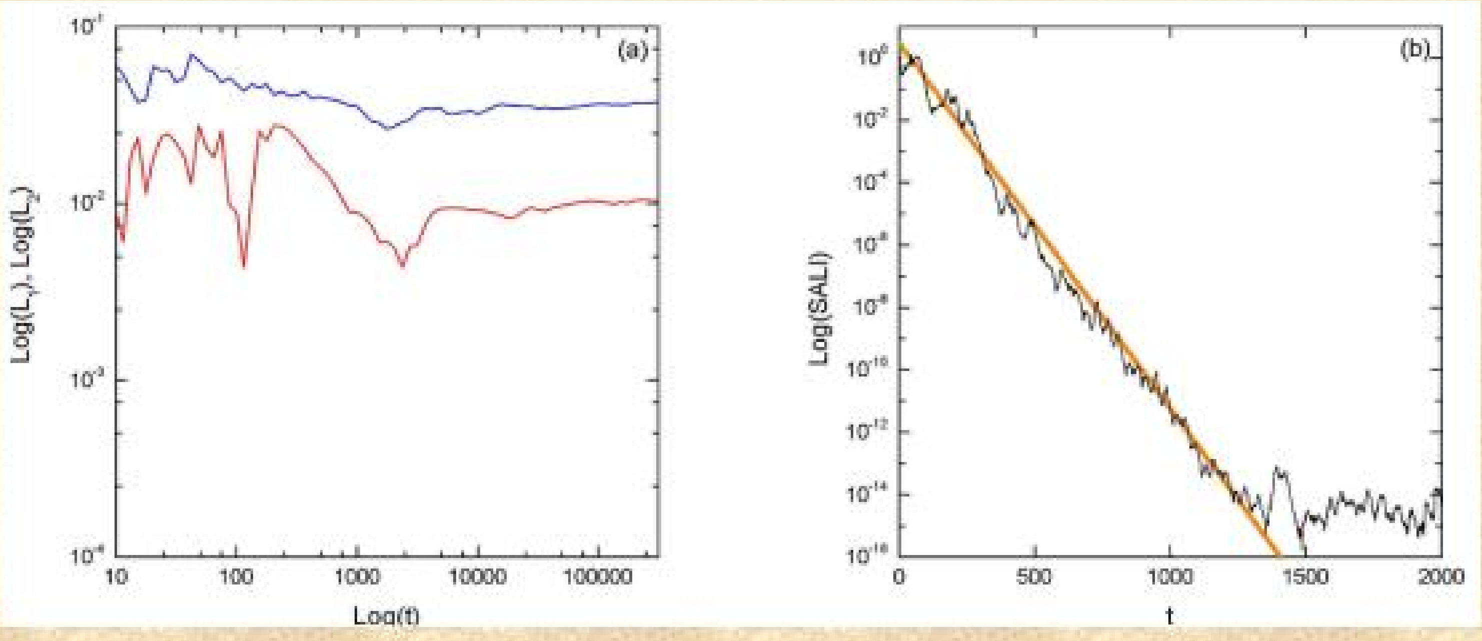
$$\mathrm{SALI}(t)=\min\left\{\left\|\frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|}+\frac{\mathbf{v}_2(t)}{\|\mathbf{v}_2(t)\|}\right\|,\left\|\frac{\mathbf{v}_1(t)}{\|\mathbf{v}_1(t)\|}-\frac{\mathbf{v}_2(t)}{\|\mathbf{v}_2(t)\|}\right\|\right\} \approx \left|\frac{c_2^{(1)}}{\left|c_1^{(1)}\right|} \pm \frac{c_2^{(2)}}{\left|c_1^{(2)}\right|}\right| e^{-(\sigma_1-\sigma_2)t}$$

Behavior of the SALI (Chaotic motion)

We test the validity of the approximation $\boxed{SALI \propto e^{-(\sigma 1-\sigma 2)t}}$ (Skokos et al., 2004, J. Phys. A, 37, 6269) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with ω_1=1, ω_2=1.4142, ω_3=1.7321, H=0.09



Behavior of the SALI

Hamiltonian flows and multidimensional maps

The ordered motion occurs on a torus and two different initial deviation vectors become tangent to the torus having different directions (SALI$\neq$0).

In chaotic cases two initially different deviation vectors tend to coincide to the direction defined by the most unstable nearby manifold (SALI$\rightarrow$0).

2D maps

Any two deviation vectors tend to coincide or become opposite for ordered and chaotic orbits (SALI$\rightarrow$0 with different time rates for each case).

Papers

- Skokos Ch. (2001) J. Phys. A, 34, 10029.
- Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2003) Prog. Theor. Phys. Supp., 150, 439.
- Skokos Ch., Antonopoulos Ch., Bountis T. C. & Vrahatis M. N. (2004) J. Phys. A, 37, 6269.

2 equal Lyapunov exponents

FPU system $$H = \frac{1}{2}\sum_{i=1}^{N} \dot{q}_i^2 + \sum_{i=1}^{N}\left(\frac{1}{2}(q_{i+1} - q_i) + \frac{1}{4}\beta(q_{i+1} - q_i)^4\right)$$

Saitô mode $$q_0(t) = q_{N+1}(t) = 0 \;\; \forall t,$$

$$q_{2i}(t) = 0, \;\; q_{2i-1}(t) = -q_{2j+1}(t) = \hat{q}(t), \;\; i = 1, 2, \ldots, \frac{N-1}{2}$$