MHD turbulence
where we were where we are and where we are going

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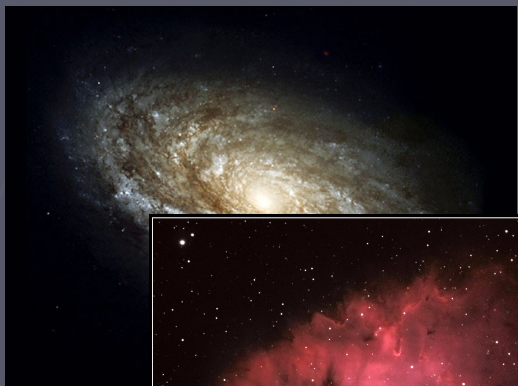
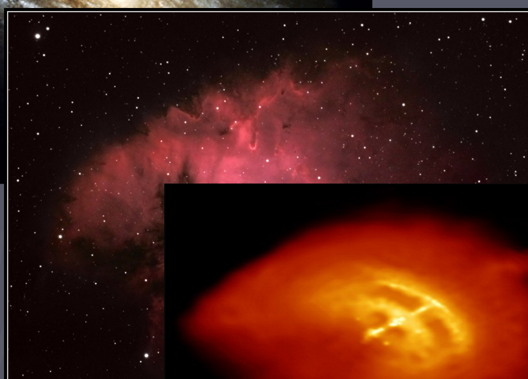
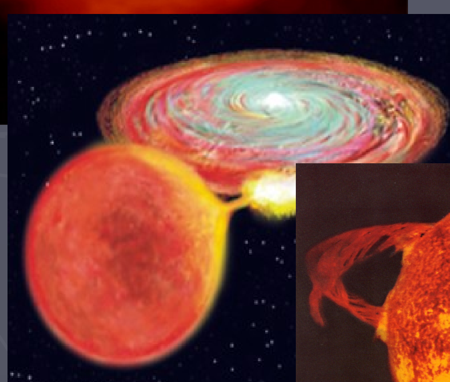
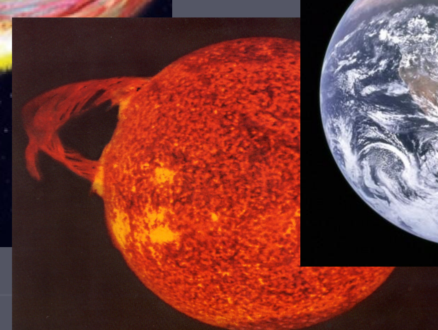
Outline

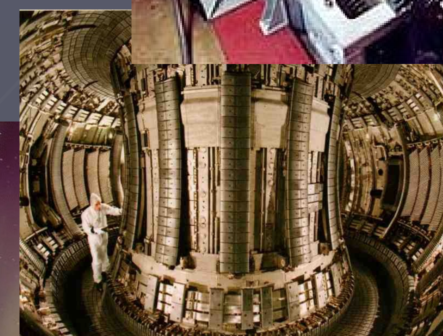
- Places of MHD turbulence
- Questions of turbulence
- The equations of motion
- Assumptions in turbulence theory
- Models and Numerical simulations
- Locality of interactions
- Conclusions

What we want to study

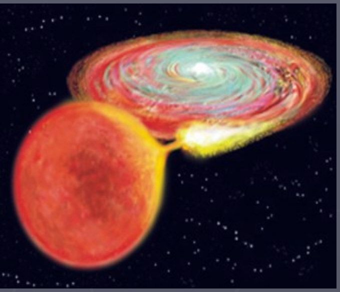
- $Re \sim 10^{3\text{-}40}$, $P_M \sim 10^{-10}$ -10^{+17}
- Compressible/Stratified
  - Open boundaries
  - Two fluid effects
    - Braginskii viscosity
      - ...

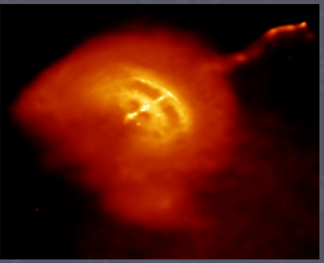




Basic questions of turbulence

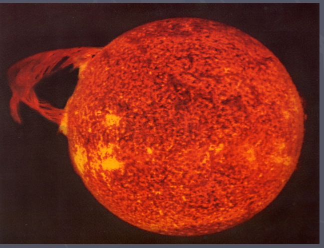


- The time scale in accretion processes in the absence of turbulence would be longer than the age of the univerce. Turbulence controls the time scale for accretion.

- Turbulence also controls the star formation time scale from molecular clouds.



- Turbulence also controls the final state of astrophysical jets.



- Turbulent convection controls the heat flux out of the sun.

A few words about turbulence

"turbulence is a state of continuous instability"

D.J.Tritton

L. Da Vinci 1452-1519

The Millennium Problems

You Do the Math, and Earn $7M in the Process

THE ASSOCIATED PRESS

NEWSDAY, THURSDAY, MAY 25, 2000

THE ASSOCIATED PRESS

Paris — A group of the world's top mathematicians is offering $7 million for solutions to seven of the world's hardest equations.

After puzzling for years over seven unsolved math problems, a U.S.-based mathematics foundation put the "Millennium Prize Problems" challenge to the world via the Internet yesterday.

Experts say solving the problems could lead to breakthroughs in encryption and aerospace — and open areas of mathematics as yet unimagined.

The Clay Mathematics Institute posted the problems on its Web site, http://www.claymath.org at the same time it unveiled the contest in Paris at its annual meeting. The group has posted a $1-million prize for each of the seven problems.

Few expect a winner anytime soon.

"There's no time limit," said Arthur Jaffe, a Harvard University math professor and president of the Clay institute, a private, nonprofit foundation based in Cambridge, Mass.

According to contest rules, solutions must be published in a renowned math journal and undergo a two-year waiting period to allow time for independent review. If the mathematics community accepts the solution, the Clay institute will then open its own review before awarding any money.

Mathematicians are quick to note that a few decades, or even a century, is not a long wait to unravel the world's toughest puzzles.

The list of problems includes the following equations, named for the mathematicians who postulated them: the Riemann Hypothesis, the Poincare Conjecture, the Hodge Conjecture, the Birch and Swinnerton-Dyer Conjecture, Navier-Stokes Equations, the Yang-Mills Theory and the P vs. NP Problem.

Physicists & Mathematicians

- Exact solutions of the Navier Stokes are of limited use since they tend to be unstable.
- We need a statistical description of turbulence.
- Mathematicians want to prove existence and uniqueness of strong solutions of the NS
- Physicists want to quantify the effect of the “random” turbulent flows on large scale measurable quantities

The equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{F}$$

$$\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} + \varepsilon\nabla\times(\mathbf{J}\times\mathbf{b}) + \eta\nabla^2\mathbf{b}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{b} = 0 \qquad \mathbf{J} = \nabla\times\mathbf{b}$$

Navier-Stokes MHD Hall

Control Parameters

HD (Hydrodynamics)

- Reynolds Number

  $Re = UL/\nu$

MHD (Magnetohydrodynamics)

- Reynolds Number

  $Re = UL/v$

- Magnetic Reynolds Number

  $R_m = UL/\eta$

(Prandtl $Pm = R_m/Re$)

- Alfven Mach number

  $M = U/B$

Ideal Invariants

HD (Hydrodynamics)

- Energy

$$E = \tfrac{1}{2}\int \mathbf{u}^2 dx^3$$

- Helicity

$$H = \tfrac{1}{2}\int \mathbf{u}\cdot\nabla\times\mathbf{u}\, dx^3$$

MHD (Magnetohydrodynamics)

- Energy

$$E = \tfrac{1}{2}\int \mathbf{u}^2 + \mathbf{b}^2 dx^3$$

- Magnetic helicity

$$H_M = \tfrac{1}{2}\int \mathbf{a}\cdot\mathbf{b}\, dx^3, \quad \mathbf{b} = \nabla\times\mathbf{a}$$

- Magnetic cross helicity

$$H_M = \tfrac{1}{2}\int \mathbf{u}\cdot\mathbf{b}\, dx^3$$

Desired resolution in a simulation

- Number of grid points = $(L / \ell)^3$
- ℓ = smallest scale of the system
- L = largest scale of the system

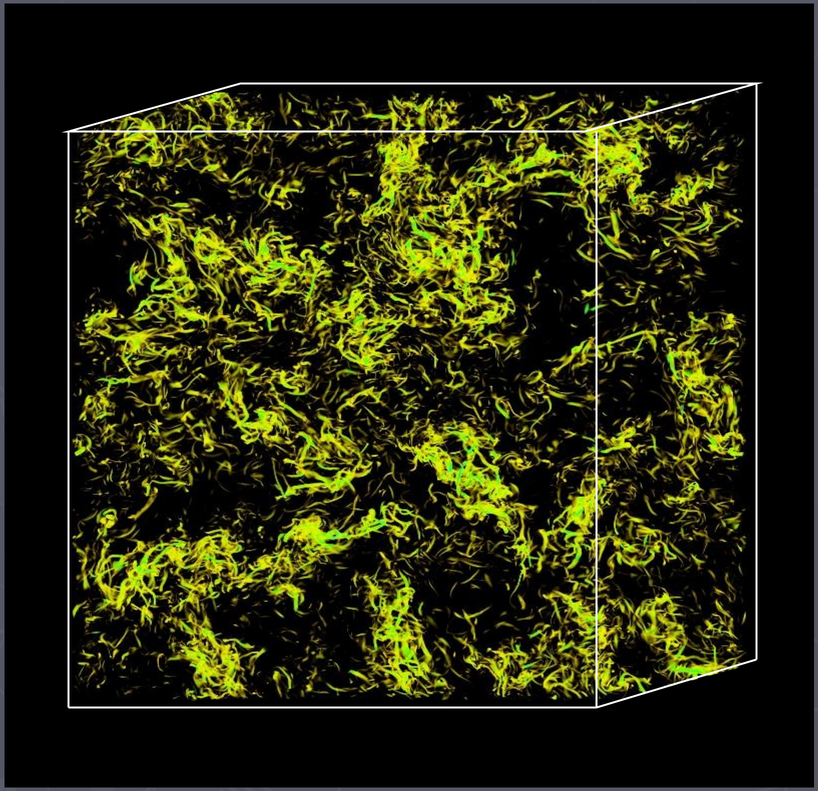
$L \sim$ size of a star $\sim 10^9$m

ℓ such that $u^2/ \ell \sim \nu\, u/ \ell^2$ --> $\ell \sim \nu/u \sim 10^{-2}$.

$$N \sim 10^{33}$$

(maximum ever achieved in $N \sim 6\ 10^{10}$)

What we study



- Triple periodic
- Incompressible
- Unstratified
- MHD approximation
- Re ~ 10^3,
- $P_M \sim 10^{\pm 2}$

Justification

- At small enough scales the behavior of the flow is independent of forcing mechanism and boundary conditions.
- For large enough Reynolds number the rate the large scale flow is loosing energy is finite and independent of the dissipation mechanism at the small scales.

Energy dissipation-Naive estimates

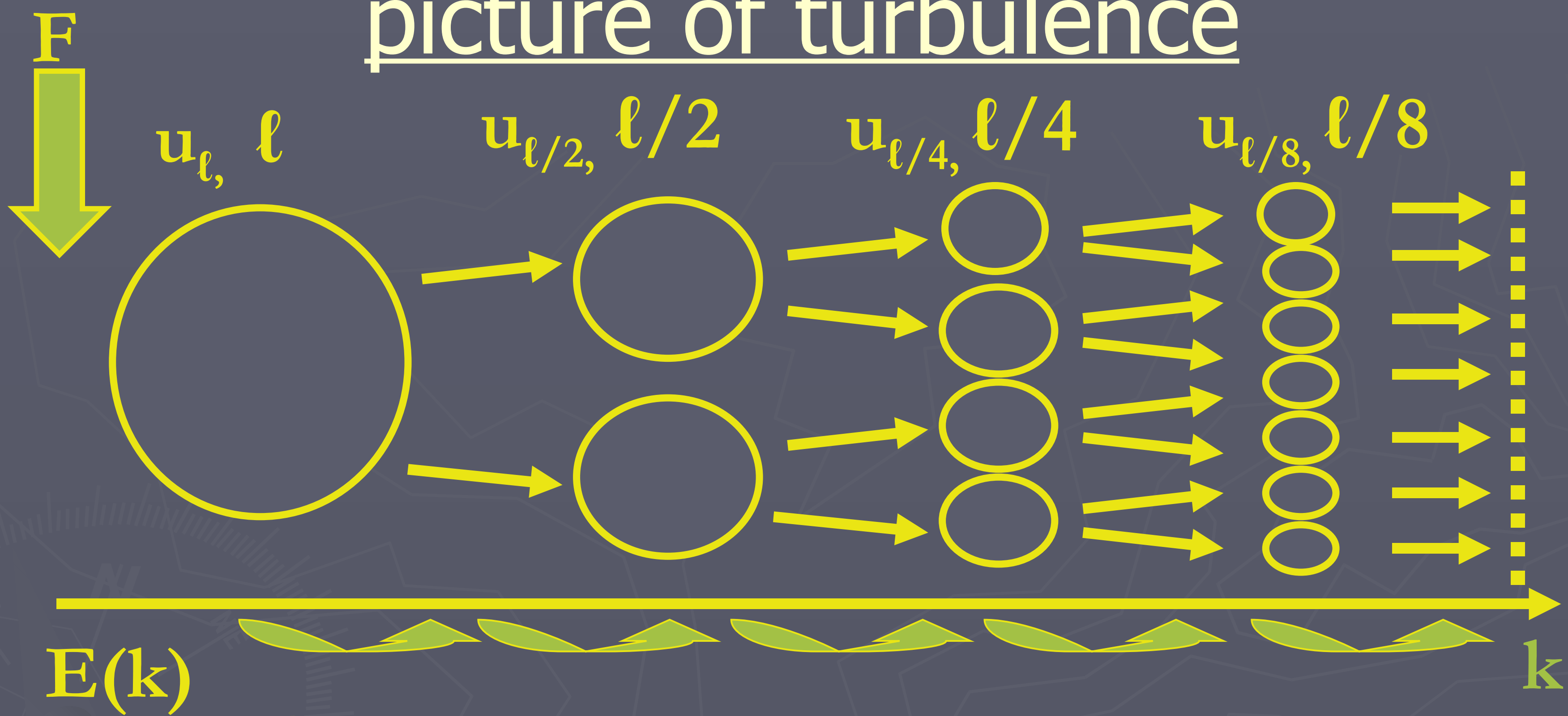
$$\frac{\partial}{\partial t}\frac{1}{2}\int \mathbf{u}^2 dx^3 = -\nu \int |\nabla \mathbf{u}|^2 dx^3$$

$$\partial_t E \sim -\nu E / \ell^2$$

This leads to extremely large timescales:

$$\tau \sim \ell^2 / \nu \sim \mathrm{Re}\, \ell / U$$

The phenomenological picture of turbulence



HD

$\tau \sim \ell / u_\ell$ $\qquad\qquad$ $\epsilon \sim u_\ell^2/\tau \sim u_\ell^3/\ell$

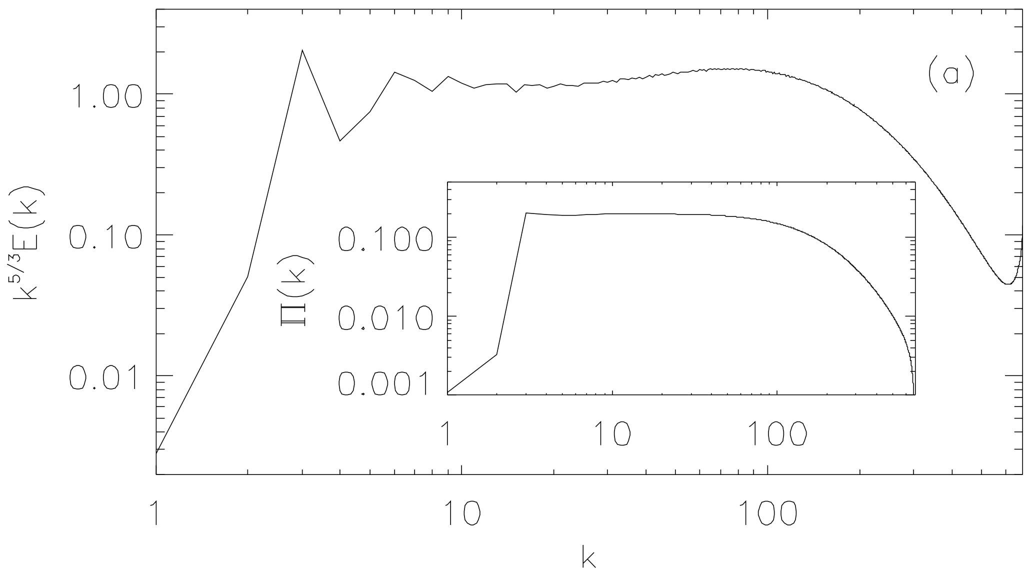
$dE/dk \sim E/k \sim u_\ell^2 \cdot \ell \sim k^{-5/3}$

Time scale for energy dissipation is $T \sim L/U$

Assumptions made

- There is a universal energy spectrum
- Constant flux of energy to the small scales
- Independent from the way it is forced
- Independent from the way energy is dissipated
- The cascade is isotropic
- Energy cascades only to small scales
- Same scale interactions control the cascade
- No role of other invariants

Yet, it seems to work for Hydrodynamic turbulence.



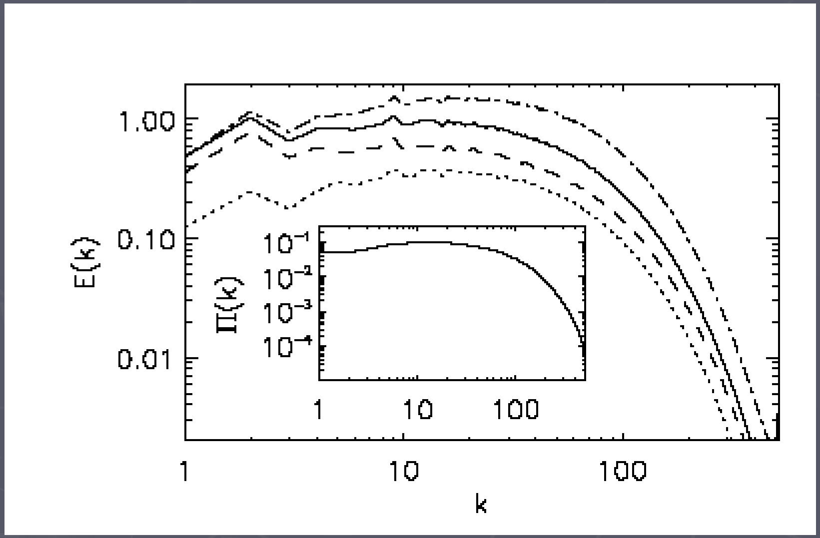
Energy spectrum from numerical simulation at 2048^3 grid point resolutions

MHD turbulence Models

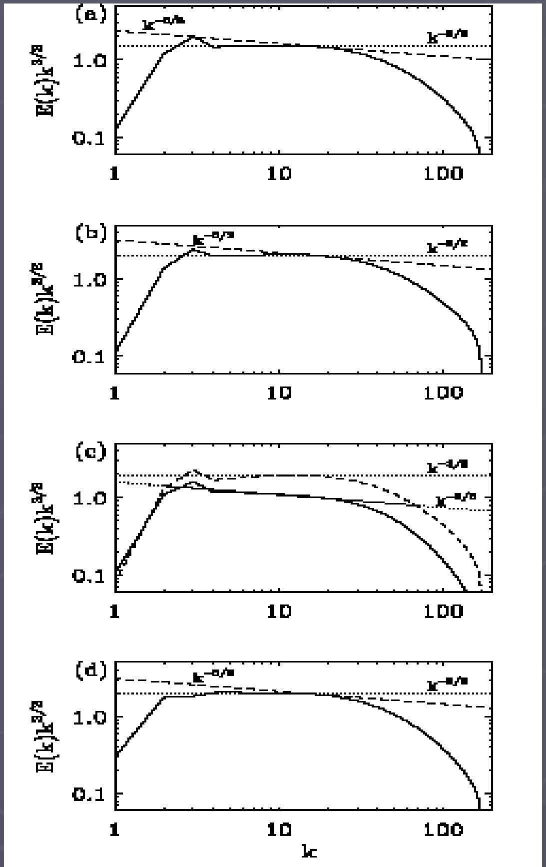
- Iroshnikov-Kraichann $E(k) \sim k^{-3/2}$
- Goldreich-Shridhar $E(k) \sim k_{\perp}^{-5/3}$
- Zhou – Matthaus $E(k) \sim k^{-a(B)}$
- Boldyrev $E(k) \sim k_{\perp}^{-3/2}$
- Galtier (weak turbulence theory) $E(k) \sim k_{\perp}^{-2}$
- Alexakis (non-universal)

Numerical Simulations

Isotropic free decay



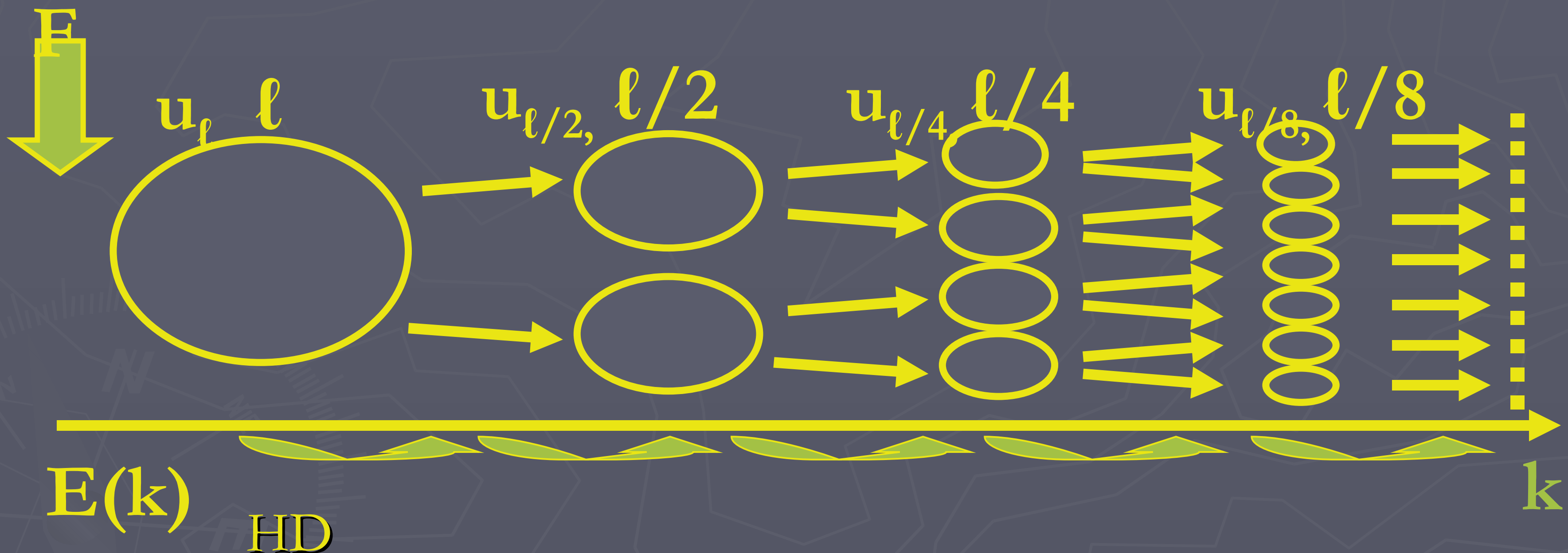
With a guiding field



Spectral exponents seem
Range between -3/2 and 5/3

Testing the assumptions

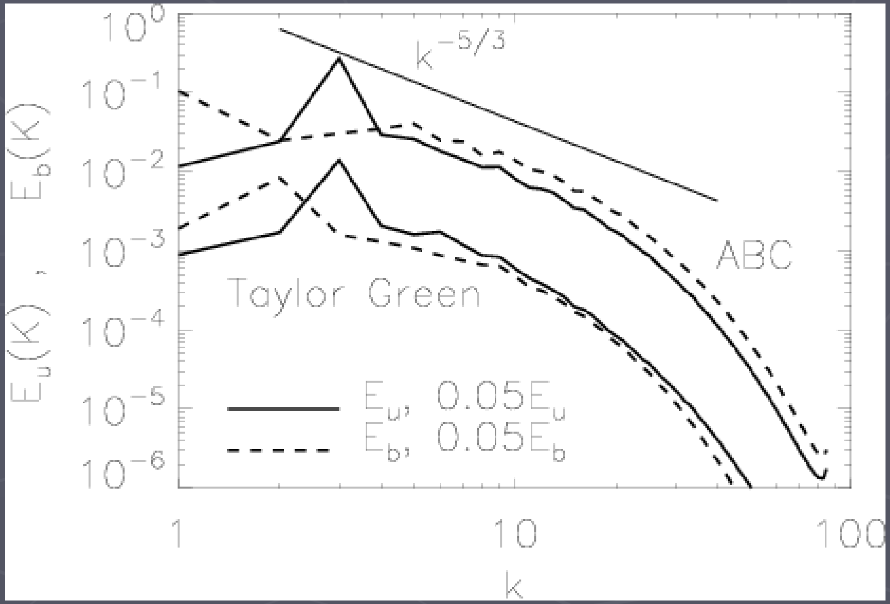
- Isotropic case – Locality of interactions



We will call "local interactions" the interactions of similar size eddies

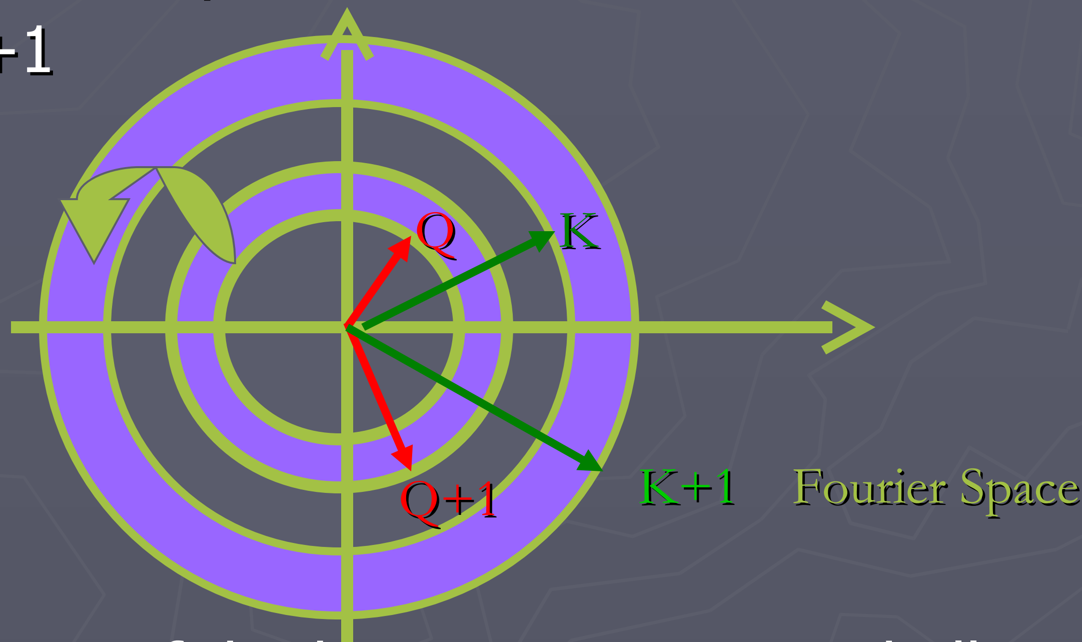
The Numerical simulations used

- 256^3 grid points
- Pseudo-spectral method with 2/3 dealiasing
- Magnetic field amplified by dynamo action
- Re=300



Energy Transfer

- Let $\mathbf{u}_K(x)$ be the velocity field with wave numbers in the range K<|k|<K+1



- Then the evolution of the kinetic energy in a shell K is given by:

$$\partial_t E(K) = -\sum_{P,Q} \int u_K (u_P \cdot \nabla) u_Q dx^3 - \nu \nabla^2 u_K + \int F \cdot u_K dx^3$$

Definitions of Transfer Functions

- The rate energy is transferred from the modes Q to the modes K due to the interaction with the shell P is given by:

$$T_3(K,P,Q) = -\int \mathbf{u}_K \left(\mathbf{u}_P \cdot \nabla \right) \mathbf{u}_Q d\mathbf{x}^3$$

- The rate energy is transferred from the modes Q to the modes K is given by:

$$T_2(K,Q) = \sum_P T_3(K,P,Q)$$

- The rate energy is changing in a shell k due to the non-linear term is given by:

$$T_1(K) = \sum_Q T_2(K,Q)$$

- The flux of energy through a wavenumber shell K is given by:

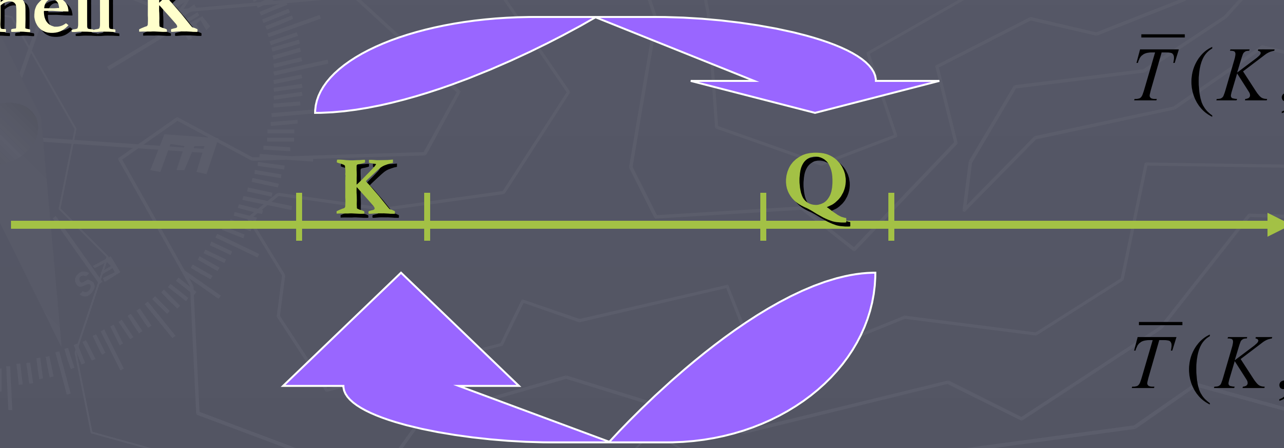
$$\Pi(K) = \sum_{K'=0}^{K} T_1(K')$$

Transfer of Energy

$$\int \mathbf{u}_K \left(\mathbf{u} \cdot \nabla \right) \mathbf{u}_Q d\mathbf{x}^3 = -\int \mathbf{u}_Q \left(\mathbf{u} \cdot \nabla \right) \mathbf{u}_K d\mathbf{x}^3$$

$$\bar{T}(K,Q) = -\bar{T}(Q,K)$$

The Energy the shell K losses to the shell Q is equal to the Energy the shell Q gains from the shell K

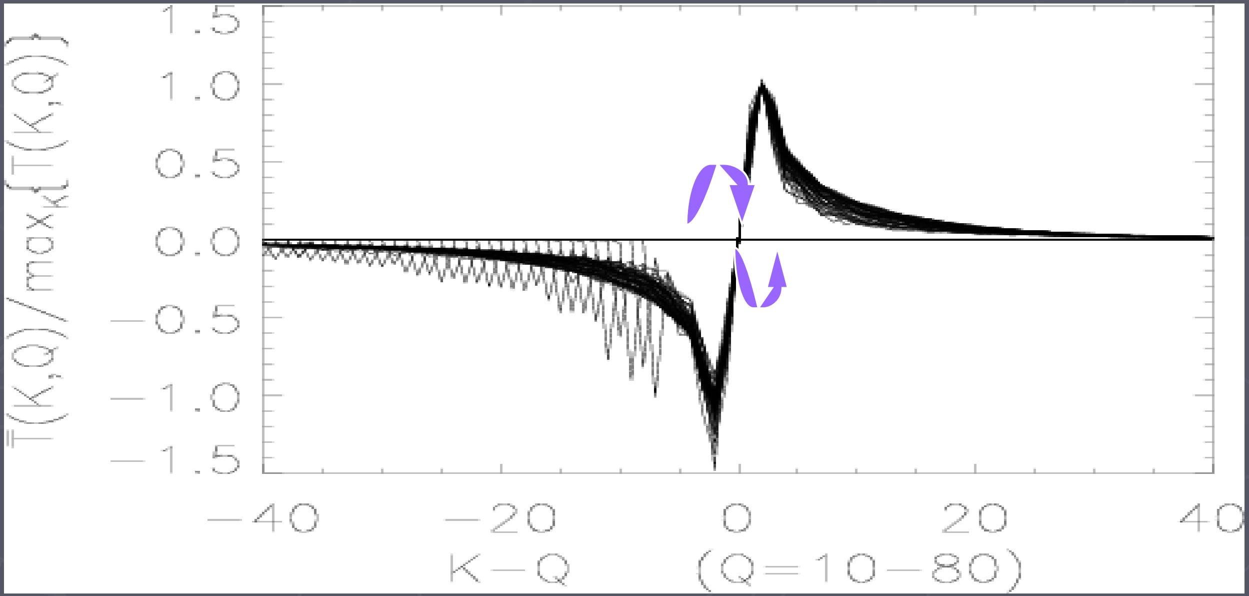


$$\bar{T}(K,Q) < 0$$

$$\bar{T}(K,Q) > 0$$

Hydrodynamics: The Transfer term T(K,Q)

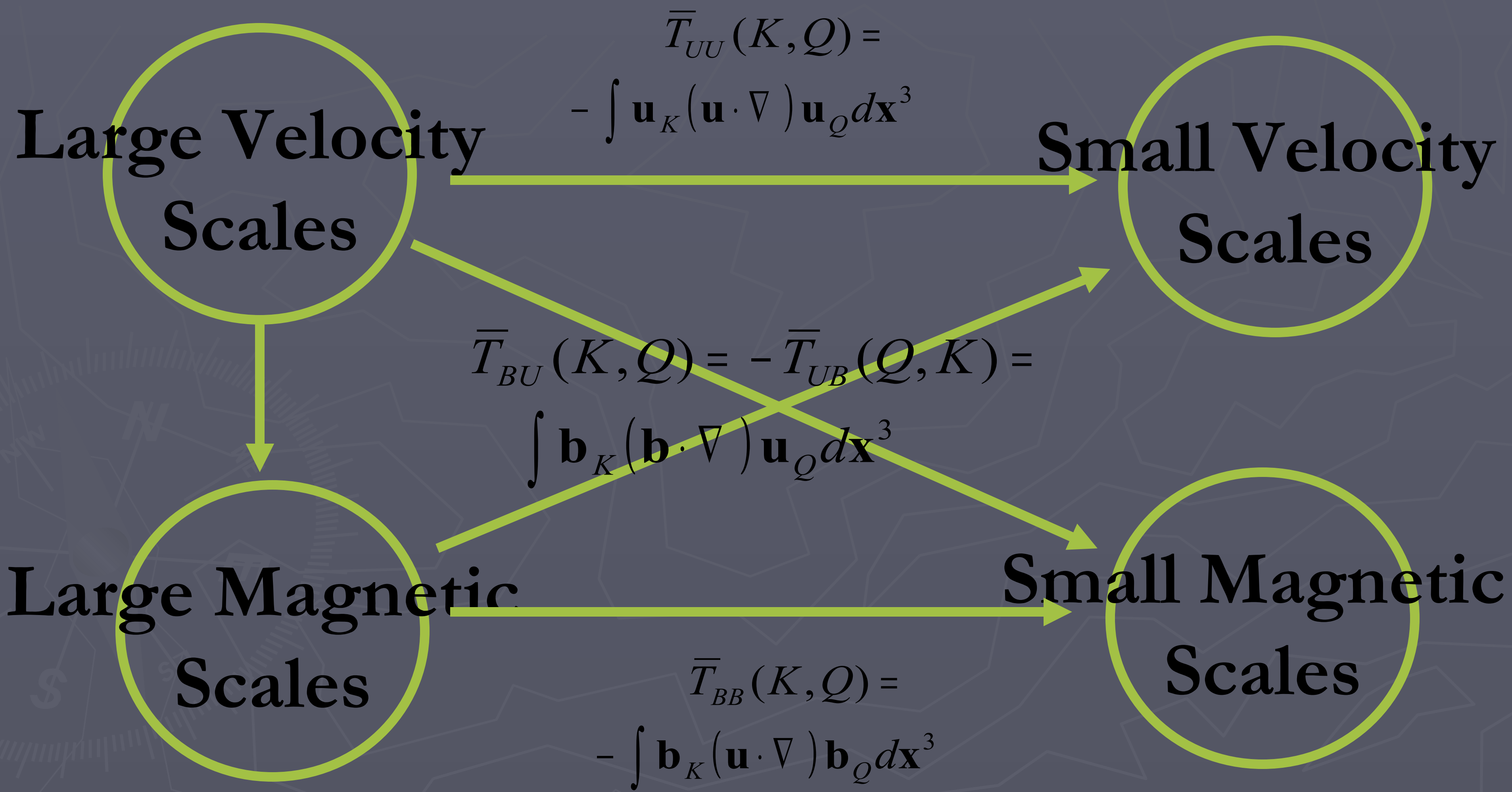
The rate that energy is transferred from the shell Q to the shell K.



K-Q for Q=10,11,12,...80

- Energy transfer T(K,Q) is local.

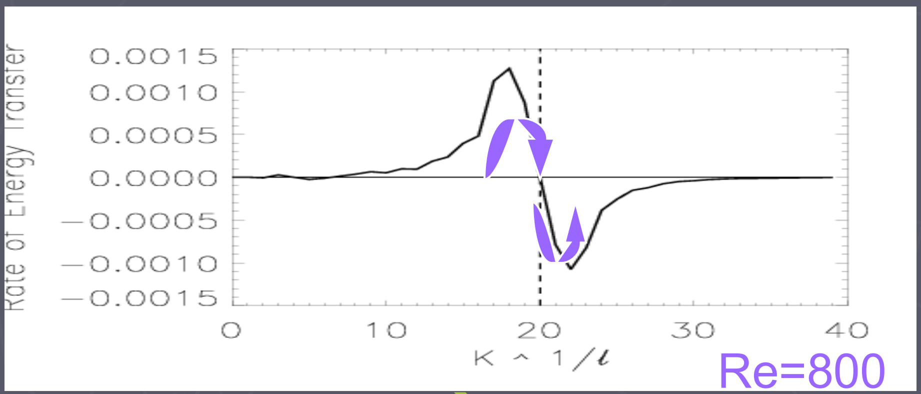
MHD -Transfer

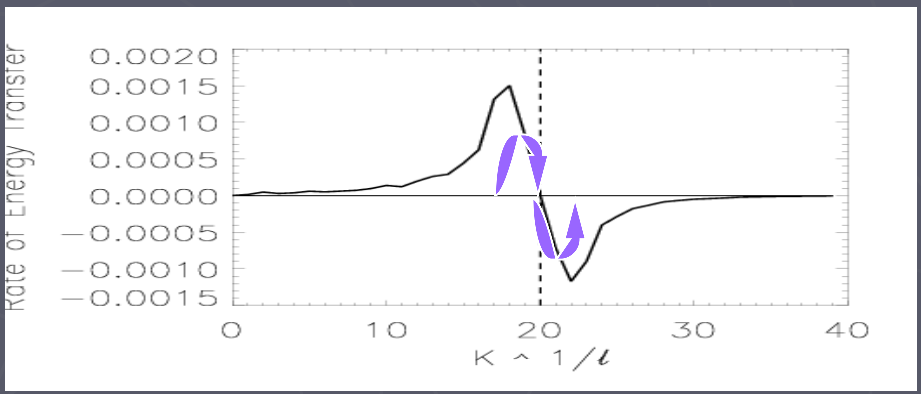


Rate of Energy transfer in MHD

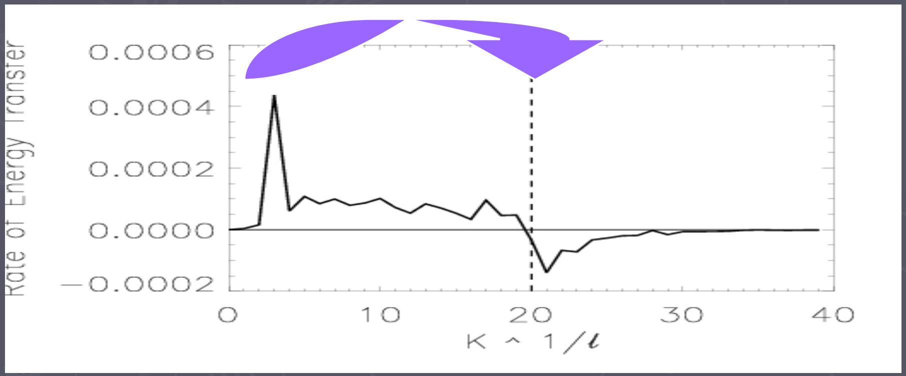
U→U

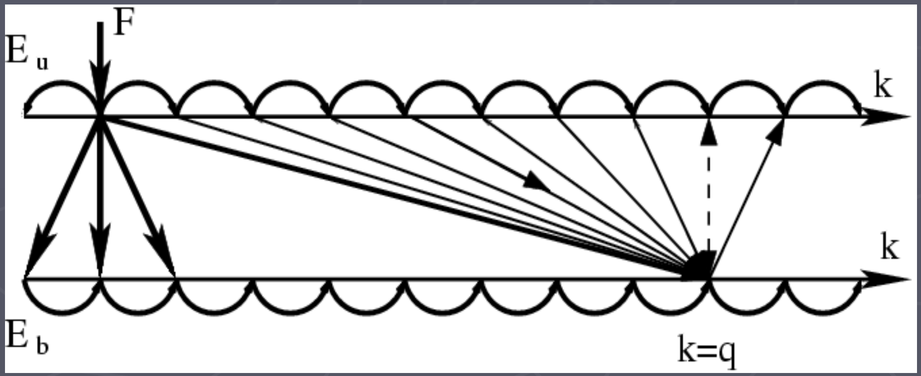
B→B



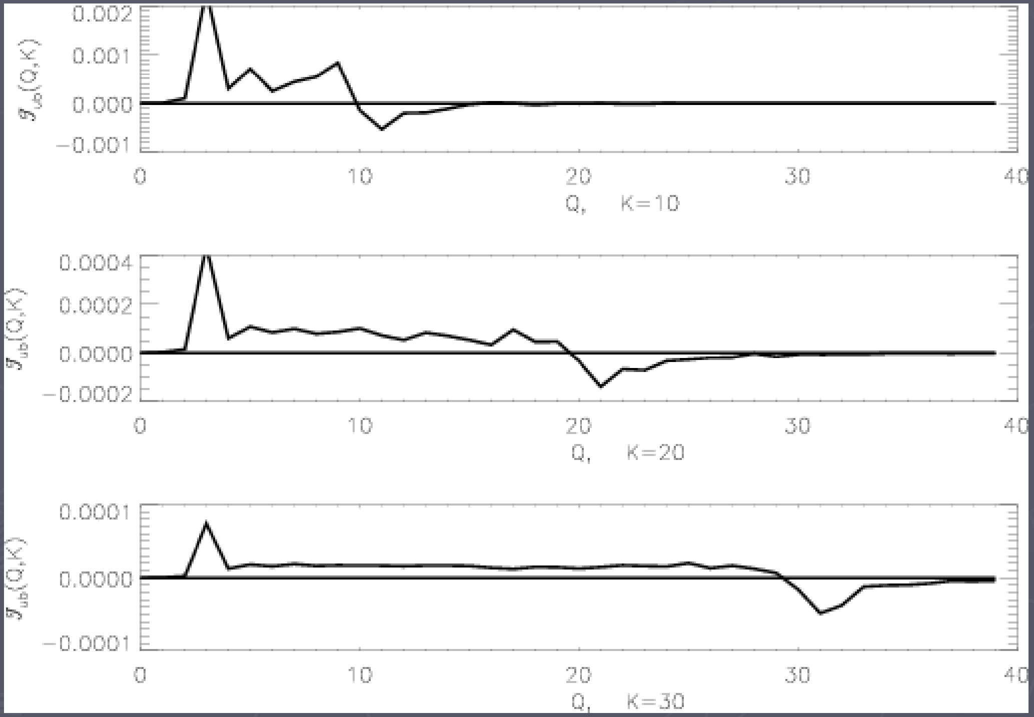


U→B





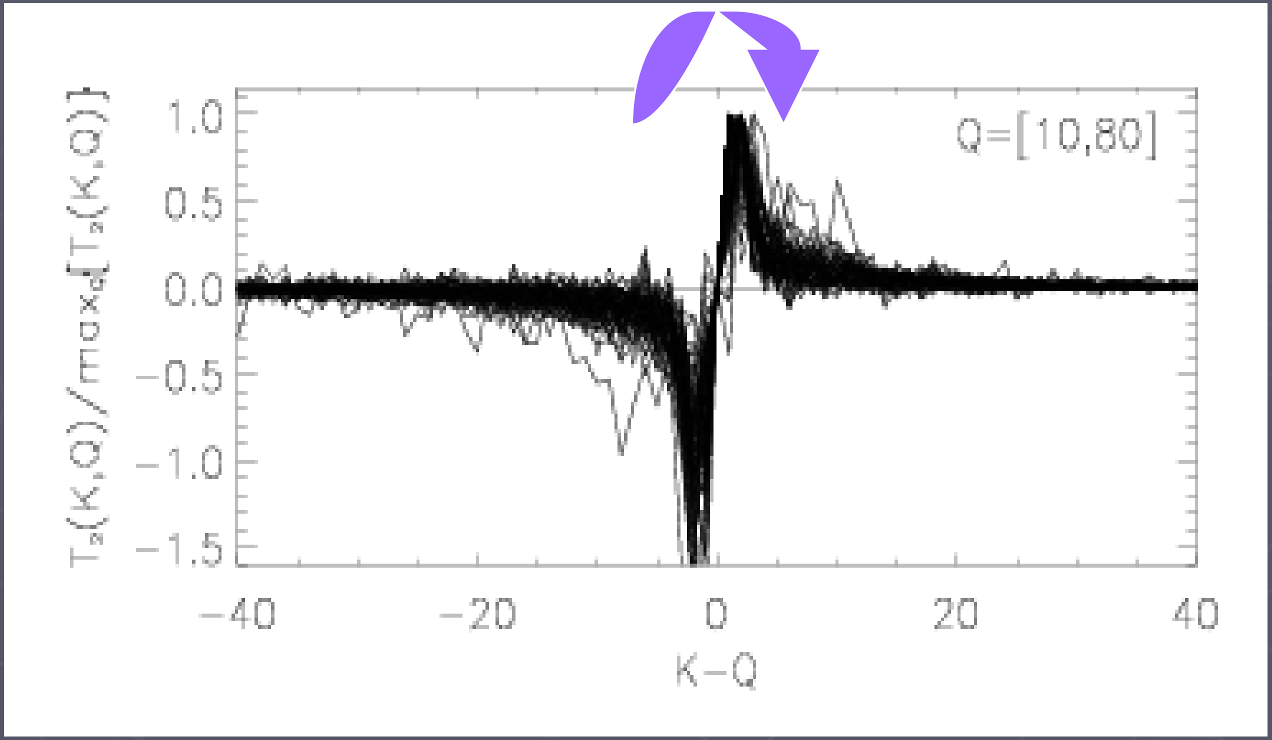
Rate of Energy transfer from u to b for different shells



The magnetic field at a given scale receives energy at the same rate from the velocity field from all larger scales!

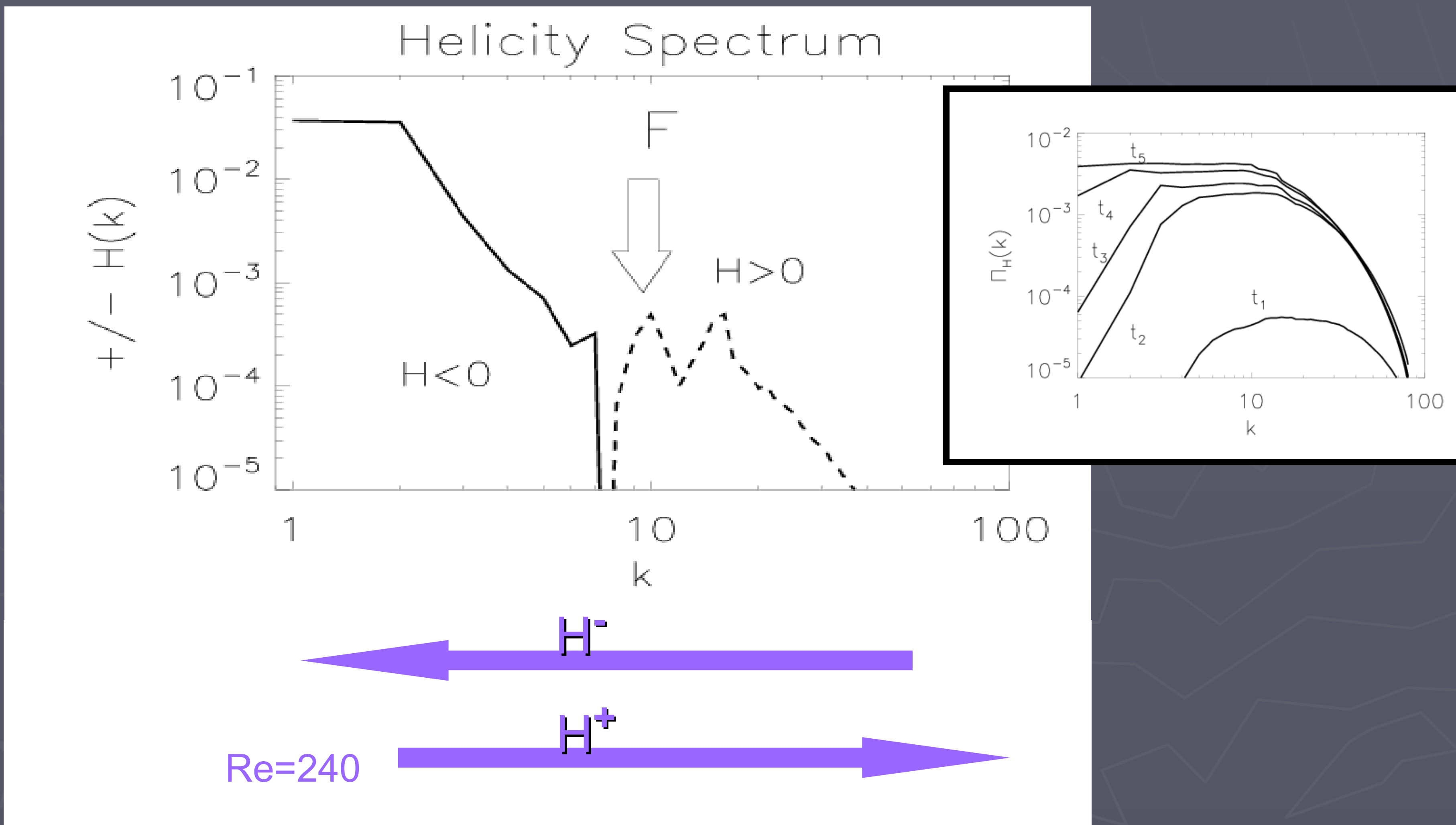
How about other Invariants?

Hydrodynamic Helicity:



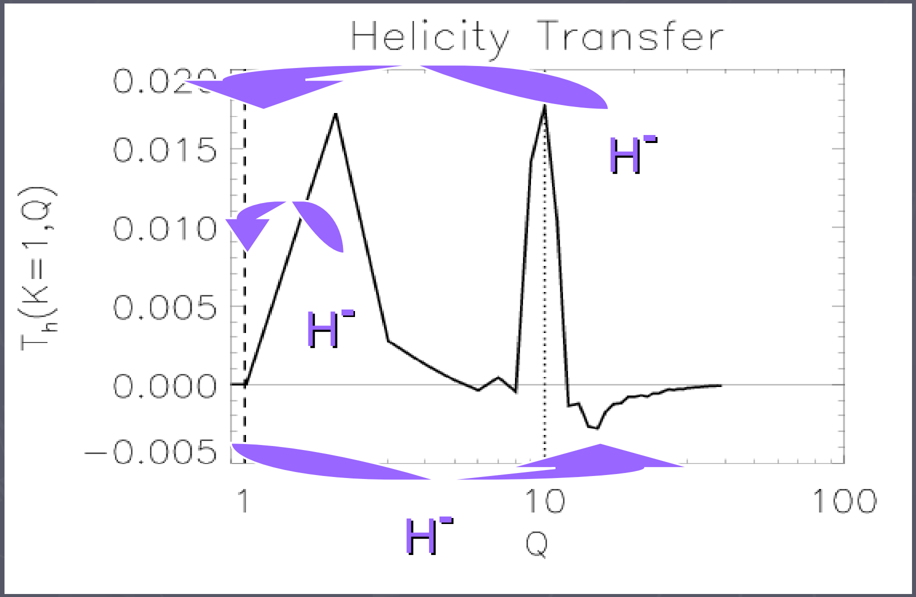
- Mostly local helicity transfer

Magnetic Helicity spectrum

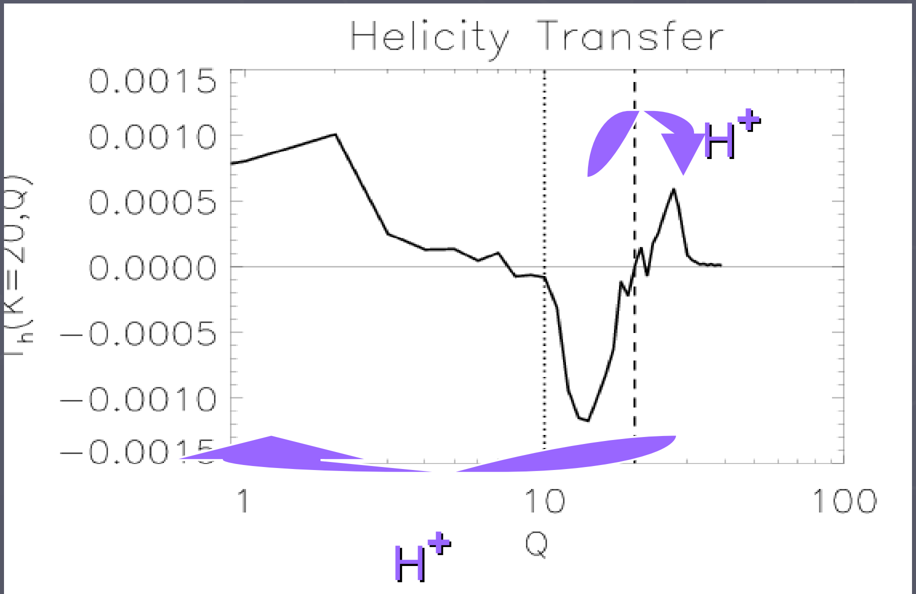


Magnetic Helicity:

T(1,Q)



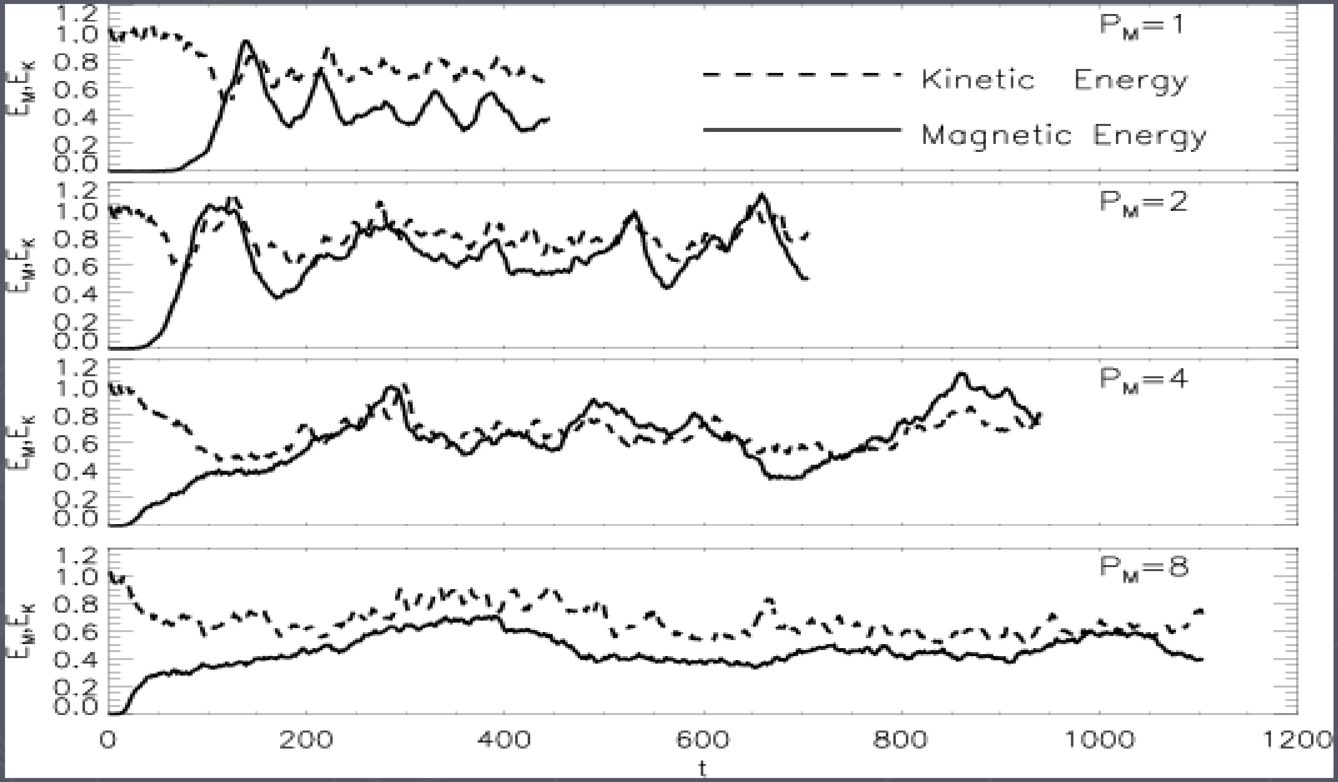
T(20,Q)



- Dominantly non-local transfer!

Some examples of the effect of non-local coupling.

Large Magnetic Prandtl Flows



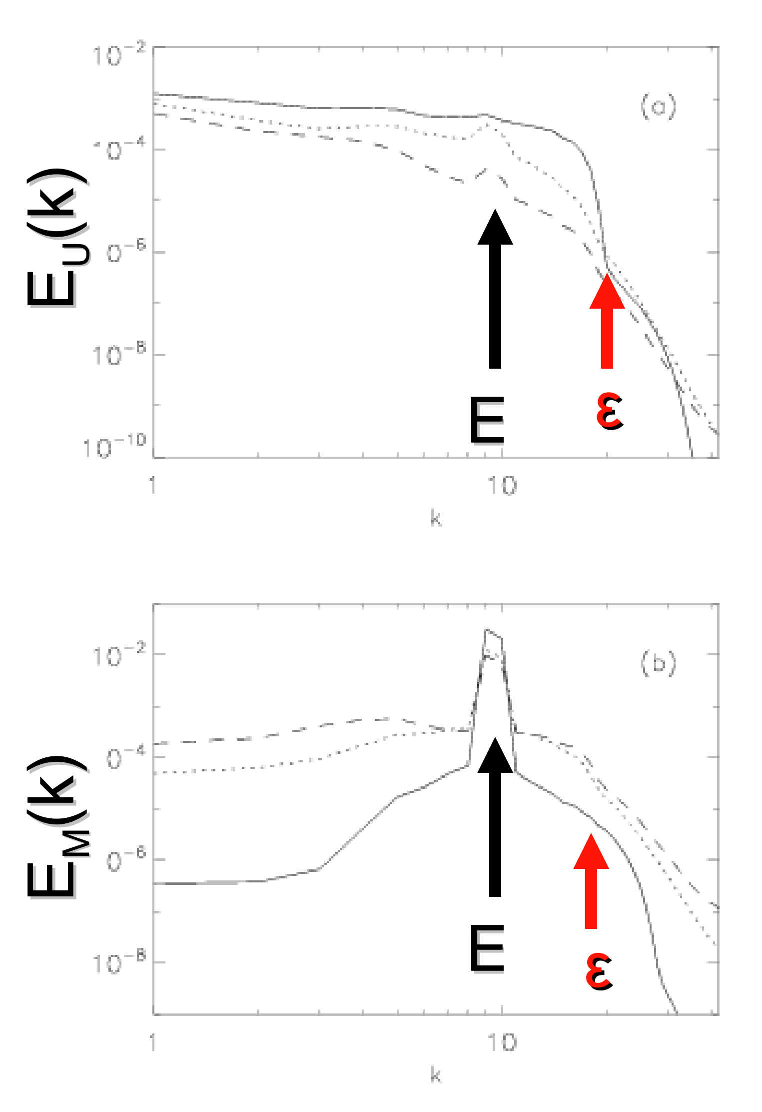
The Hall effect - A small scale term

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u}$$

$$\partial_t \mathbf{b} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} + \varepsilon\nabla\times\underline{(\mathbf{J}\times\mathbf{b})} + \eta\nabla^2\mathbf{b} + \nabla\times E$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \nabla\cdot\mathbf{b} = 0 \qquad \mathbf{J} = \nabla\times\mathbf{b}$$

ε=0.00	———
ε=0.05	········
ε=0.1	– – – –



So where are we going?

- Basic estimates are still not clear

$\varepsilon = U^3/L$ (Hydro), $\varepsilon = U^3/L\ F(Pr, M, \ldots)$ (MHD)

- Non-locality is a basic part in MHD.
- Type of forcing might play an important role
- Sub-viscous or Sub-diffusive scales can play an important role.
- There might not be a universal spectrum.

The end