

A self-organized criticality model for ion temperature gradient mode driven turbulence in confined plasma

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A new self-organized criticality (SOC) model is introduced in the form of a cellular automaton (CA) for ion temperature gradient (ITG) mode driven turbulence in fusion plasmas. Main characteristics of the model are that it is constructed in terms of the actual physical variable, the ion temperature, and that the temporal evolution of the CA, which necessarily is in the form of rules, mimics actual physical processes as they are considered to be active in the system, i.e., a heating process and a local diffusive process that sets on if a threshold in the normalized ITG R/L_T is exceeded. The model reaches the SOC state and yields ion temperature profiles of exponential shape, which exhibit very high stiffness, in that they basically are independent of the loading pattern applied. This implies that there is anomalous heat transport present in the system, despite the fact that diffusion at the local level is imposed to be of a normal kind. The distributions of the heat fluxes in the system and of the heat out-fluxes are of power-law shape. The basic properties of the model are in good qualitative agreement with experimental results. © 2010 American Institute of Physics.

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I. INTRODUCTION

Self-organized criticality (SOC) is a possible state of complex, spatially extended systems that are systematically driven and that have mechanisms to develop local instabilities and to relax them. SOC is characterized by intermittent transport events that range from very small size up to system size, so-called avalanches, by power-law distributions of variables characterizing the transport, and by power spectra of the dissipated energy that are of power-law shape. SOC models are usually constructed in the form of cellular automata (CA), i.e., by using a discrete grid and rules for the evolution of the system, with prototype the sand-pile model of Ref. 1.

In inquiries on confined plasmas and the related transport phenomena, evidence has been collected that the plasma might well be in the state of SOC, e.g., the fluctuations in density, potential, particle flux, and electron temperature show avalanchelike characteristics, such as intermittency and power spectra of power-law shape, and the probability distribution of the particle flux displays a power-law (e.g., Refs. 2–4, and see the recent discussion in Ref. 5). Also fluid simulations have been found to exhibit SOC features, such as the occurrence of avalanches and the appearance of frequency spectra of power-law shape (e.g., Refs. 6 and 7). A discrepancy between SOC models and experimental results had appeared concerning distributions of waiting-times between bursts in density fluctuation and particle-flux time-series, the distributions determined from experimental time-series were found to be of power-law shape,⁸ whereas the earlier SOC models, close in form to the model of Ref. 1, yielded distributions of exponential shape. It was though realized later

that spatial correlations in the loading process can lead to power-law distributed waiting-times,^{5,9,10} and Ref. 11 found that, depending on the threshold applied in the burst detection, the waiting time distributions can turn from exponential to power-law shape, even in the case of the classical SOC model of Ref. 1. Waiting time statistics can therefore not be considered as an appropriate tool to test for SOC behavior in physical systems.

Several SOC models in the form of CA have been suggested for fusion plasmas that address different aspects of turbulent transport and that are able to reproduce a number of observed phenomena, including the transport suppressing role of sheared poloidal flows,¹² the scaling of transport characteristics with system size,¹³ the power-law shape of waiting time distributions,^{5,10,14} the occurrence of anomalous, superdiffusive transport events,¹⁵ the appearance of enhanced confinement, edge pedestals, edge localized modes, and the L - to H -mode transition,^{16–19} and the nondiffusive energy transport observed in off-axis heating experiments.²⁰

These SOC models are all CA models of the sand-pile type, and they basically are variants of the original SOC model of Ref. 1 and its generalizations by Ref. 21, which also allows nonlocal relaxation of the instabilities, and Ref. 22, which introduces the running sand-pile model that is continuously loaded. The basic elements of the models thus are sand-grains, height and height differences of sand-columns, with a usually vague identification of these system variables with the physical variables such as density, energy density, or temperature, and with an alike vague association of the system dynamics with actual physical processes. (A different approach is followed in Ref. 23, where a partial differential

equation is constructed that exhibits characteristics of SOC, and in which natural variables can be used.)

Here, we introduce a new SOC model in the form of a CA for ion temperature gradient (ITG) mode driven turbulence in fusion plasma. A main characteristic of the model is that it is constructed in terms of the usual physical variables and that the temporal evolution of the CA, which necessarily is in the form of rules, mimics actual physical processes as they are considered to be active in the system. All elements of the model are thus consistently interpretable in the usual physical way, no sand-grain or sand-pile analogy is used.

In the low β core plasma of a tokamak, where the magnetic field dominates over pressure, turbulence is driven by two main electrostatic microinstabilities: the ITG driven modes and the trapped electron modes (e.g., Ref. 24). In principal, the two instabilities can coexist, though in the case we consider here, where ion heating dominates, the ITG modes become unstable and actually are the dominant instability. In the strong turbulence regime then, the radial ion temperature profiles stay close to marginally stable, the gradients are stuck to their critical values, which is termed profile stiffness or profile consistency (e.g., Refs. 24–26). This plasma behavior is very reminiscent of SOC, and it is a main purpose of the application shown in this article to adequately model it.

From the CA and SOC modeling point of view, the necessary conditions for a system to be able to reach the state of SOC are: (i) there must be a driving process which systematically increases the “stress” in the system, (ii) a threshold dependent instability must be defined in some way, and (iii) a relaxation process must set on if somewhere the instability threshold is exceeded. All these processes should preferably act only in a local neighborhood, and they must be formulated in the form of discrete evolution rules. ITG mode driven turbulence obviously meets the three prerequisites for SOC: the system is systematically driven, namely heated, the ITG mode instability is threshold dependent, and there is obviously a process that relaxes the instabilities, given the fact that ITG mode driven turbulence reaches a saturated stationary state (e.g., Ref. 24).

Section II presents the model and explains how it is derived from usual physical processes in the plasma, Sec. III contains the results, and a discussion and the conclusions are given in Secs. IV and V, respectively.

II. THE MODEL

We consider a one dimensional grid along the direction of the minor radius a in a toroidal confinement device, such as the tokamak, with the modeled domain being $[R-sa, R+sa]$, with R the major radius and $s < 1$, so that only the core region is taken into account.

The basic scalar grid variable at the grid sites $x_i \in [R-sa, R+sa]$, for $i=1, \dots, L$, is the local ion temperature $T_i \equiv T(x_i)$, considered as a positive real number (i.e., assuming also noninteger values, in contrast to some sand-pile models). The grid is assumed to be equispaced, with grid-spacing $\Delta x = 2sa/(L-1)$. The grid-spacing is not considered

“infinitesimal,” as in the solution of differential equations, but it is finite, of the size of some smallest scale of interest, here of the size of a typical ion Larmor radius.

A. Instability criterion

As is well known, see, e.g., Refs. 27 and 28, the ITG mode instability has a critical dependence on the normalized scale length R/L_T , where L_T is the ITG scale length, $1/L_T := |\nabla T|/T$, in the sense that ITG mode instabilities are triggered if the condition

$$R \frac{|\nabla T|}{T} > \frac{R}{L_{\text{crit}}} \quad (1)$$

is fulfilled.

In order to use the instability criterion in the CA model, we have to determine the temperature gradient ∇T at the grid sites x_i . We thereto interpolate T_i locally in the neighborhood $(i-1, i, i+1)$ around the central site i with a second order polynomial, and we differentiate the polynomial at the point x_i , which yields

$$\delta_i := \frac{1}{2\Delta x} \frac{T_{i+1} - T_{i-1}}{T_i} \quad (2)$$

as an approximation to $\nabla T/T$ (and which is identical with the expression that a central difference scheme approximation for ∇T would yield). After all, the grid site i is considered to be unstable if

$$R|\delta_i| > R/L_{\text{crit}}. \quad (3)$$

B. Redistribution rules

We assume a normal diffusive process to be triggered around the grid site where an instability occurs, i.e., we assume an evolution according to a simple Fokker–Planck equation without advective term

$$\frac{1}{2}nk_B\partial_i T = -\partial_x q, \quad (4)$$

with k_B as Boltzmann’s constant, n as the number density, and q as the classical heat flux

$$q^{(cl)}(x) = -\chi nk_B\partial_x T, \quad (5)$$

where χ is the heat diffusivity.

1. General form of the redistribution rules

In the CA, the local diffusion process is formulated in terms of a set of *redistribution rules*, which describe how, in the case of an instability, the temperature in the local neighborhood is redistributed, the latter being defined here as consisting of the central unstable site, say i , and its two nearest neighbors: $i-1$ and $i+1$. The general form of the redistribution rules can be written as

$$T_i^+ = T_i + F_0(T_{i-1}, T_i, T_{i+1}; \delta_i, R/L_{\text{crit}}), \quad (6a)$$

$$T_{i\pm 1}^+ = T_{i\pm 1} + F_{\pm}(T_{i-1}, T_i, T_{i+1}; \delta_i, R/L_{\text{crit}}), \quad (6b)$$

where the values after redistribution are denoted with a plus sign (+) as superscript, and F_0 and F_{\pm} are the changes in

temperature, here written with their possible general functional dependence. In the derivation of specific redistribution rules, we will take into account actual physical processes and restrictions, and we will also make some assumptions.

2. Energy conserving local diffusion

We demand the redistribution rules to conserve the total temperature of the three involved grid sites. From a physical point of view, the energy should be conserved, i.e., the product of the local temperature and the local density, so that we actually make here the assumption that the density is constant, which can partly be justified by the small spatial extent of the local neighborhood, which is of the order of three ion Larmor radii, and by the fact that the density in tokamaks usually is quite close to uniform. It is to note that the local conservation of the grid variable in the CA is an important property for the system to be able to reach the SOC state. From Eq. (6), the condition for local temperature conservation can be written as

$$F_- + F_0 + F_+ = 0. \quad (7)$$

Assuming the density to be constant in a local neighborhood, Eqs. (4) and (5) reduce to a simple diffusion equation

$$\frac{1}{2}\partial_t T = \chi\partial_{xx}T, \quad (8)$$

where we have also assumed that the diffusivity χ is constant over a local neighborhood. Diffusion according to Eq. (8) results, under normal conditions, in the local smoothing of the temperature profile, and if the boundary conditions are fixed around the local neighborhood, a linear asymptotic profile will be reached (in the one dimensional case we treat here), where diffusion stops since $\nabla^2 T = \partial_x^2 T = 0$. The relaxation process in the CA should thus describe such a local profile smoothing, expressed in terms of rules, and for simplicity we assume here the smoothed profile to equal the asymptotic one and thus to be linear.

The assumption for the relaxed profile to be linear together with the assumption of temperature conservation already determine the temperature T_i^+ after relaxation at the central site i of an unstable region. Since the relaxed profile is linear, T_i^+ equals the mean temperature in the neighborhood

$$T_i^+ = \frac{1}{2}(T_{i-1}^+ + T_{i+1}^+), \quad (9)$$

into which we insert Eq. (6)

$$T_i + F_0 = \frac{1}{2}(T_{i-1} + F_- + T_{i+1} + F_+) \quad (10)$$

which we rearrange

$$2F_0 - F_- - F_+ = -2T_i + T_{i-1} + T_{i+1}. \quad (11)$$

Now we note that $2F_0 - F_- - F_+ = 3F_0 - [F_0 + F_- + F_+]$, the sum in the square bracket is though zero due to energy conservation [Eq. (7)], and Eq. (11) determines the increment at the central site

$$F_0 = -\frac{2}{3}\left[T_i - \frac{1}{2}(T_{i-1} + T_{i+1})\right]. \quad (12)$$

For convenience, we define

$$\tau_i := T_i - \frac{1}{2}(T_{i-1} + T_{i+1}). \quad (13)$$

In the following, the increments F_\pm of the nearest neighbors will be derived from a condition on the local heat fluxes.

3. Heat fluxes

In the relaxation events described by Eq. (6), heat is transported in the local neighborhood, and we quantify this heat transport by determining the associated heat fluxes. The starting point is the definition of the dynamic heat flux

$$q^{(dy)}(x) = \frac{1}{2}n(x)k_B T(x)\langle v(x) \rangle, \quad (14)$$

where $\langle v(x) \rangle$ is a local average velocity with which heat is flowing.

We consider a local neighborhood $i-1, i, i+1$ around the grid site i , which is unstable. The mean location s_i of the total heat content within the neighborhood can be defined by using the temperature values T_j as weights

$$s_i := \frac{\sum_{j=i-1}^{i+1} j \Delta x T_j}{\sum_{j=i-1}^{i+1} T_j} \quad (15)$$

and where we have made use of the assumption of constant density. Generally, s_i will not coincide with a grid point. During the relaxation event, the local heat content will move to a new mean location s_i^+ , so that the heat displacement is $s_i^+ - s_i$. This displacement takes place in one time-step of duration Δt , so that the velocity with which heat is transported is

$$v_i := (s_i^+ - s_i)/\Delta t. \quad (16)$$

According to Eq. (14), we can now define the heat flux of a relaxation event as

$$q_i := \frac{1}{2}nk_B v_i \sum_{j=i-1}^{i+1} T_j, \quad (17)$$

or, with Eqs. (16) and (15) and using the conservation of heat in relaxation events

$$\sum_{j=i-1}^{i+1} T_j = \sum_{j=i-1}^{i+1} T_j^+,$$

$$q_i^{(dy)} := \frac{1}{2\Delta t}nk_B \sum_{j=i-1}^{i+1} j \Delta x (T_j^+ - T_j). \quad (18)$$

Inserting the redistribution rules in the form of Eqs. (6) into Eq. (18) yields

$$q_i^{(dy)} = \frac{\Delta x}{2\Delta t}nk_B[(i-1)F_- + iF_0 + (i+1)F_+], \quad (19)$$

where the sum in the square brackets equals $i(F_- + F_0 + F_+) + F_+ - F_-$, which, with Eq. (7), reduces to $F_+ - F_-$, so that the heat flux finally writes as

$$q_i^{(dy)} = \frac{\Delta x}{2\Delta t} nk_B (F_+ - F_-). \quad (20)$$

4. Local diffusion of normal nature

In order for the local diffusion process to be of normal nature, we demand that heat can only be transported from a hotter to a colder site, since else we would introduce anomalous diffusive effects at the local scale. To ensure this condition, we demand that the dynamic heat flux $q_i^{(dy)}$, Eq. (20), equals the classical heat flux $q_i^{(cl)}$ of Fourier's law [Eq. (5)]. In this way, heat is transported always in the direction of the driving gradient ("downhill").

We discretize the gradient in Eq. (5) with the same method as used in the derivation of the instability criterion in Sec. II A

$$\partial_x T \approx \frac{T_{i+1} - T_{i-1}}{2\Delta x},$$

so that $q^{(cl)}(x) = -\chi nk_B \partial_x T$ is approximated as

$$q_i^{(cl)} = -\frac{1}{2\Delta x} \chi nk_B (T_{i+1} - T_{i-1}). \quad (21)$$

With Eqs. (20) and (21), the condition

$$q_i^{(dy)} = q_i^{(cl)} \quad (22)$$

writes as

$$\frac{\Delta x}{2\Delta t} nk_B (F_+ - F_-) = -\frac{1}{2\Delta x} \chi nk_B (T_{i+1} - T_{i-1}), \quad (23)$$

in which, due to energy conservation [Eq. (7)], we can replace $-F_-$ by $F_0 + F_+$,

$$(2F_+ + F_0) = -\frac{\Delta t}{(\Delta x)^2} \chi (T_{i+1} - T_{i-1}), \quad (24)$$

where F_0 is given by Eq. (12), so that the final form of the increment F_+ is

$$F_+ = -\frac{1}{2} \left[F_0 + \frac{\Delta t}{(\Delta x)^2} \chi (T_{i+1} - T_{i-1}) \right]. \quad (25)$$

With Eq. (7), the increment F_- follows as $F_- = -F_0 - F_+$ from Eq. (25),

$$F_- = -\frac{1}{2} \left[F_0 - \frac{\Delta t}{(\Delta x)^2} \chi (T_{i+1} - T_{i-1}) \right] \quad (26)$$

and, for convenience, we define the normalized heat diffusivity σ as

$$\sigma \equiv \frac{\Delta t}{(\Delta x)^2} \chi. \quad (27)$$

We note that we do have physical units for Δx , which equals the ion Larmor radius, Δt though is in arbitrary units, as explained in Sec. II E below, so that also σ necessarily is in arbitrary units.

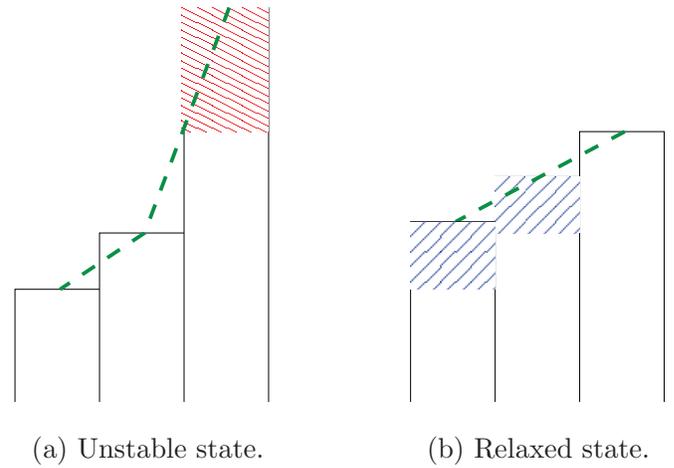


FIG. 1. (Color online) Sketch of the relaxation process. The hatched region in the unstable configuration (a) marks the excess amount of heat to be moved, and the hatched region in the relaxed state (b) marks the amount of heat received in the relaxation event. The dashed line connects the temperatures at the grid sites and represents the local temperature profile.

5. Summary of the relaxation rules

After all, the relaxation rules are [see Eqs. (6), (12), (13), and (25)–(27)]

$$T_i^+ = T_i - \frac{2}{3} \tau_i, \quad (28a)$$

$$T_{i\pm 1}^+ = T_{i\pm 1} + \frac{1}{3} \tau_i \mp \frac{\sigma}{2} (T_{i+1} - T_{i-1}), \quad (28b)$$

with free parameter the normalized heat diffusivity $\sigma > 0$, and where the increments F_0 and F_{\pm} in Eq. (6) are given as

$$F_0 = -\frac{2}{3} \tau_i, \quad (29a)$$

$$F_{\pm} = \frac{1}{3} \tau_i \mp \frac{\sigma}{2} (T_{i+1} - T_{i-1}). \quad (29b)$$

A sketch of the relaxation process is shown in Fig. 1.

The main effect of the local diffusion process in Eq. (4) should be to remove the cause of the diffusion, i.e., instabilities should be relaxed, and we numerically find that the redistribution rules we derived indeed relax the instabilities, i.e., it holds that $R|\delta_i^+| < R/L_{\text{crit}}$, as long as the normalized diffusivity is positive and not too small, $\sigma \gtrsim 0.01$.

Last, we note that the redistribution rules in Eq. (28), without though the parameter σ (i.e., formally for $\sigma=0$), were used in Refs. 29–31 in the astrophysical context of solar flares [in combination with an instability criterion different from Eq. (3)].

C. Boundary conditions

We assume a constant value T_b of the pedestal temperature outside the domain of the cellular automaton. T_b is used in the calculation of the instability criterion, Eq. (3), and in the redistribution rules, Eq. (28), at the edge of the system, formally entering the equations as $T_0 = T_{L+1} = T_b$.

The redistribution rules together with the initial conditions (see below) ensure that there is no heat in flow from the boundaries, since always $T_b < T_1$ and $T_b < T_L$, the temperature is always lower outside than inside the grid.

D. Driving process

The system undergoes a driving (heating) process, in which the temperature is increased, simulating the effect of heat injection. The loading process as such is not further specified, and it is understood to represent Ohmic heating as well as external heating.

Concerning the temporal characteristics of the loading process, we apply two variants: (i) we apply continuous loading, where the system is heated in every time-step (see the next section for a definition of the time-step), and (ii) we consider loading only in stable configurations, where there are no instabilities present in the system.

Moreover, we consider different spatial heating patterns: global heating everywhere in the system and heating only in localized regions. The latter case includes heating only in the central region, reminiscent of Ohmic heating, heating only in a region off-axis, as in the case of rf heating, and a combination of central and off-axis heating, representing Ohmic and off-axis heating applied simultaneously.

In either case of heating pattern, a grid site is chosen at random from the region that undergoes heating, and the temperature at this site is increased by a constant, predefined amount ΔS_j . The value of the parameter ΔS_j is chosen small enough to avoid overdriving the system, as usual for SOC models.

E. Description of the algorithm

The time evolution of the model is as follows: (i) In a first step, the grid is scanned, and a list of the unstable sites is created. (ii) If there are unstable sites, the instabilities are relaxed during a second scan of the grid. Prior to redistribution, each site is checked again for stability and it is relaxed only if it is still found to be unstable. The reason for checking again is the following: it may happen that two neighboring sites are simultaneously unstable, the redistribution rules are then first applied to just one of them, whose relaxation may cause the other site not to be unstable anymore, and it would be physically unmotivated to redistribute its temperature.

If the system is loaded only in the absence of instabilities, the steps (i) and (ii) are repeated until there are no instabilities in the system anymore, and only then the system is loaded again. Each completion of steps (i) and (ii) is considered a time-step, whose duration we consider to be $\Delta t = 1$ in arbitrary units. The repetition of steps (i) and (ii), from the first appearance of an instability until the system has reached an everywhere stable state, is termed an avalanche, as usual in the context of SOC models. Continuous loading of the system means that after the completion of a time-step the system is also loaded, independent of whether there are instabilities present in the system or not. In this case, there is no practical definition of avalanches possible.

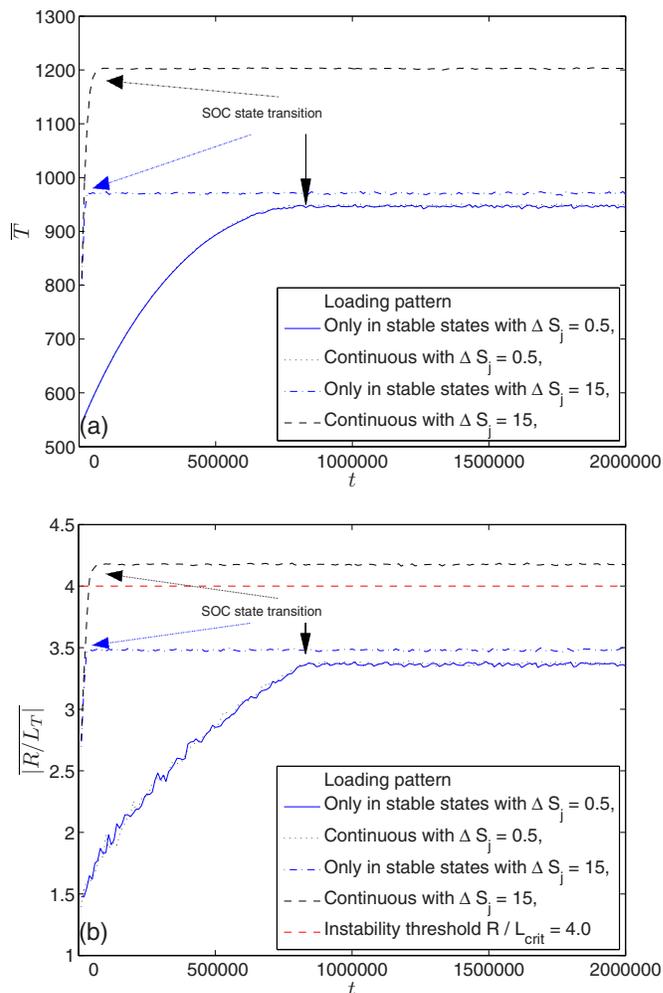


FIG. 2. (Color online) Spatial mean of (a) the temperature \bar{T} and of (b) the absolute normalized scale length $|R/L_T|$ as a function of time, for heating only in stable states and for continuous heating, and with two different heating intensities. The horizontal dashed line in (b) shows the instability threshold R/L_{crit} , while the arrows mark the transition of the system to the SOC state.

III. RESULTS

A. Parameters

The parameters of the model are chosen to coincide with those of the Joint European Torus³² (JET) experiment. We thus assume a minor radius $a = 1.25$ m and a major radius $R = 2.95$ m, and since the ITG instabilities are considered to dominate turbulence only in the core, we restrict the spatial domain used in the simulations to $[R - 0.8a, R + 0.8a]$. The grid-size Δx is of the same order as the ion Larmor radius ρ_i , so that, with $\rho_i \approx 0.49$ cm under typical conditions for JET, we use a grid of $L = 401$ sites with grid-size $\Delta x = 0.5$ cm.

According to Ref. 28, the threshold R/L_{crit} of the ITG mode instability can assume values in the range of 3.5–5. In the following, and unless stated otherwise, the instability threshold is set to $R/L_{crit} = 4$, the normalized diffusivity to $\sigma = 0.5$, the heating increment to $\Delta S_j = 0.5$, and we apply a constant value of the pedestal temperature $T_b = 500$ outside

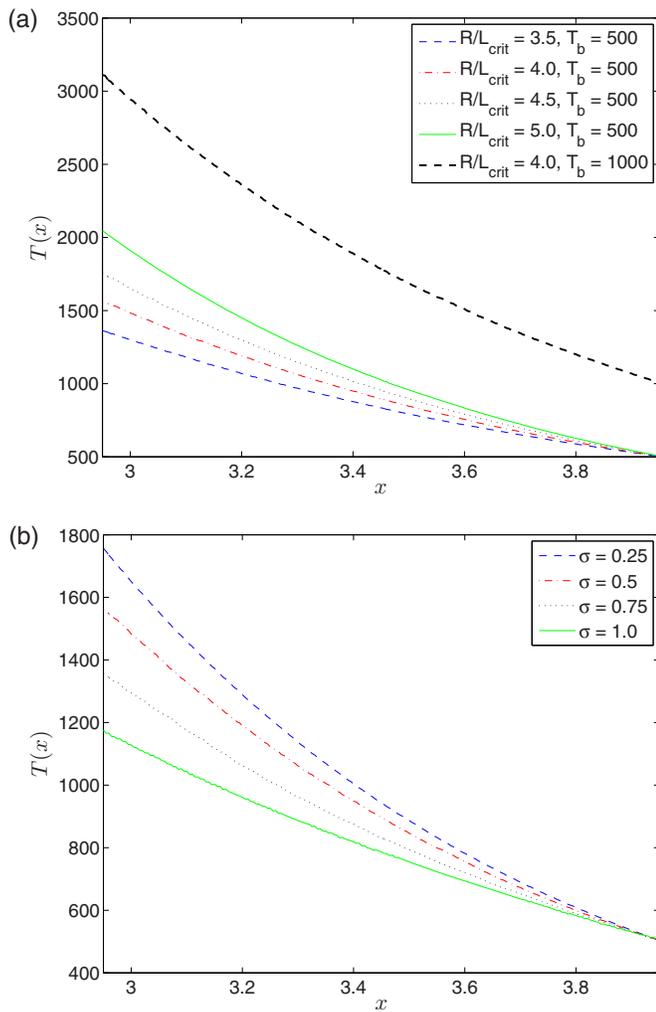


FIG. 3. (Color online) Temperature profiles during the SOC state for (a) different values of R/L_{crit} and T_b , and (b) for different values of σ , all with loading only in stable configurations.

the domain of the model as boundary condition. For the initial configuration we let the temperature everywhere equal T_b .

The basic free parameters of the model, which mainly determine the results and which can be used to adjust the model to experimental data, are the threshold R/L_{crit} , the pedestal temperature at the boundaries T_b , and the normalized diffusivity σ .

B. The SOC state

Figure 2 shows the spatial mean of the temperature $\bar{T}(t)$ and of the absolute local normalized scale length $|R/L_T|$ for typical simulations with global heating, one with loading only in stable configurations and one with continuous loading, and for two different heating intensities. There is a transient phase, in which both mean values are growing, and after which there is a turnover to a dynamic equilibrium state, where the mean values fluctuate around an asymptotic value. Once the asymptotic state is reached, avalanches of widely varying sizes start to appear, from very small ones, bounded to a local region, to avalanches that sweep through

the entire system. The reaching of an asymptotic value of $|R/L_T|$ and $\bar{T}(t)$, together with the appearance of avalanches of all sizes, are indicative for the system to have reached the SOC state. The duration of the transient phase depends on the intensity of the driving process. It is to note that, in the case of intense continuous loading, the mean normalized scale length can also be above critical, whereas in the case of loading only in stable configurations it is always below critical.

Temperature profiles in the SOC state are shown in Fig. 3(a) for different values of the threshold R/L_{crit} and pedestal temperature T_b (here and in the following, the right halves of the symmetric profiles are shown). In all cases, the profiles are of exponential shape, and the threshold, together with the value T_b of the pedestal temperature at the boundaries, determine the maximum value that is reached in the center: the central temperature increases with increasing threshold as well as with increasing pedestal temperature, the latter moreover causing a shift of the entire temperature profile toward higher values.

During the SOC state, the dynamically evolving temperature profiles stay very close to the characteristic exponential shape, even the distortions caused by large avalanches would hardly be visible in the scales of Fig. 3. This implies that the total energy content of the system is, within small fluctuations, constant, heat is ejected from the system at the boundaries at the same statistical rate it is injected by heating.

In Fig. 3(b), we show the temperature profiles during SOC state for different values of the normalized diffusivity σ , keeping the pedestal temperature and the threshold fixed. The central peak temperature decreases with increasing diffusivity, a large σ causes heat to be redistributed faster in the system and also to be more efficiently ejected at the edge.

C. Central and off-axis heating

In order to investigate the impact of off-axis heating on the temperature profile, we apply two spatial heating patterns: one for pure central heating and one for simultaneous central heating and equally strong off-axis heating (the central region being defined as the grid sites $i \in [151, 251]$, and the off-axis region as the sites $i \in [351, 401]$). In Fig. 4, the temperature profiles yielded by the two simulations are shown during the SOC state. Also the profile from off-axis heating is clearly peaked at the center, there is no sign of the off-axis heat source visible, and the difference between the two profiles is actually very small. This implies that the temperature profiles exhibit a very high degree of stiffness; they are basically not influenced by the applied spatial heating pattern. This also implies that there is strong anomalous transport, where heat is transported against the driving gradient (uphill), despite the fact that diffusion at the local level is imposed to be of a normal kind.

D. Continuous loading

In the cases of temperature profiles shown so far, heating was applied only when there were no instabilities present in the system. This is justified physically only if the ITG mode

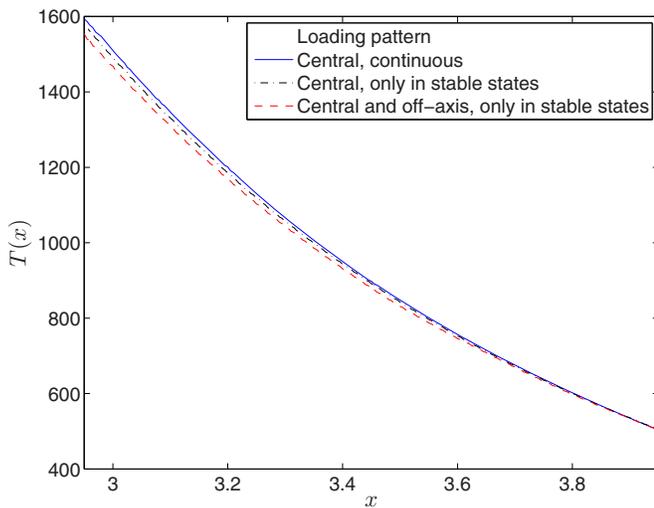


FIG. 4. (Color online) Temperature profiles for central heating (dotted-dashed), and for simultaneous central and off-axis heating (dashed), both with heating only in stable states, and for central, continuous heating (solid).

instabilities together with their relaxation have a much faster time-scale than the heating process. When the two time-scales are comparable, loading should take place more frequently.

In Fig. 4, we also show a temperature profile for continuous heating in the central region, here heating in every time-step. The profile is also of exponential shape, and it is similar to the one with heating only in stable states, the temperature values though are slightly larger because there is more heat injected into the system in this case, and the system has less time to eject heat at the edges (see also Fig. 2). In the case of continuous heating, the value of the central peak temperature depends also on the heating increments ΔS_j , it increases with increasing ΔS_j , whereas in the case of loading only in stable configurations it is almost independent of ΔS_j .

E. Distribution of heat out-fluxes

The local heat flux per relaxation event is given by $q_i^{(dy)}$ in Eq. (20), so that the total heat flux in a time-step (see Sec. II E) is

$$Q_t := \sum_{i=1}^L q_i(t) \quad (30)$$

and the total internal heat flux of an avalanche q_{int} follows as

$$q_{int} := \sum_{t \in \text{avalanche}} Q_t. \quad (31)$$

Correspondingly, the total heat out-flux per avalanche at the right edge $q_{0.8}$ is given by setting $i=L$ in Eq. (20) (i.e., considering the radial position $x=0.8a$) and summing over the avalanche

$$q_{0.8} := \sum_{t \in \text{avalanche}} q_L(t). \quad (32)$$

For the case of global heating in stable configurations, Fig. 5 shows the normalized distributions of the internal

fluxes q_{int} and the heat out-fluxes $q_{0.8}$ from a large number of avalanches, and for different values of the parameters ΔS_j , R/L_{crit} , and σ . All the distributions are of clear single or double power-law shapes in the intermediate range, extending over roughly three decades for q_{int} and 1.5 decades for $q_{0.8}$. The larger extent of the power-laws of q_{int} reflects the fact that, from its definition, q_{int} has a larger dynamic range. The power-laws we find, together with Fig. 2, are indicative for the system to be in the state of SOC.

The indices α and β of the fitted power-laws [$p(q_{int}) \propto q_{int}^{-\alpha}$ and $p(q_{0.8}) \propto q_{0.8}^{-\beta}$, respectively] are summarized in Table I. (1) The power-law indices are basically independent of the threshold R/L_{crit} . (2) They depend on the heating intensity ΔS_j , the dependence is though very weak if the ΔS_j are small, i.e., for low-level heating, the latter being defined in the sense that the system, when in SOC state and just having become stable, must on the average be heated with much more than just one heat increment ΔS_j in order to get unstable again. (3) The power-law indices clearly depend on the diffusivity, they decrease and the distributions become flatter with increasing σ , a high diffusivity at the local level thus favors large heat fluxes at the macroscopic scale, as one also would intuitively expect.

IV. DISCUSSION

A. Marginally stable profiles

From Eq. (1), we can analytically determine the marginally stable configuration, defined through $|\nabla T(x)|/T(x) = 1/L_{crit}$, which has the general solution $T(x) = ce^{-|x|/L_{crit}}$, and where c depends on the boundary conditions we apply, $T(x=\ell) = T_b$ (with $\ell := 0.8a$), from where $c = T_b e^{\ell/L_{crit}}$, so that

$$T(x) = T_b e^{-(|x|-\ell)/L_{crit}}. \quad (33)$$

(An exponential, marginally stable profile is also derived in Ref. 33, in the frame of a critical gradient transport model.) The dynamic profile of the model in the SOC state follows the shape of the marginally stable profile, with a smaller decay constant though, i.e., reaching lower temperature and thus being submarginal when the system is loaded only during stable states, and with a larger than critical decay constant in the case of intense enough continuous loading (see Fig. 2). The maximum temperature in the center is $T(x=0) = T_b e^{\ell/L_{crit}}$ according to Eq. (33), and in the numerical simulations we indeed find the peak temperature to be proportional to T_b and $1/L_{crit}$ [see Fig. 3(a)], and to be dependent on ℓ such that larger systems reach larger temperatures in the center.

It is worthwhile noting that Eq. (33) does not give a complete qualitative description of the system dynamics, it does not contain any information on the local relaxation process and therewith on the third free parameter, the normalized diffusivity σ . The latter has though a decisive influence on, e.g., the central peak temperature reached, as Fig. 3 shows, the peak temperature increases with decreasing σ , and the profile actually approaches the marginally stable profile for very small σ .

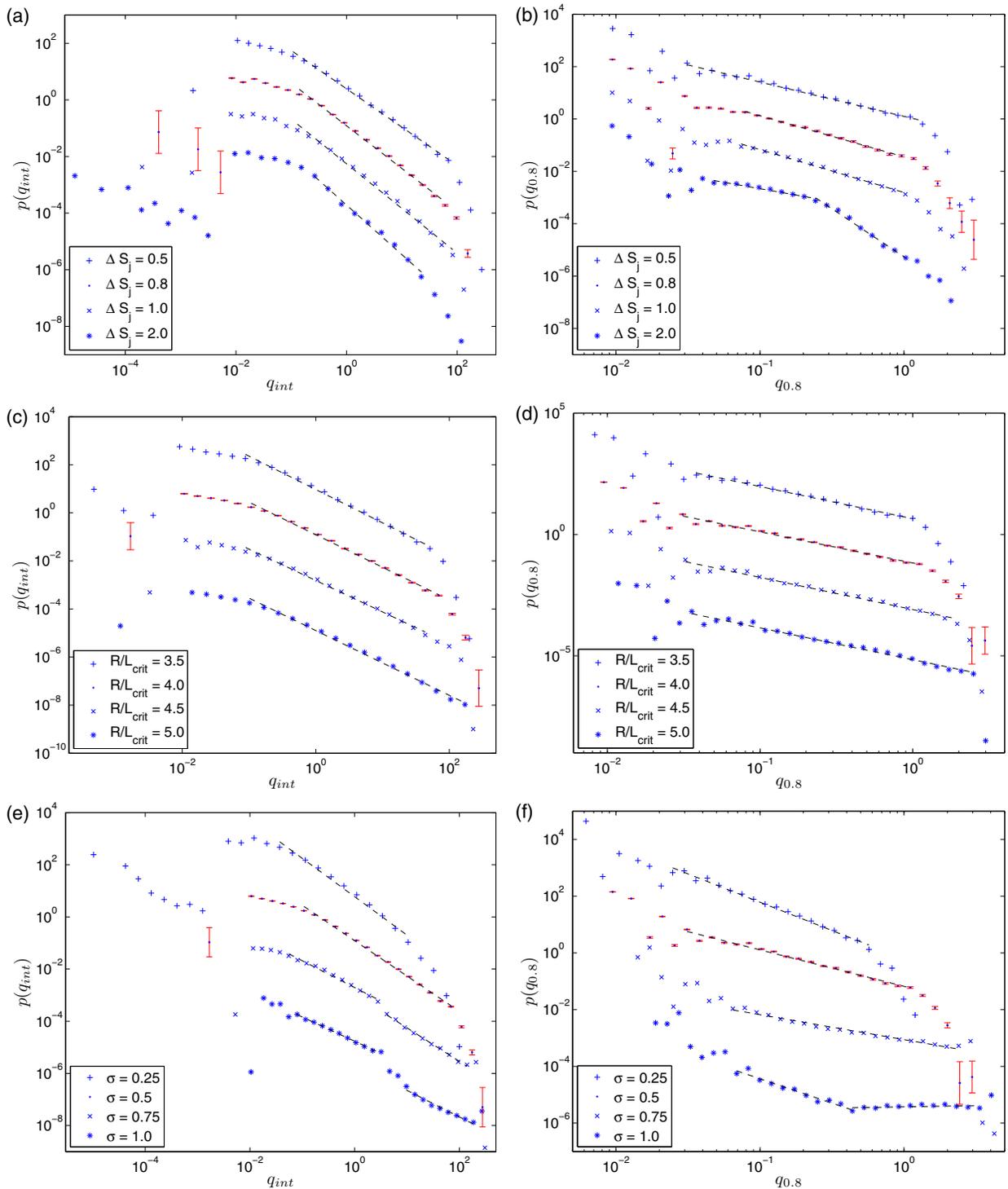


FIG. 5. (Color online) Distributions (normalized histograms) of the total heat fluxes per avalanche q_{int} [(a), (c), and (e)] and total heat-out-fluxes per avalanche $q_{0.8}$ [(b), (d), and (f)], for different loading intensities ΔS_j [(a) and (b)], for different thresholds R/L_{crit} [(c) and (d)], and for different diffusivities σ [(e) and (f)], together with power-law fits (dashed lines). [The distributions are shifted in the vertical direction for better visualization, and in each subfigure error bars (red) are shown for reference in one case.]

B. Comparison with experimental data

Experimental ion temperature profiles in tokamaks are well known to be very stiff and of exponential shape in the inner core region (see, e.g., Ref. 33). Examples of published, exponentially shaped ion temperature profiles include different devices and confinement modes, e.g., the L -mode in

Tokamak Fusion Test Reactor³⁴ (Fig. 1 in Ref. 27), the H-mode in ASDEX Upgrade³⁵ (Fig. 2 in Ref. 36), the L -mode in ASDEX Upgrade (Fig. 2 in Ref. 37), and the L -mode in DIII-D (Ref. 38) tokamak [Fig. 1(c) in Ref. 39], and the high degree of stiffness in experimental ion temperature profiles is presented and analyzed in e.g., Ref. 28. Moreover, Ref. 3 experimentally finds power-law distributions for

TABLE I. Power-law indices α and β from the power-law fits to the distributions of the total internal fluxes q_{int} and the total out-fluxes $q_{0.8}$, respectively, in Fig. 5.

ΔS_j	α	β
0.5	1.36	1.30
0.8	1.51	1.57
1.0	1.58	1.63
2.0	1.75	0.97 and 3.67
R/L_{crit}	α	β
3.5	1.39	1.30
4.0	1.36	1.30
4.5	1.30	1.30
5.0	1.35	1.30
σ	α	β
0.25	1.47	2.00
0.50	1.36	1.30
0.75	1.03 and 1.28	0.90
1.00	0.93 and 1.02	1.70 and -0.90

particle fluxes (with power-law index 1), which can be related to heat fluxes under the assumption that the heat flux is convective.

We thus can conclude that generally a good qualitative agreement of the model with experiments can be achieved with respect to (i) the exponential profile shape (Sec. III B), (ii) the high profile stiffness (Sec. III C), and to some degree there is also a qualitative agreement with respect to (iii) the power-law distribution of out-fluxes (Sec. III E). Last, we note that, having used parameters of the JET experiment (concerning the system size, ion Larmor radius, and pedestal temperature T_b) and assuming T_b to be in units of eV, (iv) we naturally find a dynamic range of the ion temperature that is comparable to the one experimentally seen at JET, see, e.g., Fig. 2(e) in Ref. 28 in comparison with our Fig. 3. For a quantitative comparison, we would need to find a way to calibrate the time and therewith the normalized diffusivity in the model.

C. Remarks on the SOC modeling approach

The purpose of the study presented here was to introduce a SOC model for the ion temperature dynamics in the core region of tokamaks, with the elements of the model being physically interpretable in a sound and consistent way. This implies that in the derivation of the instability criterion, relaxation rules, and loading process, we had to be guided by and take into account the basic physical variables and processes that are active in the system (to the degree that they are known).

The SOC model was constructed from four basic pieces of information: (i) the form of the instability criterion, Eq. (1), which was taken from experiments on ITG mode driven turbulence (Fig. 1 in Ref. 28), (ii) a simplified Fokker-Planck equation from general transport theory, Eqs. (4) and (8), (iii) the definition of the classical [Fourier's law, Eq. (5)]

and dynamical [Eq. (14)] heat flux, and (iv) the concept of SOC in its operational definition (basically following Ref. 1, and also Ref. 29).

The operational definition of SOC actually expresses the necessary conditions for an extended system to reach the SOC state, which partly were already mentioned in Sec. I. An extended system disposed to SOC must (a) systematically be driven, (b) allow local, threshold-dependent instabilities, (c) possess a mechanism that relaxes the instabilities locally, and (d) the system variable considered must be conserved in the relaxation events. In the model construction, the concept of SOC played the guiding role, in that it determines the elements of the model and their form, it contains the formal prerequisites for the final model to be a realization of the SOC concept.

In the classical sand-pile model of Ref. 1 and in many of its descendants mentioned in Sec. I, instabilities occur if the height difference between neighboring sand-columns exceeds a threshold, where the height difference can be interpreted as a very rough approximation to the gradient ∇h of the local sand-column height h . In the tokamak core plasma though, instabilities occur when $\nabla T/T$ exceeds a threshold (see Sec. II A), not ∇T . The core plasma thus belongs to a formally different class of unstable systems than the sand-pile, which implies that we cannot make use of the sand-pile paradigm in core plasma SOC modeling, as long as it is our explicit aim to incorporate the actual physics of the problem under consideration as close as possible into the SOC model.

It is worthwhile noting that SOC models that belong to different instability classes exhibit different characteristic spatial profiles of the grid variable. The sand-pile models, with an instability criterion formulated in terms of an approximation to the gradient ∇h of the sand-column height, exhibit linear profiles (see Ref. 1). In an astrophysical context, Refs. 29–31 use an approximation to the Laplacian, $\nabla^2 \mathbf{A}$, in the instability criterion, with \mathbf{A} the vector potential of the magnetic field, and they find parabolic spatial profiles for the components of \mathbf{A} . The instability criterion used here, an approximation to $\nabla T/T$, leads to exponential spatial profiles, as shown in Sec. III. The reason for these characteristic differences is that, in all cases, the spatial profiles in SOC state follow the marginally stable profiles in shape, which are different for the different classes of instabilities.

Last, we note that our model has a high computational efficiency, which is a great advantage over, e.g., the gyrokinetic approach, it allows for fast interpretation and analysis of experimental results, and, after some further development, it even might be of potential use for plasma control. For example, the temperature profiles can easily be modeled globally, in the entire core region, heating patterns of any kind can easily be applied and explored, and the statistics of the heat fluxes can be determined very accurately, since the model can be monitored over very long times.

V. CONCLUSION

We introduced a CA model that implements the basic physics of ITG driven turbulence, as relevant for the core of fusion plasmas in tokamak devices, in L -mode as well as in

H-mode. The model is formulated in terms of evolution rules, which makes it computationally very efficient. The rules implement two processes: heating by simple heat deposition, and local diffusion if a threshold in the normalized ITG R/L_T is exceeded, whereby it is ensured that energy is conserved and that heat is never transported from a cooler to a hotter site. The model is formulated in terms of the usual physical variables, also in what the threshold is concerned, and the actual physical processes are directly mimicked in terms of rules.

The model also represents an implementation of SOC, which it always reaches after an initial transient phase. In the SOC state then, the model yields symmetric ion temperature profiles of exponential shape. These profiles exhibit very high stiffness, in that they basically are independent of the loading pattern applied (central and off-axis heating yield the same profile). This implies that there is anomalous heat transport (“uphill” heat transport, against the driving gradient) present in the system, despite the fact that diffusion at the local level is imposed to be of a normal kind.

In a qualitative comparison of the model’s basic properties with experimental data, we find good agreement, at least for time instances where the experimental profiles also exhibit an exponential shape. We thus can conclude that the physical system we investigate, i.e., ITG driven turbulence, is qualitatively compatible with the SOC state of our model.

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