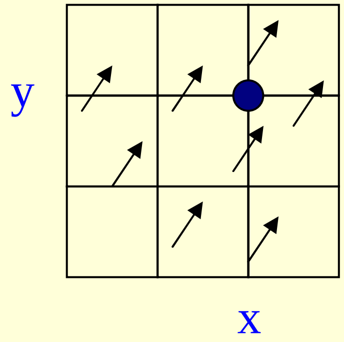
Eulerian vs. Lagrangian Description

The flow of a fluid can be described in two different, but equivalent ways:

A) *Eulerian description*:

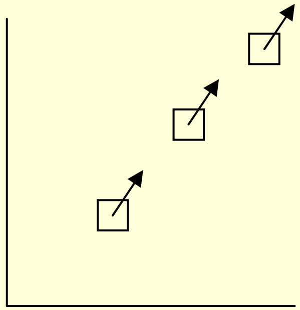
All fluid properties are measured with respect to a *fixed coordinate system*. Time variations are described by *local derivatives* at a given coordinate location, e.g.



$$\frac{\partial \rho}{\partial t}$$

B) *Lagrangian description*:

All fluid properties are measured with respect to a *moving control mass* of variable volume *V(t)* occupied by the same particles at all times. Time variations are described by *total derivatives* at the location of the control mass, e.g.



$$\frac{D\rho}{Dt}$$

Relation between the two derivatives:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$$

Reynold's Transport Theorem (I)

For a *scalar field* ϕ along a moving control mass:

$$\boxed{\frac{D}{Dt}\int_{V(t)} \phi dV = \int_{V(t)} \frac{\partial \phi}{\partial t} dV + \int_{A(t)} \phi \vec{v} \cdot \hat{n} dS}$$

where:

$V(t)$	volume of control mass
$A(t)$	surface area of volume *V(t)*
$\hat{n}$	unit normal vector to surface *A(t)*

Using Gauss' theorem: $\int_A \vec{u} \cdot \hat{n} dS = \int_V \vec{\nabla} \cdot \vec{u} dV$ we obtain:

$$\boxed{\frac{D}{Dt}\int_{V(t)} \phi dV = \int_{V(t)} \left(\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot (\phi \vec{v}) \right) dV}$$

On the other hand, if V_{cv} is a *fixed control volume* (Eulerian description), then one can show that:

$$\boxed{\frac{D}{Dt}\int_{V(t)} \phi dV = \frac{d}{dt}\int_{V_{cv}} \phi dV + \int_{A_{cv}} \phi \vec{v} \cdot \hat{n} dS}$$

Reynold's Transport Theorem (II)

For a *vector field* $\vec{\Phi}$ along a moving control mass:

$$\begin{aligned}\frac{D}{Dt}\int_{V(t)} \vec{\Phi}\, dV &= \int_{V(t)} \frac{\partial \vec{\Phi}}{\partial t} dV + \int_{A(t)} \vec{\Phi} \otimes \vec{v} \cdot \hat{n}\, dS \\ &= \int_{V(t)} \left(\frac{\partial \vec{\Phi}}{\partial t} + \vec{\nabla} \cdot (\vec{\Phi} \otimes \vec{v}) \right) dV \\ &= \frac{d}{dt}\int_{V_{cv}} \vec{\Phi}\, dV + \int_{A_{cv}} \vec{\Phi} \otimes \vec{v} \cdot \hat{n}\, dS\end{aligned}$$

where $\vec{\Phi} \otimes \vec{v}$ is the *tensor product*:

$$\vec{\Phi} \otimes \vec{v} = \begin{bmatrix} \Phi_1 v_1 & \Phi_1 v_2 & \Phi_1 v_3 \\ \Phi_2 v_1 & \Phi_2 v_2 & \Phi_2 v_3 \\ \Phi_3 v_1 & \Phi_3 v_2 & \Phi_3 v_3 \end{bmatrix} \Rightarrow (\vec{\Phi} \otimes \vec{v})_{ij} = \Phi_i v_j$$

Conservation of Mass

The *total mass* of the fluid is

$$M = \int_V \rho dV$$

and it is conserved, if there is no mass production inside the fluid.

By definition, the mass within the volume *V(t)* of a moving control mass is conserved:

$$\frac{D}{Dt} \int_{V(t)} \rho dV = 0$$

Using Reynold's transport theorem, we find that:

$$\boxed{\frac{d}{dt} \int_{V_{cv}} \rho dV = - \int_{V_{cv}} \vec{\nabla} \cdot (\rho \vec{v}) dV}$$

integral form

or, equivalently

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0}$$

differential form

which is called the *continuity equation*.

Conservation of Momentum (I)

The *total momentum* of the fluid is

$$\vec{P} = \int_V \rho \vec{v} dV$$

and it is conserved, if there are *no external forces or torques* acting on the fluid.

However, the momentum contained within the volume V(t) of a moving control mass changes with time, due to *volume* and *surface forces* acting on this control mass.

Volume forces:

$$\vec{f}_V = \int_{V(t)} \rho \vec{g} dV$$

where e.g. $\vec{g}$ is the *acceleration of gravity*.

Surface forces:

$$\vec{f}_S = \int_{A(t)} \mathbf{S} \cdot \hat{n} dS = \int_{V(t)} \vec{\nabla} \cdot \mathbf{S} dV$$

where $\mathbf{S}$ is the stress tensor:

$$\mathbf{S} = -p\mathbf{I} + \mathbf{\Pi}$$

Here, *p* is the *isotropic pressure*, **I**=diag[1,1,1] is the *unit tensor* and **Π** is the *viscous part* of the stress tensor.

Conservation of Momentum (II)

According to Newton's 2nd law, the rate of change of the momentum of the moving control mass is equal to the sum of all forces acting on it:

$$\frac{D}{Dt}\int_{V(t)} \rho\vec{v}dV = \vec{f}_V + \vec{f}_S$$

Using Reynold's transport theorem:

$$\frac{d}{dt}\int_{V_{cv}} \rho\vec{v}dV + \int_{A_{cv}} \rho\vec{v}\otimes\vec{v}\cdot\hat{n}dS = \int_{V_{cv}} \rho\vec{g}dV + \int_{A_{cv}} (-p\mathbf{I}+\mathbf{\Pi})\cdot\hat{n}dS$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}\int_{V_{cv}} \rho\vec{v}dV = \int_{V_{cv}} \left[\vec{\nabla}\cdot(-\rho\vec{v}\otimes\vec{v} - p\mathbf{I} + \mathbf{\Pi}) + \rho\vec{g}\right] dV}$$

integral form

or, equivalently:

$$\int_{V(t)} \left(\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}\otimes\vec{v})\right) dV = \int_{V(t)} \rho\vec{g}dV + \int_{V(t)} \vec{\nabla}\cdot(-p\mathbf{I}+\mathbf{\Pi})dV$$

$$\Rightarrow \quad \boxed{\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}\otimes\vec{v} + p\mathbf{I} - \mathbf{\Pi}) = \rho\vec{g}}$$

differential form

Conservation of Energy (I)

The *total energy density* is the sum of the *kinetic energy density* and the *internal energy density*:

$$\boxed{E = \frac{1}{2}\rho v^2 + \rho e}$$

The *total energy* contained within the volume of a moving control mass is

$$E_V = \int_{V(t)} E dV$$

During a fluid flow, the total energy of a moving control mass can change due to several reasons:

a) *Volume forces*: The *work done per unit time* by the volume forces

$$\vec{f}_V = \int_{V(t)} \rho \vec{g} dV$$

is

$$\boxed{W_V = \int_{V(t)} \rho \vec{v} \cdot \vec{g} dV}$$

b) *Surface forces*: The *work done per unit time* by the surface forces

$$\vec{f}_S = \int_{A(t)} \mathbf{S} \cdot \hat{n} dS$$

is

$$\boxed{W_S = \int_{A(t)} \vec{v} \cdot \mathbf{S} \cdot \hat{n} dS}$$

Conservation of Energy (II)

c) *Energy flow*: If energy is flowing into or out of the volume (e.g. due to heat conduction and $\vec{Q}$ is the energy flow vector field (energy per unit area, per unit time), then the energy flowing into or out of the volume per unit time is

$$W_Q = -\int_{A(t)} \vec{Q} \cdot \hat{n} dS$$

d) *Addition of heat*: If, in addition, heat is also added directly to each particle, e.g. using microwaves, at a rate q per unit mass, per unit time, then the corresponding energy added per unit time is

$$W_q = \int_{V(t)} \rho q dV$$

The rate of change of the total energy of a moving control mass is then

$$\frac{DE_V}{Dt} = W_V + W_S + W_Q + W_q$$

and using Reynold's transport theorem:

$$\Rightarrow \quad \frac{d}{dt}\int_{V_{cv}} E dV + \int_{A_{cv}} E\vec{v}\cdot\hat{n} dS = W_V + W_S + W_Q + W_q$$

Conservation of Energy (III)

$$\Rightarrow \quad \frac{d}{dt}\int_{V_{cv}} E dV + \int_{A_{cv}} E\vec{v}\cdot\hat{n} dS = \int_{V_{cv}} \rho\vec{v}\cdot\vec{g} dV + \int_{A_{cv}} \vec{v}\cdot\mathbf{S}\cdot\hat{n} dS - \int_{A_{cv}} \vec{Q}\cdot\hat{n} dS + \int_{V_{cv}} \rho q dV$$

$$\Rightarrow \quad \frac{d}{dt}\int_{V_{cv}} E dV = \int_{V_{cv}} \left[-\vec{\nabla}\cdot(E\vec{v}) + \rho\vec{v}\cdot\vec{g} + \vec{\nabla}\cdot(\vec{v}\cdot\mathbf{S}) - \vec{\nabla}\cdot\vec{Q} + \rho q\right] dV$$

$$\Rightarrow \quad \boxed{\frac{d}{dt}\int_{V_{cv}} E dV = \int_{V_{cv}} \left[\vec{\nabla}\cdot\left(-(E+p)\vec{v} + \vec{v}\cdot\mathbf{\Pi} - \vec{Q}\right) + \rho(\vec{v}\cdot\vec{g} + q)\right] dV}$$

integral form

The corresponding differential form is:

$$\boxed{\frac{\partial E}{\partial t} + \vec{\nabla}\cdot\left((E+p)\vec{v} - \vec{v}\cdot\mathbf{\Pi} + \vec{Q}\right) = \rho(\vec{v}\cdot\vec{g} + q)}$$

differential form

Summary of Equations

The system of differential equations governing fluid dynamics is:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v} + p\mathbf{I} - \mathbf{\Pi}) = \rho \vec{g}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \left((E + p)\vec{v} - \vec{v} \cdot \mathbf{\Pi} + \vec{Q} \right) = \rho(\vec{v} \cdot \vec{g} + q)$$

This system is in *first-order hyperbolic form*:

$$\boxed{\partial_t \vec{U} + \partial_x \vec{F}(\vec{U}) + \partial_y \vec{G}(\vec{U}) + \partial_z \vec{H}(\vec{U}) = \vec{S}(\vec{U})}$$

where $\vec{U} = \{\rho, \rho v_x, \rho v_y, \rho v_z, E\}$ is the *state vector* of unknowns

$\vec{F}, \ \vec{G}, \ \vec{H}$ are the *flux vectors*, and

$\vec{S}$ is the *source vector*.

Viscous Stresses

In the *Newtonian approximation*, one assumes that viscous stresses still obey Hooke's linear law (deformation is linearly proportional stress). In this approximation, the *viscous part of the stress tensor* is written as:

$$\mathbf{\Pi} = 2\eta\mathbf{D} + \left(\eta_b - \frac{2}{3}\eta\right)\vec{\nabla}\cdot\vec{v}\mathbf{I}$$

where

$$\mathbf{D} = \frac{1}{2}\left[\vec{\nabla}\otimes\vec{v} + (\vec{\nabla}\otimes\vec{v})^T\right]$$

is the *deformation tensor*, and

η	coefficient of *shear viscosity*
η_b	coefficient of *bulk viscosity*

For monatomic gases $\eta_b = 0$, while for polyatomic gases $\eta_b \neq 0$.

The shear viscosity depends strongly on temperature and only slightly on pressure. When approximating $\eta = \eta(T)$, *Sutherland's formula* holds:

$$\eta = C_1\left(1 + \frac{C_2}{T}\right)^{-1}\sqrt{T}$$

Heat Flow

Heat flow can be the result of:

1) Heat flow due to *temperature gradients*
2) *Diffusion processes* in gas mixtures
3) *Radiation*

In the case of temperature gradients, it has been found that *Fourier's law* holds:

$$\boxed{\vec{Q} = -k\vec{\nabla}T}$$

where k is the coefficient of thermal conductivity, depending strongly on T and only slightly on pressure, just as the shear viscosity coefficient. From molecular theory:

$$k \propto \eta$$

If c_p is constant, then one can define the *Prandtl number*

$$P_r \equiv \frac{c_p \eta}{k} = \frac{4\gamma}{9\gamma - 5} \qquad \text{(Eucken)}$$

Since P_r is of order unity, this means that heat flow and viscosity must both be treated consistently.

Entropy Equation

An *alternative form* of the energy equation, in terms of the *specific internal energy*, is:

$$\rho \frac{De}{Dt} - \vec{\nabla} \otimes \vec{v} \cdot \mathbf{S} + \vec{\nabla} \cdot \vec{Q} = \rho q$$

Exercise 2a

Using the first law of thermodynamics, we can derive an equation for the dynamical evolution of the *specific entropy*:

$$T \frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt}$$

$$\Rightarrow \quad T \frac{Ds}{Dt} = \frac{\vec{\nabla} \otimes \vec{v} \cdot \mathbf{\Pi}}{\rho} - \frac{\vec{\nabla} \cdot \vec{Q}}{\rho} + q$$

Exercise 2b

Inviscid flow (perfect fluid):

$$\mathbf{\Pi} = 0$$

Adiabatic flow of perfect fluid is *isentropic:*

$$\vec{Q} = 0, \quad q = 0 \quad \Rightarrow \quad \frac{Ds}{Dt} = 0$$ in smooth flow

$\Rightarrow s =$ constant along particle paths

Isentropic and Homentropic Flows

Along a particle path in an isentropic flow,

$$p = K(s_0)\rho^{\gamma}$$ Exercise 3a

where $K(s_0)$ is a constant, depending on the specific entropy of the particle path at t=0. Alternatively, one can show that

$$T\rho^{1-\gamma} = \text{const.}$$ Exercise 3b

If the entropy is constant everywhere, at all times, the flow is called *homentropic*.

Entropy

If c_v, c_p = constant, then the specific entropy can be expressed as

$$s = s_0 + c_v \ln\left(\frac{p}{\rho^{\gamma}}\right)$$ Exercise 4a

where s_0 is a reference value.

Barotropic and Incompressible Flows

If the temperature or the specific entropy are constant everywhere, e.g.

$$T = T_0 = \text{ constant (isothermal flow)}$$

$$s = s_0 = \text{ constant (homentropic flow)}$$

then the pressure depends only on the density

$$\boxed{p = p(\rho)}$$

and the flow is called *barotropic*. Then:

$$\boxed{\frac{1}{\rho}\vec{\nabla}p = \vec{\nabla}h}$$

Exercise 4b

where $h = \int \frac{dp}{\rho}$ is the specific enthalpy.

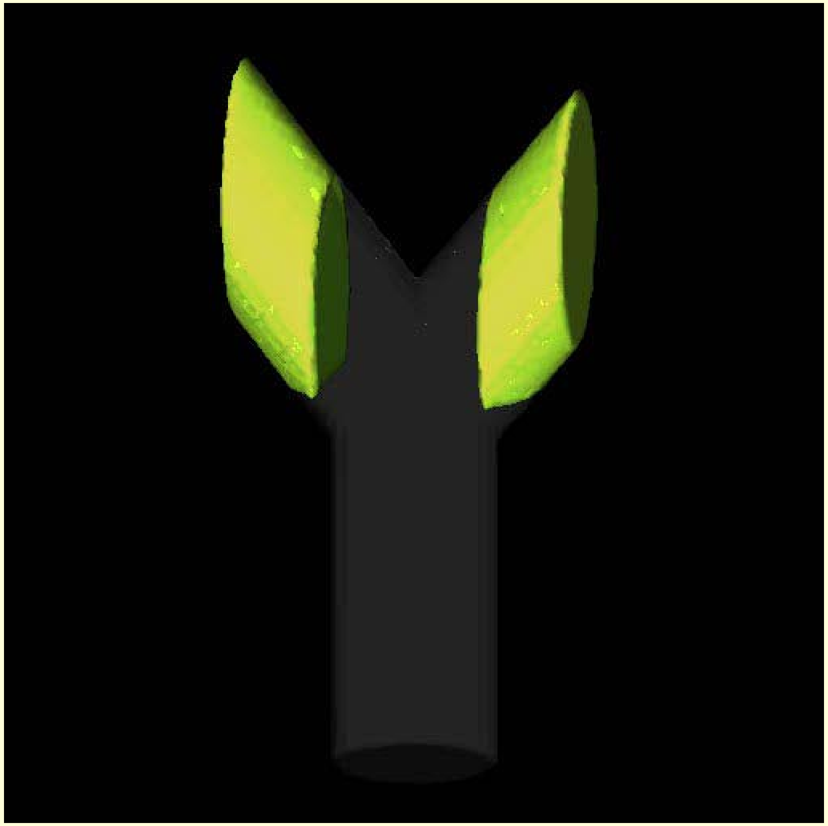
If the density of the fluid is constant everywhere,

$$\rho = \rho_0 = \text{ constant}$$

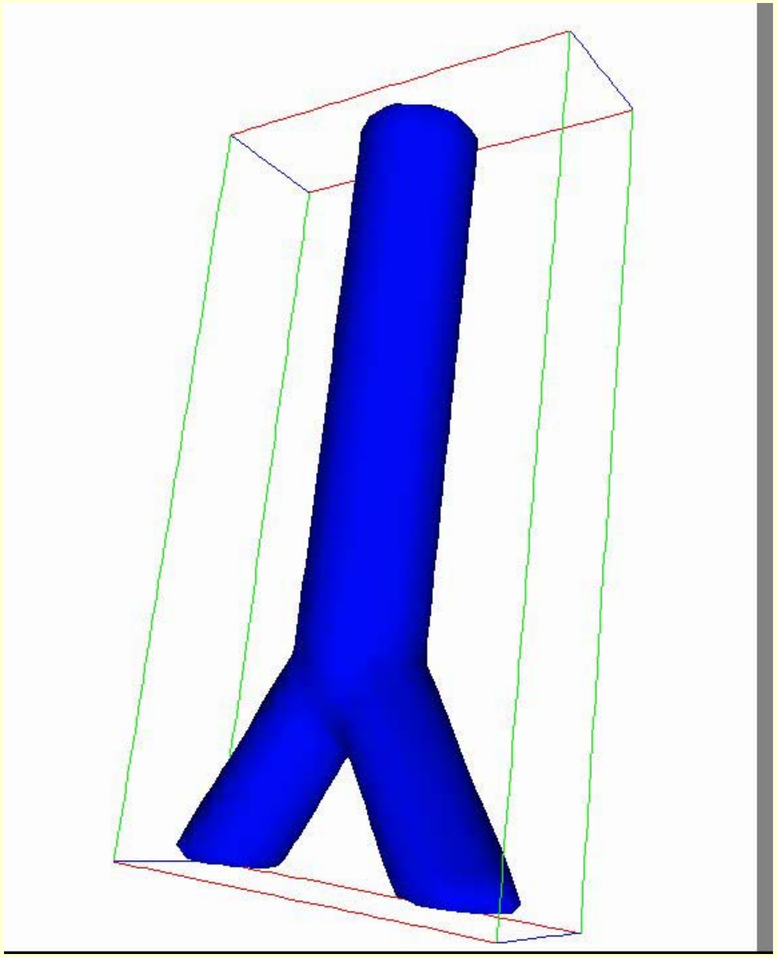
then the flow is called *incompressible*, and the continuity equation implies:

$$\boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

Flow at Junctions

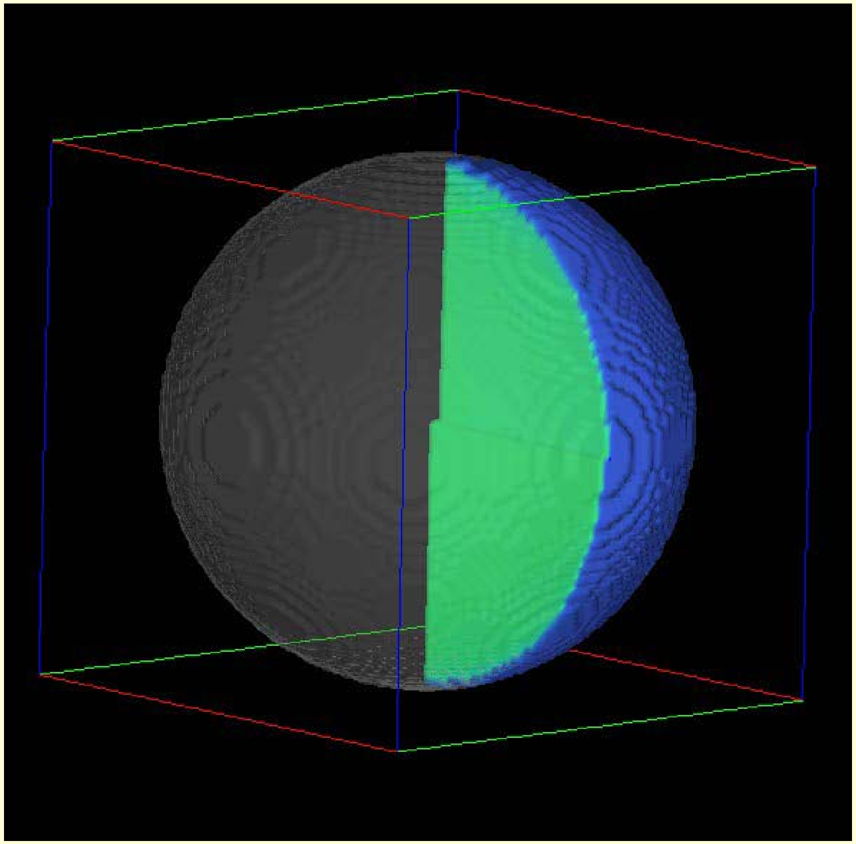


Two falling blobs of liquid in a pipe.

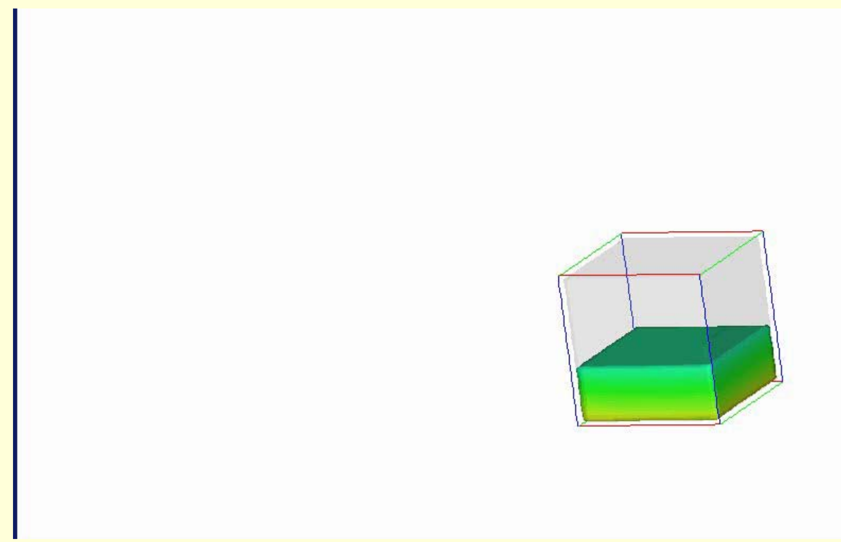


Wall shear stress in an elastic blood vessel.

Flow in Containers

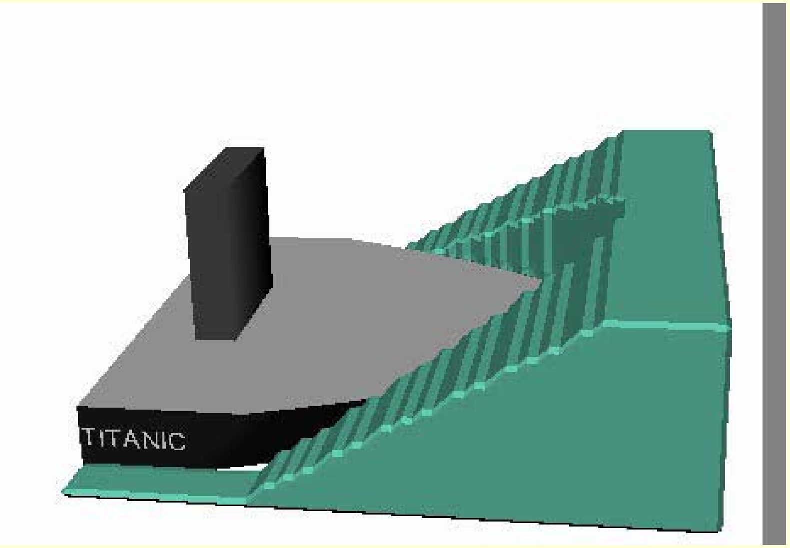


Dambreak in sphere.

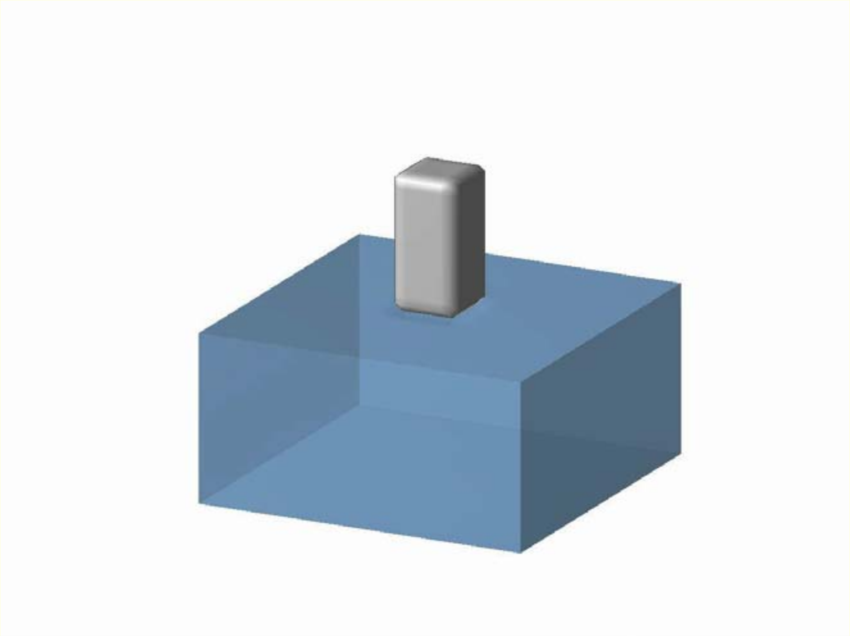


Marching cube partially filled with water.

Maritime Applications

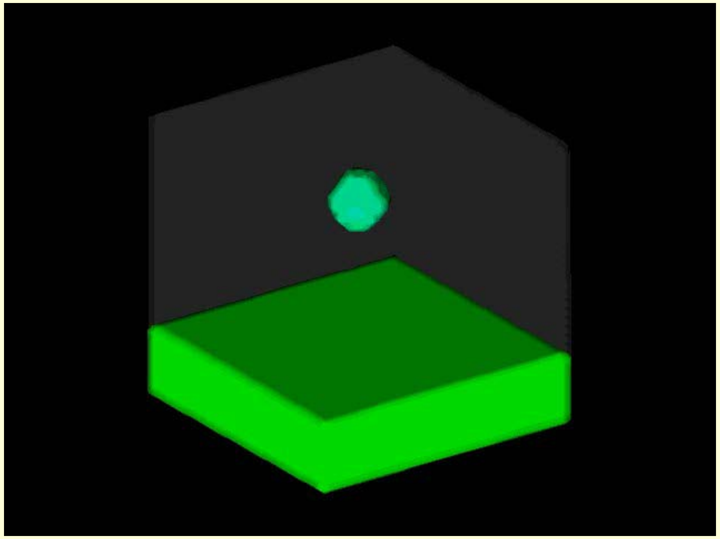


Water clashing over a ship's deck with a rectangular obstacle.



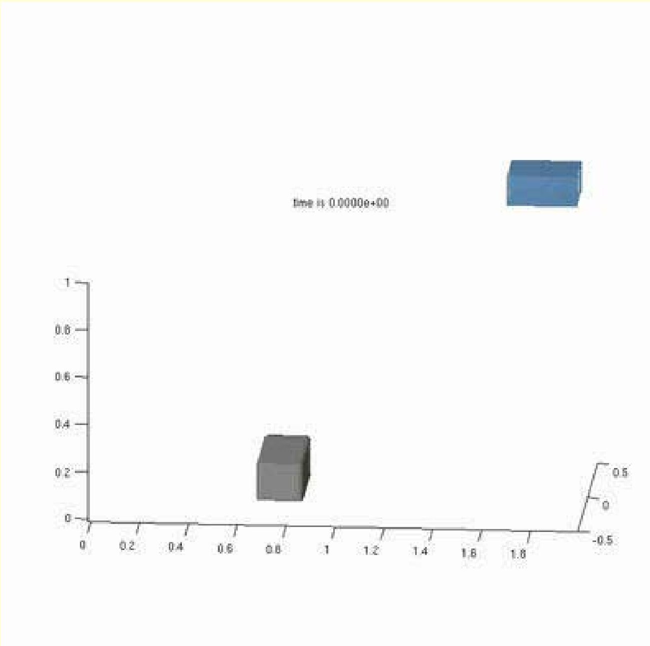
Rectangular block falling into water.

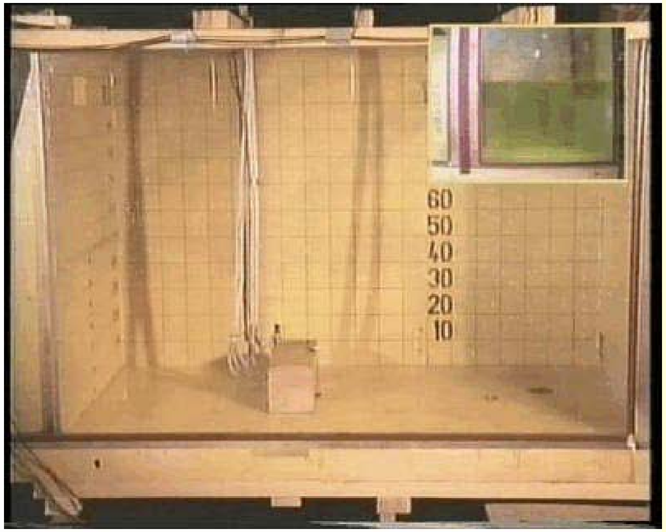
Drops and Dambreak



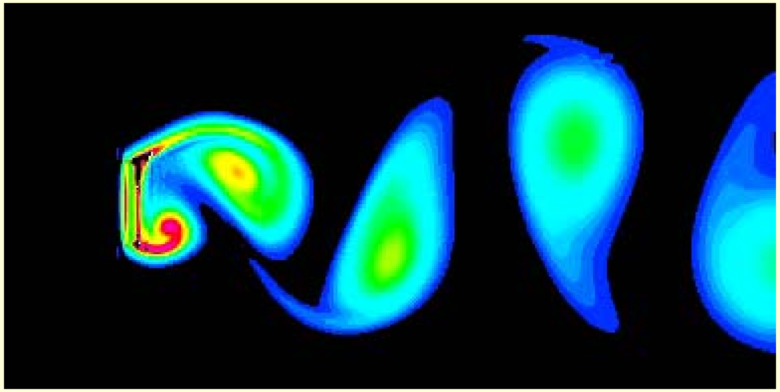
A drop falling in a pool of water.

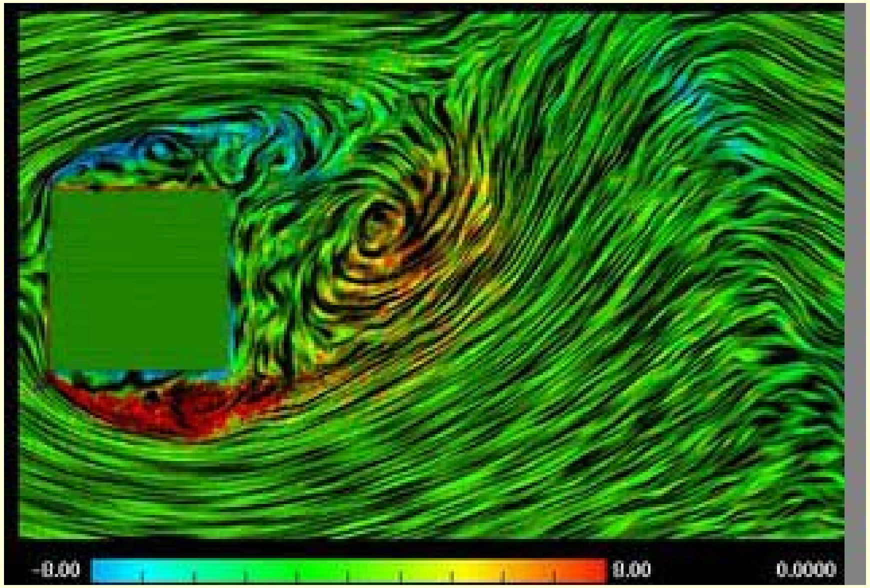
Dambreak with box.

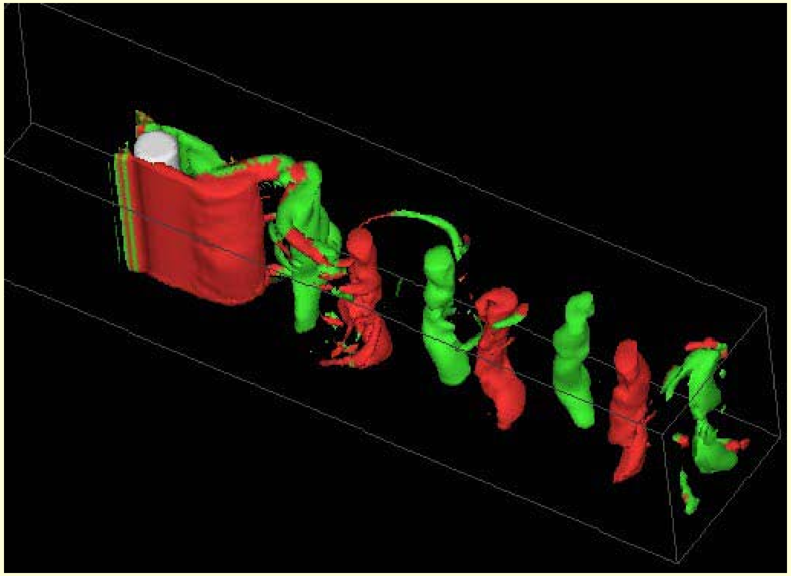




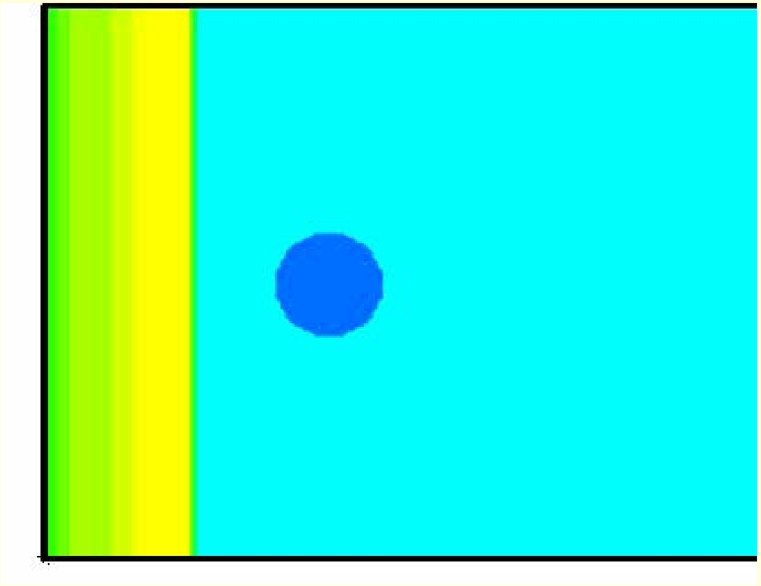
Turbulent Flow Past an Obstacle

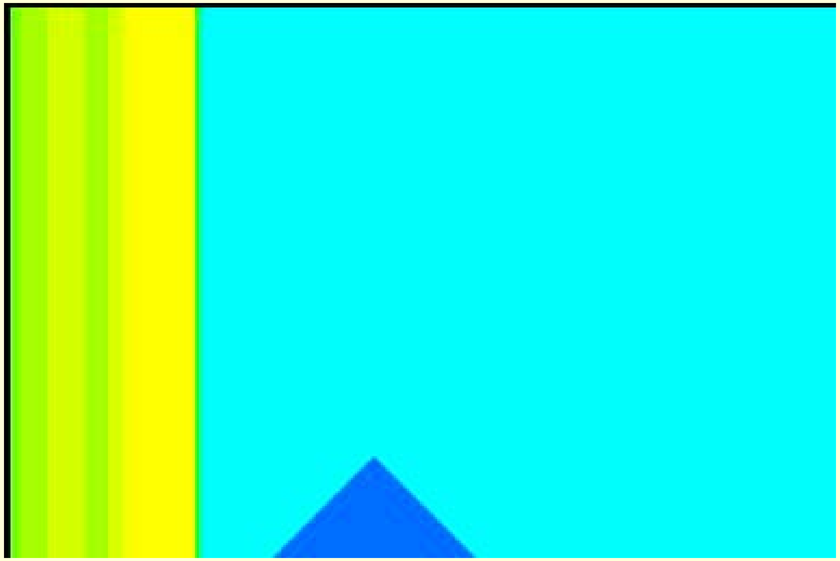




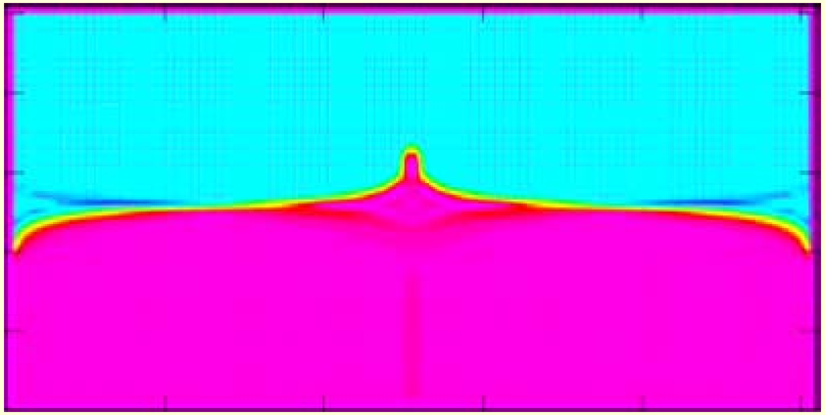
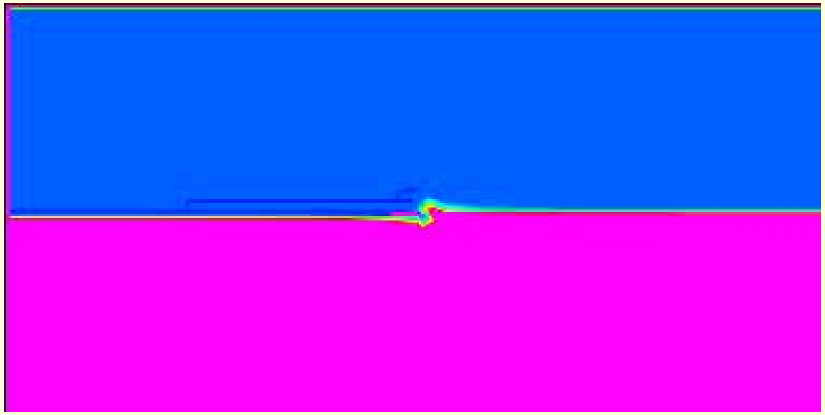
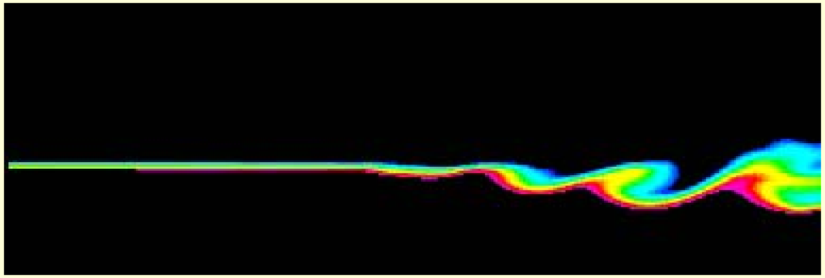
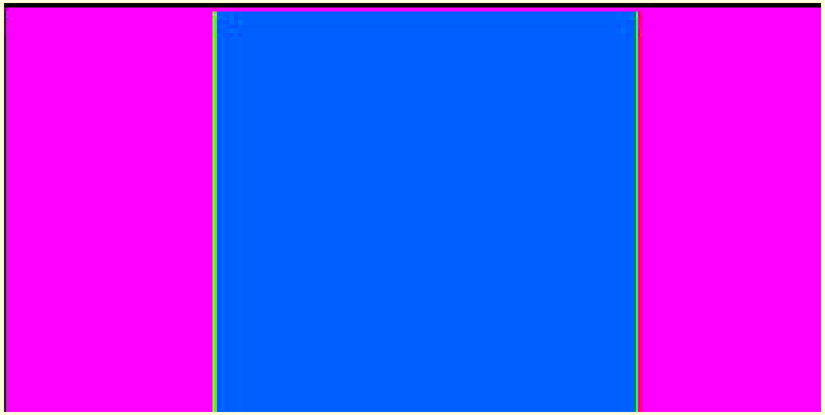


Shocks





Instabilities in Flows

Industrial Applications