Integral Form of Conservation Laws

Differential Form:

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}(\vec{U})}{\partial x} = 0$$

Integral Form I:

$$\frac{d}{dt}\int_{x_L}^{x_R} \vec{U}(x,t)dx = \vec{F}(\vec{U}(x_L,t)) - \vec{F}(\vec{U}(x_R,t))$$

Integral Form II:

$$\int_{x_L}^{x_R} \vec{U}(x,t_2)dx - \int_{x_L}^{x_R} \vec{U}(x,t_1)dx = \int_{t_1}^{t_2} \vec{F}(\vec{U}(x_L,t)) - \int_{t_1}^{t_2} \vec{F}(\vec{U}(x_R,t))$$

More generally, for any domain *V* in the *x*-*t* space, the following relation holds for the *closed line integral*:

$$\int [\vec{U}dx - \vec{F}(\vec{U})dt] = 0$$

Nonlinear Scalar Conservation Law

Differential form:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \lambda(u)\frac{\partial u}{\partial x} = 0$$

where:

$$\lambda(u) = \frac{\partial f}{\partial u}$$

Initial Data: $u(x,0) = u_0(x)$

Convex / Concave flux:

$\lambda'(u) > 0$ *Convex flux*

$\lambda'(u) < 0$ *Concave flux*

if for some u: $\lambda'(u) = 0$ *Non-Convex, Non-Concave flux*

Characteristic Solution

The characteristic problem is:

$$\frac{dx}{dt} = \lambda(u), \quad x(0) = x_0$$

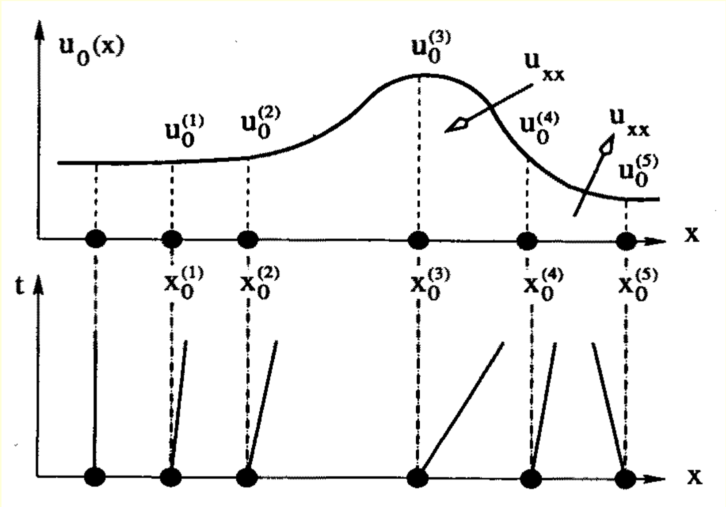
Since both *u* and *x* are functions of *t*:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \lambda(u)\frac{\partial u}{\partial x} = 0$$

i.e. *u is constant along characteristic curves, which must be straight lines.*
The solution is:

$$\boxed{u(x,t) = u_0(x_0)}$$

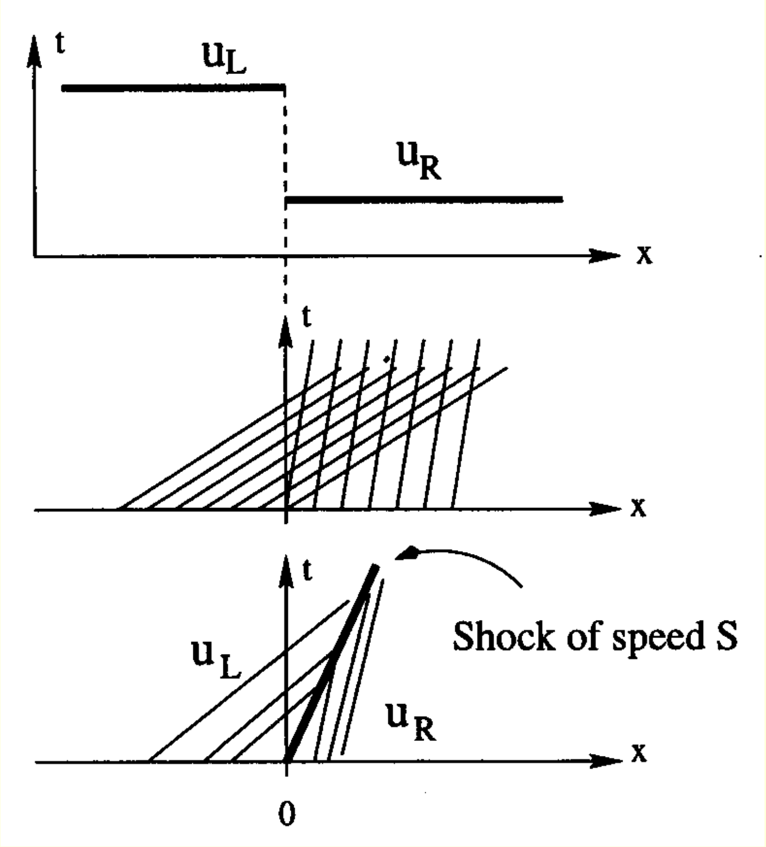
Example:



Shock Waves

Consider the *Riemann problem* for a *convex flux*, with *compressive initial data* ($u_L > u_R$). Then, the characteristic speeds are

$$\lambda(u_L) > \lambda(u_R)$$



The characteristics cross, forming a *shock wave*.

Shock Wave Conditions

From the integral form of the conservation law, one can easily show that the *shock speed* is given by the *Rankine-Hugoniot condition* across the shock:

$$S = \frac{\Delta f}{\Delta u} = \frac{f(u(x_R,t)) - f(u(x_L,t))}{u(x_R,t) - u(x_L,t)}$$

The shock speed satisfies the *entropy condition*:

$$\lambda(u_L) > S > \lambda(u_R)$$

e.g. for Burger's equation ($f=u^2/2$) :

$$S = \frac{1}{2}(u_L + u_R)$$

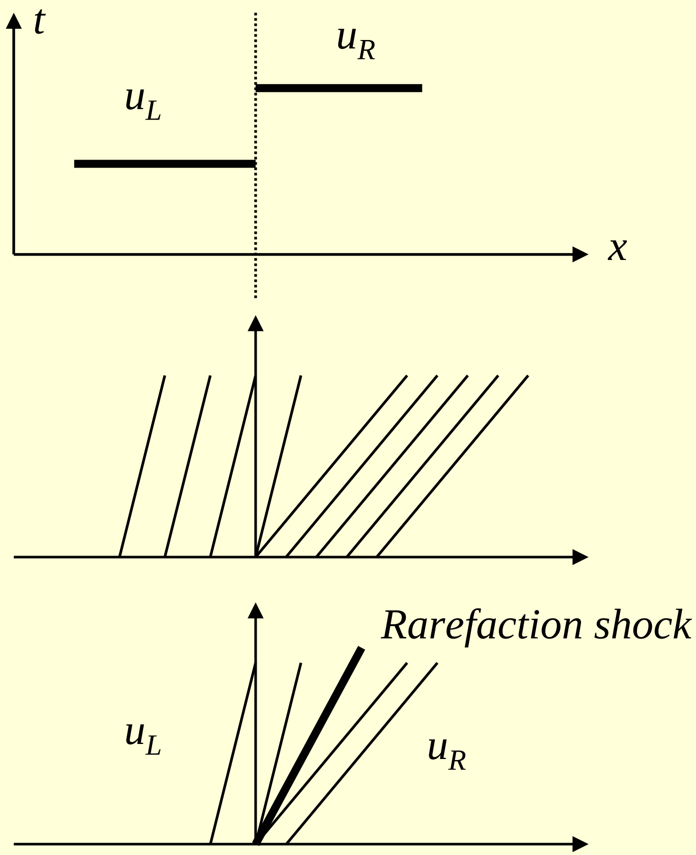
The shock solution of the convex, compressive Riemann problem is:

$$u(x,t) = \begin{cases} u_L, & x - St < 0 \\ u_R, & x - St > 0 \end{cases}$$

(Unphysical) Rarefaction Shock

Consider a convex Riemann problem with expansive data. There exists a mathematical solution of the same form as the shock solution:

$$u(x,t) = \begin{cases} u_L, & x - St < 0 \\ u_R, & x - St > 0 \end{cases}$$



with S given as before. However, since now

$$\lambda_L < \lambda_R$$

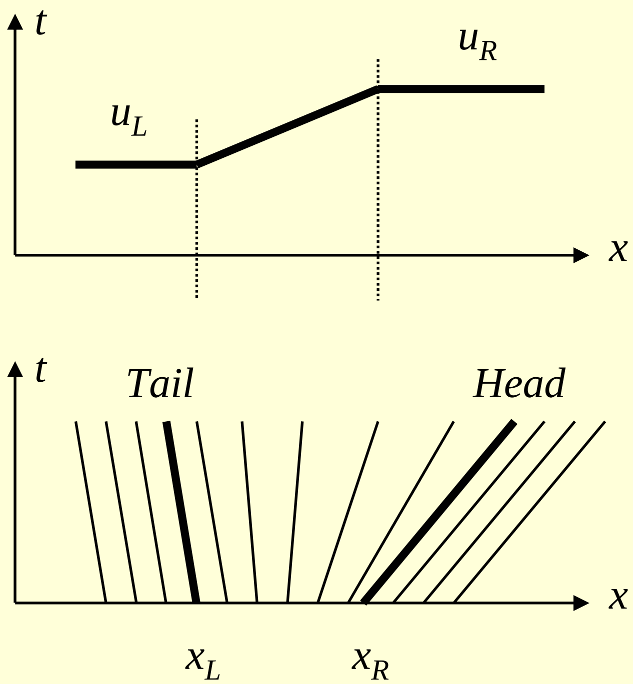
the entropy condition is not satisfied. Therefore, this mathematical solution (called a rarefaction shock) is unphysical, since it is entropy-violating.

Moreover, this solution is *unstable*.

Rarefaction Waves (I)

If a small continuous part is add in-between the discontinuity, then the solution changes and becomes

$$u(x,t) = \begin{cases} u_L, & x - x_L < \lambda_L t \\ (x - x_L)/t, & \lambda_L t < x - x_L < \lambda_R t \\ u_R, & x - x_R \geq \lambda_R \end{cases}$$



This is a *non-centered rarefaction wave.*

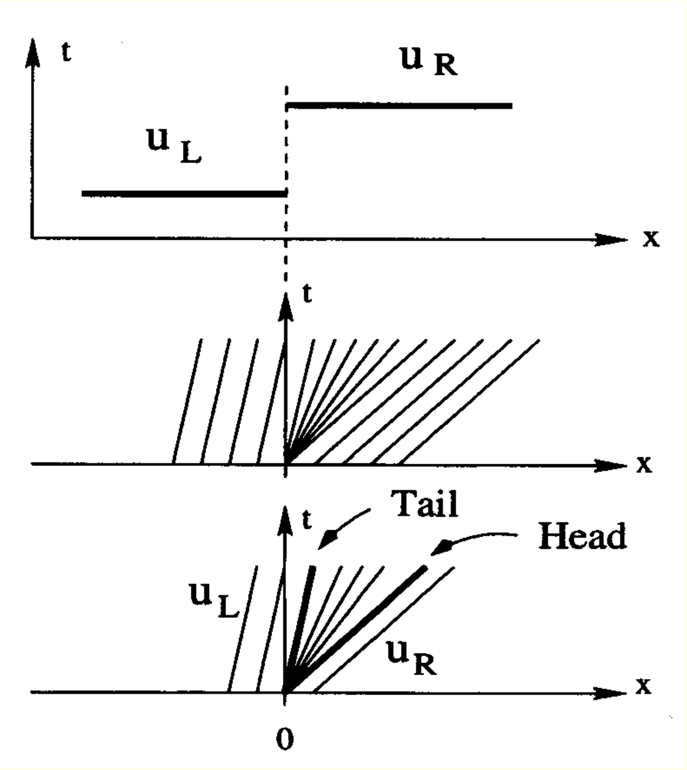
Rarefaction Waves (II)

If we let

$$\Delta x = x_R - x_L \to 0$$

the solution becomes

$$u(x,t) = \begin{cases} u_L, & x < \lambda_L t \\ x/t, & \lambda_L t < x < \lambda_R t \\ u_R, & x \geq \lambda_R \end{cases}$$



This is a *centered rarefaction wave*.

Thus, the convex Riemann problem with expansive initial data has two possible solutions, a rarefaction shock and a centered rarefaction wave. Of these two solutions, only the latter is physical, satisfying the entropy condition.