Dynamic Cosmic Strings I

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The field equations for a time dependent cylindrical cosmic string coupled to gravity are reformulated in terms of geometrical variables defined on a $2+1$-dimensional spacetime by using the method of Geroch decomposition. Unlike the 4-dimensional spacetime the reduced case is asymptotically flat. A numerical method for solving the field equations which involves conformally compactifying the space and including null infinity as part of the grid is described. It is shown that the code reproduces the results of a number of vacuum solutions with one or two degrees of freedom. In the final section the interaction between the cosmic string and a pulse of gravitational radiation is briefly described. This is fully analyzed in the sequel.

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I. INTRODUCTION

Cosmic strings are topological defects that formed during phase transitions in the early universe. They are important because they are predicted by grand unified theories and produce density perturbations in the early universe that might be important in the formation of galaxies and other large scale structures [1]. They are also important since they are thought to be sources of gravitational radiation due to rapid oscillatory motion [2]. In the simplest case of a string moving in a fixed background one can take the thin string limit and the dynamics are given by the Nambu–Goto action [3] which is known to admit oscillatory solutions. However in order to fully understand the behavior of cosmic strings one should study the field equations for a cosmic string coupled to Einstein's equations.

A cosmic string is described by a U(1) gauge vector field A_μ coupled to a complex scalar field $\Phi = 1/\sqrt{2}Se^{i\phi}$. The Lagrangian for these coupled fields is given by

$$L_M = \frac{1}{2}\nabla_\mu S \nabla^\mu S + \frac{1}{2}S^2(\nabla_\mu \phi + eA_\mu)(\nabla^\mu \phi + eA^\mu) - \lambda(S^2 - \eta^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, e, λ are positive coupling constants and η is the vacuum expectation value.

The Einstein-scalar-gauge field equations for an infinitely long static cosmic string have been investigated by Laguna and Garfinkle [4]. Some geometrical techniques related to those used in this paper have also been used in the analysis of cosmic strings by Peter and Carter [5] and by Carter [6]. In the present paper (and its sequel) we will investigate the behavior of a time dependent cylindrical cosmic string coupled to gravity. In particular we investigate the effect of a pulse of gravitational radiation on an initially static cosmic string and the corresponding gravitational radiation that is emitted as a result of oscillations in the string. Since we are interested in the gravitational radiation produced by the string it is desirable to measure this at null infinity where the gravitational flux is unambiguously defined and one does not need to impose artificial outgoing radiation conditions at the edge of the numerical grid. However the infinite length of the string in the z-direction prevents the spacetime from being asymptotically flat. We therefore follow the approach of Clarke et al. [7] and use a Geroch decomposition with respect to the Killing vector in the z-direction to reformulate the problem in $2+1$ dimensions. Note however that unlike Clarke et al. [7] we apply the Geroch decomposition to the entire problem not just to the exterior characteristic region. The gravitational degrees of freedom of the $3+1$ problem are then encoded in two geometrically defined scalar fields defined on the $2+1$-dimensional spacetime. These are the norm of the Killing vector ν and the Geroch potential τ for the rotation. We show in section III that the energy-momentum tensor of these fields describes the gravitational energy of the original

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cylindrical problem. An important feature of the reduced $2+1$ spacetime is that it is asymptotically flat and this allows us to conformally compactify the spacetime and include null infinity as part of the numerical grid.

In section II we briefly describe the Geroch decomposition and the field equations that one obtains in the cylindrical case. We also show how it is possible to rescale t, ρ and the matter variables to simplify the equations. In order to demonstrate the numerical accuracy of the full code it is useful to compare it with either an exact solution or else with some other independently produced numerical results. Unfortunately this is not possible for the full code where we must satisfy ourselves with the internal consistency of the code as demonstrated by convergence testing for example. If one considers only the gravitational part of the code it is possible to specialize to the case of a vacuum solution. The simplest of these is the Weber–Wheeler solution [8], but we also consider a rotating vacuum solution due to Xanthopoulos [9] which describes a thin cosmic string and a second rotating vacuum solution due to Piran et al. [10]. In order to compare our results with these exact solutions we must first write them in terms of the Geroch potential, this is done in section IV. This description is useful not just for numerical purposes but also in interpreting these solutions since the two polarization states for the gravitational radiation have a simple interpretation in terms of the Geroch variables ν and τ. The numerical code used in this case, the convergence analysis and the comparison between the numerical results and the exact solution is given in section IV. As far as the matter part of the code is concerned there are no exact solutions and one must compare the code with other numerical results. The simplest special case involves decoupling the matter variables from the metric variables and considering the equations of motion in Minkowski space. There do not seem to be other results available in the dynamic case, but the static solutions have been investigated by a number of authors [11], [12] and [13]. Another solution which has been investigated previously is a static string which is coupled to the gravitational field. Finding such solutions is much harder than one might suppose due to the asymptotic behavior of the matter variables. As well as the physical solution to the equations there is an exponentially diverging non-physical solution [13] which must be suppressed. By compactifying the radial coordinate we can control the behavior of the solution at infinity in the static case by using a relaxation scheme. This allows us to obtain solutions for all values of the radius rather than the fairly restricted range of ρ that had been previously obtained, and also permits us to use the proper boundary conditions at infinity. The static string is described in further detail in sections V and VI. Finally we briefly describe the numerics for the dynamic string coupled to gravity. Here again the asymptotics of the matter variables make it hard to write a stable code, but by using an implicit scheme we are able to produce a code with long term stability and second order convergence which agrees with the previous results in the special cases described earlier. From a physical point of view the most interesting feature of this code is that it is able to describe the interaction of a gravitational field with two degrees of freedom with the full non-linear cosmic string equations. So far we have only investigated the interaction of the cosmic string with an incoming Weber–Wheeler type pulse of gravitational radiation with just one degree of freedom. We find that the pulse excites the cosmic string and causes the scalar and vector fields to vibrate with a frequency which is roughly proportional to their respective masses. This oscillation slowly decays and the string eventually returns to its previous static state. This is briefly described in section VII and in detail in the sequel [14], which we henceforth will refer to as paper II, where comparisons with other results and a full convergence analysis is given.

II. THE GEROCH DECOMPOSITION

As we explained above it is not possible for a cylindrically symmetric cosmic string to be asymptotically flat due to the infinite extent of the string in the z-direction. By factoring out this direction we can obtain a 3-dimensional spacetime that is asymptotically flat. If the Killing vector in the z-direction is hypersurface orthogonal then one can simply project onto the surfaces $\mathcal{S}$ given by $z = $ const. However we wish to consider cylindrical solutions which also have a rotating mode and in this case the Killing vector ξ^μ is not hypersurface orthogonal. Geroch [15] has shown how to factor out the Killing direction in this more general case. The idea is to identify points which lie on the same integral curves of the Killing vector field and thus obtain $\mathcal{S}$ as a quotient space rather than as a subspace. There is then a one-to-one correspondence between tensor fields on $\mathcal{S}$ and tensor fields on the 4-dimensional manifold M which have vanishing contraction with the Killing vector and also vanishing Lie derivative along ξ^μ. One may therefore use the four dimensional metric $g_{\mu\nu}$ to define a metric $h_{\mu\nu}$ on $\mathcal{S}$ according to the equation

$$h_{\mu\nu} = g_{\mu\nu} + (\xi^\sigma \xi_\sigma)^{-1} \xi_\mu \xi_\nu. \tag{2}$$

The extra information in the 4-metric is encoded in two new geometric variables; the norm of the Killing vector

$$\nu = -\xi^\mu \xi_\mu \tag{3}$$

(where we have introduced the minus sign to make ν positive in the spacelike case) and the twist

$$\tau_\mu = -\epsilon_{\mu\nu\tau\sigma}\xi^\nu\nabla^\tau\xi^\sigma. \tag{4}$$

Geroch then showed how it is possible to rewrite Einstein's equations in terms of the 3-dimensional Ricci curvature of $(\mathcal{S}, h)$ and equations involving the 3-dimensional covariant derivatives of ν and the twist. If we let D_μ define the covariant derivative with respect to $h_{\mu\nu}$ then one can show that

$$D_{[\rho}\tau_{\sigma]} = \epsilon_{\rho\sigma\mu\nu}\xi^\mu R^\nu_\tau\xi^\tau, \tag{5}$$

where R^ν_τ is the 4-dimensional Ricci tensor. It is clear that this vanishes in vacuum so that τ_σ is curl free and may be defined in terms of a potential. It is a remarkable fact that this remains true for spacetimes with a cosmic string energy-momentum tensor so that even in this case we may write

$$\tau_a = D_a\tau, \tag{6}$$

where we have introduced the convention of using Latin indices to describe quantities defined on $\mathcal{S}$.

We may now write Einstein's equations for the 4-dimensional spacetime (M, g) in terms of the Ricci curvature of $(\mathcal{S}, h)$ and the two scalar fields ν and τ defined on $\mathcal{S}$. We obtain

$$\mathcal{R}_{ab} = \frac{1}{2}\nu^{-2}[(D_a\tau)(D_b\tau) - h_{ab}(D_m\tau)(D^m\tau)] + \frac{1}{2}\nu^{-1}D_aD_b\nu - \frac{1}{4}\nu^{-2}(D_a\nu)(D_b\nu)$$

$$+8\pi h_a{}^\mu h_b{}^\nu(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \tag{7}$$

$$D^2\nu = \frac{1}{2}\nu^{-1}(D_m\nu)(D^m\nu) - \nu^{-1}(D_m\tau)(D^m\tau) + 16\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)\xi^\mu\xi^\nu, \tag{8}$$

$$D^2\tau = \frac{3}{2}\nu^{-1}(D_m\tau)(D^m\nu). \tag{9}$$

The transformation to the 3-dimensional description is not only mathematically convenient but is physically meaningful. If the Killing vector is hypersurface orthogonal then τ vanishes and the gravitational radiation has only one polarisation which may be defined in a simple way in terms of ν. If there are both polarisations present then the $+$ mode is given in terms of ν while the $\times$ mode is given in terms of τ [see equation (19) below]. One can simplify things by making a conformal transformation and using the metric $\tilde{h}_{ab} = \nu h_{ab}$ in which case we can write the vacuum Einstein–Hilbert Lagrangian on M in terms of 3-dimensional variables on $\mathcal{S}$

$$I_G = \int_M R\sqrt{-g}d^4x \tag{10}$$

$$= \int_{\mathcal{S}}\{\tilde{\mathcal{R}} - \frac{1}{2}\nu^{-2}[\tilde{h}^{ab}(\tilde{D}_a\tau)(\tilde{D}_b\tau) + \tilde{h}^{ab}(\tilde{D}_a\nu)(\tilde{D}_b\nu)]\}\sqrt{\tilde{h}}d^3x \tag{11}$$

and we see that the 4-dimensional gravitational field is described in three dimensions by the two scalar fields ν and τ conformally coupled to the 3-dimensional spacetime with metric $\tilde{h}_{ab}$. Since the 3-dimensional spacetime has no Weyl curvature it is essentially non-dynamic and we see that ν and τ encode the two gravitational degrees of freedom in the original spacetime. The corresponding 'energy-momentum' tensor for these fields is

$$\hat{T}_{ab} = \frac{1}{2}\nu^{-2}[\tilde{D}_a\tau\tilde{D}_b\tau - \frac{1}{2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c\tau)(\tilde{D}_d\tau) + \tilde{D}_a\nu\tilde{D}_b\nu - \frac{1}{2}\tilde{h}_{ab}\tilde{h}^{cd}(\tilde{D}_c\nu)(\tilde{D}_d\nu)] \tag{12}$$

and if there is matter present in four dimensions there are also additional matter terms in three dimensions. As shown in equation (23) this 3-dimensional 'energy-momentum' tensor for ν and τ gives the correct expression for the 4-dimensional gravitational energy.

III. THE FIELD EQUATIONS

For the case of a cylindrically symmetric vacuum spacetime one can write the metric in Jordan, Ehlers, Kundt and Kompaneets (JEKK) form [16], [17]

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - \rho^2e^{-2\psi}d\phi^2 - e^{2\psi}(\omega d\phi + dz)^2. \tag{13}$$

However this form of the metric is not compatible with the cosmic string energy momentum tensor so we follow Marder [18] by introducing an extra variable μ into the metric and writing it in the form

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi} d\phi^2 - e^{2(\psi+\mu)}(\omega d\phi + dz)^2. \tag{14}$$

This form of the line element enables us to make easy comparisons with the JEKK vacuum solutions previously considered numerically by Dubal et al. [19] and d'Inverno et al. [20]. The field equation for μ decouples from those for the other metric variables and it has a source term given by $T_{00} - T_{11}$. The physical interpretation of μ is briefly discussed by Marder [18]. In the static case one can show that C^2 regularity on the axis implies that $\mu = \ln(\nu) - \gamma$. The metric given by (14) has zero shift and lapse determined by the condition $g_{tt} = -g_{\rho\rho}$. In this gauge the null geodesics are given by the simple conditions $u = t - \rho =$ const. and $v = t + \rho =$ const. The remaining coordinate freedom is given by the freedom to relabel the radial null surfaces: $u \to f(u)$ and $v \to g(v)$ where f and g are arbitrary functions. We may fix this by specifying the initial values of μ and its derivative. For example we can choose μ to be equal to its static value and $\mu_{,t}$ to vanish, but due to time dependent matter source terms in the evolution equation for μ [see equation (28) below] this does not make μ constant in time.

In terms of these variables we find the norm of the Killing vector in the z-direction $\xi^\mu = \delta^\mu_3$ to be given by

$$\nu = e^{2(\psi+\mu)} \tag{15}$$

and the twist potential is related to ω by

$$D_\sigma \tau = \rho^{-1} e^{4\psi+3\mu}(\omega_{,\rho}, \omega_{,t}, 0, 0). \tag{16}$$

Finally the conformal 3-metric $\tilde{h}_{ab}$ is given by

$$d\tilde{\sigma}^2 = e^{2(\gamma+\mu)}(dt^2 - d\rho^2) - \rho^2 e^{2\mu} d\phi^2. \tag{17}$$

It is also of interest to calculate $\hat{T}_{ab}$ in terms of these variables. We find

$$\hat{T}_{ab} t^a t^b = \frac{1}{8} e^{-2(\gamma+\mu)} \left[\left(\frac{\nu_{,u}}{\nu}\right)^2 + \left(\frac{\nu_{,v}}{\nu}\right)^2 + \left(\frac{\tau_{,u}}{\nu}\right)^2 + \left(\frac{\tau_{,v}}{\nu}\right)^2 \right], \tag{18}$$

where t^a is a unit timelike vector proportional to $\frac{\partial}{\partial t}$. Note that the quantities

$$A = \left(\frac{\nu_{,u}}{\nu}\right)^2, \qquad B = \left(\frac{\nu_{,v}}{\nu}\right)^2, \qquad C = \left(\frac{\tau_{,u}}{\nu}\right)^2, \qquad D = \left(\frac{\tau_{,v}}{\nu}\right)^2, \tag{19}$$

are exactly the same as the quantities A, B, C and D which are given (by more complicated expressions) in terms of ψ and ω in equations (4a)–(4d) of Piran et al. [21] and describe the two polarizations of the cylindrical gravitational field. Furthermore if we consider the special case of vacuum solutions and integrate $\hat{T}_{ab} t^a t^b$ over the region $V = \{0 \le \rho \le \rho_0,\, t = t_0\}$ with respect to the volume form dV on $t = t_0$ we find

$$E(t_0, \rho_0) = \int\int_V \hat{T}_{ab} t^a t^b dV \tag{20}$$

$$= \frac{\pi}{4} \int_0^{\rho_0} e^{-\gamma} \left[\left(\frac{\nu_{,u}}{\nu}\right)^2 + \left(\frac{\nu_{,v}}{\nu}\right)^2 + \left(\frac{\tau_{,u}}{\nu}\right)^2 + \left(\frac{\tau_{,v}}{\nu}\right)^2 \right] \rho d\rho \tag{21}$$

$$= 2\pi \int_0^{\rho_0} \gamma_{,\rho} e^{-\gamma} d\rho \tag{22}$$

$$= 2\pi[1 - e^{-\gamma(t_0,\rho_0)}], \tag{23}$$

where we have used the vacuum field equations for $\gamma_{,\rho}$ [see equation (31) below]. Note that this is the same as the energy obtained by Ashtekar et al. [22] but does not require the Killing vector to be hypersurface orthogonal. It differs from the C-energy in general but agrees with it in the linearized case.

As far as the matter variables are concerned we make the obvious generalization of the form used by Garfinkle [26] and write

$$\Phi = \frac{1}{\sqrt{2}} S(t,\rho) e^{i\phi}, \tag{24}$$

$$A_\mu = \frac{1}{e}[P(t,\rho) - 1]\nabla_\mu \phi. \tag{25}$$

We may now write the field equations for the complete system. Since we are working in three dimensions we have three independent evolution equations. After some algebra these may be written as

$$
\begin{aligned}
&\Box\nu - \nu^{-1}(\tau_{,t}^2 - \tau_{,\rho}^2 - \nu_{,t}^2 + \nu_{,\rho}^2) - \mu_{,t}\nu_{,t} + \mu_{,\rho}\nu_{,\rho} \\
&= -8\pi\nu[2\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2)^2 + e^{-2}\rho^{-2}\nu e^{-2\mu}(P_{,t}^2 - P_{,\rho}^2)], && (26)\\
&\Box\tau + 2\nu^{-1}(\tau_{,t}\nu_{,t} - \tau_{,\rho}\nu_{,\rho}) - (\mu_{,t}\tau_{,t} - \mu_{,\rho}\tau_{,\rho}) = 0, && (27)\\
&\Box\mu + \rho^{-1}\mu_{,\rho} - \mu_{,t}^2 + \mu_{,\rho}^2 = -8\pi[2\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2)^2 + \rho^{-2}e^{2\gamma}S^2P^2], && (28)
\end{aligned}
$$

where $\Box$ represents the flat spacetime d'Alembertian which in cylindrical coordinates is given by

$$
\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2} + \rho^{-1}\frac{\partial}{\partial \rho}. \tag{29}
$$

There are also three constraint equations, one of which vanishes identically due to the rotational symmetry. The remaining equations give

$$
\begin{aligned}
\gamma_{,t} &= \frac{\rho}{1+\rho\mu_{,\rho}}\left[\mu_{,t\rho} - \mu_{,t}(\gamma_{,\rho}+\mu_{,\rho}) + \frac{1}{2}\nu^{-2}(\tau_{,t}\tau_{,\rho} + \nu_{,t}\nu_{,\rho}) + 8\pi(S_{,t}S_{,\rho} + e^{-2}\rho^{-2}\nu e^{-2\mu}P_{,t}P_{,\rho})\right], && (30)\\
\gamma_{,\rho} &= \frac{\rho}{\rho^2\mu_{,t}^2 - (1+\rho\mu_{,\rho})^2}\Big((1+\rho\mu_{,\rho})\Big\{-4\pi[2\nu^{-1}e^{2(\gamma+\mu)}\lambda(S^2-\eta^2)^2 + (S_{,t}^2 + S_{,\rho}^2) \\
&\quad + e^{-2}\rho^{-2}\nu e^{-2\mu}(P_{,t}^2 + P_{,\rho}^2) + \rho^{-2}e^{2\gamma}S^2P^2] - \frac{1}{4}\nu^{-2}(\tau_{,t}^2 + \tau_{,\rho}^2 + \nu_{,t}^2 + \nu_{,\rho}^2) - \rho^{-1}\mu_{,\rho} - \mu_{,\rho\rho}\Big\} \\
&\quad + \rho\mu_{,t}\left[\frac{1}{2}\nu^{-2}(\tau_{,t}\tau_{,\rho} + \nu_{,t}\nu_{,\rho}) + \rho^{-1}\mu_{,t} + \mu_{,t\rho} + 8\pi(S_{,t}S_{,\rho} + e^{-2}\rho^{-2}\nu e^{-2\mu}P_{,t}P_{,\rho})\right]\Big). && (31)
\end{aligned}
$$

Finally there are the equations for the matter variables S and P which may be derived from the Euler–Lagrange equations for L_M or alternatively from the contracted Bianchi identities.

$$
\begin{aligned}
&\Box S - S_{,t}\mu_{,t} + S_{,\rho}\mu_{,\rho} = S[4\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2) + \rho^{-2}e^{2\gamma}P^2], && (32)\\
&\Box P - P_{,t}(\nu^{-1}\nu_{,t} - \mu_{,t}) + P_{,\rho}(\nu^{-1}\nu_{,\rho} - \mu_{,\rho} - 2\rho^{-1}) = e^2\nu^{-1}e^{2(\gamma+\mu)}PS^2. && (33)
\end{aligned}
$$

We also need to supplement these equation by boundary conditions on the axis. For the 4-dimensional metric variables the simplest condition is to require the metric to be C^2 on the axis so that we have a well defined curvature tensor. This gives the conditions

$$
\begin{aligned}
\psi(t,\rho) &= a_1(t) + O(\rho^2), && (34)\\
\omega(t,\rho) &= O(\rho^2), && (35)\\
\mu(t,\rho) &= a_2(t) + O(\rho^2), && (36)\\
\gamma(t,\rho) &= O(\rho). && (37)
\end{aligned}
$$

In terms of ν and τ this gives

$$
\begin{aligned}
\nu(t,\rho) &= a_3(t) + O(\rho^2), && (38)\\
\tau(t,\rho) &= O(\rho^2), && (39)
\end{aligned}
$$

where we have chosen the additive constant in the definition of the potential τ so that it vanishes on the axis. In certain situations C^2 regularity is too strong and one must impose the weaker condition of elementary flatness [23]. However even this is too strong for the Xanthopoulos solution which has a conical singularity on the axis.

The boundary conditions for S and P on the axis are [26]

$$
\begin{aligned}
S(t,\rho) &= O(\rho), && (40)\\
P(t,\rho) &= 1 + O(\rho^2). && (41)
\end{aligned}
$$

In order to consider the behavior of the solution at null infinity we first transform from t to a null time coordinate $u = t - \rho$. We also wish to compactify the region and following Clarke et al. [7] we make the transformation

$$
y = \frac{1}{\sqrt{\rho}}. \tag{42}
$$

In terms of these variables the equations become

$$\Box\nu + y^3\nu^{-1}(\tau_{,u}\tau_{,y} - \nu_{,u}\nu_{,y}) + \frac{1}{4}y^6\nu^{-1}(\tau_{,y}^2 - \nu_{,y}^2) + \frac{1}{2}y^3(\mu_{,y}\nu_{,u} + \mu_{,u}\nu_{,y} + \frac{1}{2}y^3\mu_{,y}\nu_{,y})$$
$$= 8\pi\nu[-2\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2)^2 + \frac{1}{2}e^{-2}y^7\nu e^{-2\mu}(\frac{1}{2}y^3P_{,y}^2 + 2P_{,u}P_{,y})], \tag{43}$$

$$\Box\tau - y^3\nu^{-1}(\tau_{,u}\nu_{,y} + \tau_{,y}\nu_{,u} + \frac{1}{2}y^3\tau_{,y}\nu_{,y}) + \frac{1}{2}y^3(\mu_{,y}\tau_{,u} + \mu_{,u}\tau_{,y} + \frac{1}{2}y^3\mu_{,y}\tau_{,y}) = 0, \tag{44}$$

$$\Box\mu - y^2(\mu_{,u} + \frac{1}{2}y^3\mu_{,y}) - \frac{1}{2}y^3\mu_{,y}(\frac{1}{2}y^3\mu_{,y} - 2\mu_{,u})$$
$$= -8\pi[2\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2)^2 + y^4e^{2\gamma}S^2P^2], \tag{45}$$

$$\gamma_{,u} = y^2\big((1-\frac{1}{2}y\mu_{,y})[-\frac{1}{2}y^3\mu_{,uy} - \mu_{,uu} + \mu_{,u}^2 - \frac{1}{2}\nu^{-2}(\tau_{,u}^2 + \frac{1}{2}y^3\tau_{,u}\tau_{,y} + \nu_{,u}^2 + \frac{1}{2}y^3\nu_{,u}\nu_{,y})]$$
$$+\frac{1}{8}y^3\mu_{,u}[\mu_{,y}(1-y\mu_{,y}) - y\mu_{,yy}] - \frac{1}{16}y^4\nu^{-2}\mu_{,u}(\tau_{,y}^2 + \nu_{,y}^2)$$
$$-8\pi\{(1-\frac{1}{2}y\mu_{,y})[S_{,u}^2 + \frac{1}{2}y^3S_{,u}S_{,y} + e^{-2}y^4\nu e^{-2\mu}(P_{,u}^2 + \frac{1}{2}y^3P_{,u}P_{,y})]$$
$$-\frac{1}{8}y^4\mu_{,u}(S_{,y}^2 + e^{-2}y^4\nu e^{-2\mu}P_{,y}^2)\}\big)/\{[y^2 - (\mu_{,u} + \frac{1}{2}\mu_{,y})]^2 - \mu_{,u}^2\}, \tag{46}$$

$$\gamma_{,y} = \big[-\frac{1}{4}(3y^2\mu_{,y} + y^3\mu_{,yy}) + \frac{1}{4}y^3\mu_{,y}^2 - \frac{1}{8}y^3\nu^{-2}(\tau_{,y}^2 + \nu_{,y}^2) - 2\pi y^3(S_{,y}^2 + e^{-2}y^4\nu e^{-2\mu}P_{,y}^2)\big]$$
$$/\big(y^2 - \frac{1}{2}y^3\mu_{,y}\big), \tag{47}$$

$$\Box S + \frac{1}{2}y^3S_{,u}\mu_{,y} + \frac{1}{2}y^3S_{,y}(\mu_{,u} + \frac{1}{2}y^3\mu_{,y}) = S[4\lambda\nu^{-1}e^{2(\gamma+\mu)}(S^2-\eta^2) + y^4e^{2\gamma}P^2], \tag{48}$$

$$\Box P + P_{,u}[\frac{1}{2}y^3(\nu^{-1}\nu_{,y} - \mu_{,y}) + 2y^2] + \frac{1}{2}y^3P_{,y}[\nu^{-1}(\nu_{,u} + \frac{1}{2}y^3\nu_{,y}) - (\mu_{,u} + \frac{1}{2}y^3\mu_{,y}) + 2y^2]$$
$$= e^2\nu^{-1}e^{2(\gamma+\mu)}PS^2, \tag{49}$$

where in these coordinates $\Box$ is given by

$$\Box = \frac{y^2}{4}\left(4y\frac{\partial^2}{\partial u\partial y} + y^4\frac{\partial^2}{\partial y^2} + y^3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial u}\right). \tag{50}$$

It is worth remarking that one can have solutions to these equations which are regular at $y = 0$ and which represent 4-dimensional metrics in which ω diverges. Thus the notion of the 3-dimensional spacetime being asymptotically flat is weaker than might first be supposed. The asymptotic behavior of S and P at null infinity is given by

$$S(u,y) = \eta + O(y), \tag{51}$$
$$P(u,y) = O(y). \tag{52}$$

These are discussed in more detail in section V.

The field equations are solved numerically in two ways. Firstly an explicit second order Cauchy-Characteristic Matching (CCM) scheme similar to that employed by Dubal et al. [19] and d'Inverno et al. [20] is used, but using a Geroch decomposition in the whole spacetime (not just the characteristic portion) which allows one to use the geometrically defined variables in both the interior and exterior regions. This scheme works very well when compared to the exact vacuum solutions but is less satisfactory when the matter terms are included (see below). An alternative scheme with similar accuracy but with long term stability is a fully characteristic second order implicit scheme. This has the advantage that the scheme naturally controls the growth of the derivatives at infinity and hence automatically selects the physical rather than the non-physical solutions. The details of this scheme are described in paper II.

IV. EXACT VACUUM SOLUTIONS

In this section we describe the exact vacuum solutions which are used to test the codes. The solutions we will consider are the Weber–Wheeler gravitational wave [8] which just has the + polarization mode and two solutions due to Xanthopoulos [9] and Piran et al. [10] which have both the + and × polarization mode.

The first exact solution we consider is the Weber–Wheeler gravitational wave originally investigated by Einstein and Rosen [24]. It consists of a cylindrically symmetric vacuum wave with one radiational degree of freedom corresponding to the + polarization mode [21]. It describes a gravitational pulse originating from past null infinity and moving toward the z-axis. After imploding on the axis, it emanates to future null infinity.

This solution has no rotation so that ω and hence τ vanish and the solution may be described in terms of ψ which satisfies the wave equation. A solution to the wave equation in cylindrical coordinates may be given in terms of Bessel functions and by superposing such solutions we may write

$$\psi(t,\rho) = 2b \int_0^\infty e^{-a\Omega} J_0(\Omega\rho)\cos(\Omega t)d\Omega, \tag{53}$$

where $a > 0$. For convenience we let

$$X = a^2 + \rho^2 - t^2 \tag{54}$$

and one may show that (53) may be written in the alternative form

$$\nu(t,\rho) = \exp\left[2b\sqrt{\frac{2(X+\sqrt{X^2+4a^2t^2})}{X^2+4a^2t^2}}\right]. \tag{55}$$

The corresponding value of γ is obtained by integrating $\gamma_{,\rho} = \rho(\psi_{,t}^2 + \psi_{,\rho}^2)$ and using $\gamma(t,0) = 0$ and is found to be

$$\gamma(t,\rho) = \frac{b^2}{2a^2}\left[1 - 2a^2\rho^2\frac{X^2-4a^2t^2}{(X^2+4a^2t^2)^2} - \frac{a^2+t^2-\rho^2}{\sqrt{X^2+4a^2t^2}}\right]. \tag{56}$$

The next solution we consider is one due to Xanthopoulos [9] which has a conical singularity on the z-axis and therefore describes a rotating vacuum solution with a cosmic string type singularity. Xanthopoulos derived the spacetime by finding a solution to the Ernst equation in prolate spheroidal coordinates. To compare this with our numerical result we must transform to cylindrical coordinates and also find the Geroch potential. It is convenient to first define the following quantities

$$Q = \rho^2 - t^2 + 1, \tag{57}$$

$$X = \sqrt{Q^2 + 4t^2}, \tag{58}$$

$$Y = \frac{1}{2}[(2a^2+1)X + Q] + 1 - a\sqrt{2(X-Q)}, \tag{59}$$

$$Z = \frac{1}{2}[(2a^2+1)X + Q] - 1, \tag{60}$$

where $0 < |a| < \infty$. The solution derived by Xanthopoulos then becomes

$$\psi(t,\rho) = \frac{1}{2}\ln\frac{Z}{Y}, \tag{61}$$

$$\omega(t,\rho) = \frac{\sqrt{a^2+1}(X+Q-2)(Z-Y)}{2aZ}, \tag{62}$$

$$\gamma(t,\rho) = \frac{1}{2}\ln\frac{Z}{a^2X}, \tag{63}$$

where we have imposed $\gamma(t,0) = 0$. A straightforward but rather tedious calculation shows that this satisfies Einstein's field equations. (This and a number of other calculations in the paper were checked using the algebraic computing package GRTensor II [25]).

The norm of the Killing vector in the z-direction is given by

$$\nu(t,\rho) = \frac{Z}{Y}. \tag{64}$$

The Geroch potential is easily obtained from the Ernst potential and is found to be

$$\tau(t,\rho) = -\frac{\sqrt{2(a^2+1)}\sqrt{X+Q}}{Y}. \tag{65}$$

Note that ν and τ satisfy (26)–(27) in vacuum, i.e.

$$\Box\nu - \nu^{-1}(\tau_{,t}^2 - \tau_{,\rho}^2 - \nu_{,t}^2 + \nu_{,\rho}^2) = 0, \tag{66}$$

$$\Box\tau + 2\nu^{-1}(\tau_{,t}\nu_{,t} - \tau_{,\rho}\nu_{,\rho}) = 0. \tag{67}$$

Expressions for all these quantities may be obtained in the exterior characteristic region by transforming to the (u,y) variables. Although ψ tends to zero as one approaches null infinity, ω diverges so that the 4-dimensional metric is not asymptotically flat even along null geodesics lying in the planes $z =$ const. However by contrast the Geroch potential τ vanishes as one approaches null infinity in the 3-dimensional spacetime. This is an example of the fact that the Geroch potential can be well behaved even if ω in the JEKK form of the 4-metric diverges as one goes outward in a null direction. We also give an expression for the gravitational flux at infinity which is given by $E_{,u}$ where $E(u,y) = 2\pi[1 - e^{-\gamma(u,y)}]$

$$\lim_{y\to 0} E_{,u} = -\frac{4\pi}{(1+u^2)[(1+2a^2)\sqrt{1+u^2} - u]} < 0. \tag{68}$$

Thus the string is losing energy through gravitational radiation.

To plot the solution for $0 \le \rho < \infty$ we introduce the radial variable

$$w = \begin{cases} \rho & \text{for } 0 \le \rho \le 1 \\ 3 - 2/\sqrt{\rho} & \text{for } \rho > 1 \end{cases}, \tag{69}$$

thus $0 \le w \le 3$ where the infinite value of ρ is mapped to $w = 3$. This choice is slightly different from that of Dubal et al. [19] and avoids discontinuities in the radial derivatives at the interface due to the square root in the definition of y. Plots of ν, τ and γ as given by (63)–(65) are shown in Fig. 1. The error in the numerical results as computed using the CCM code are shown in Fig. 4.

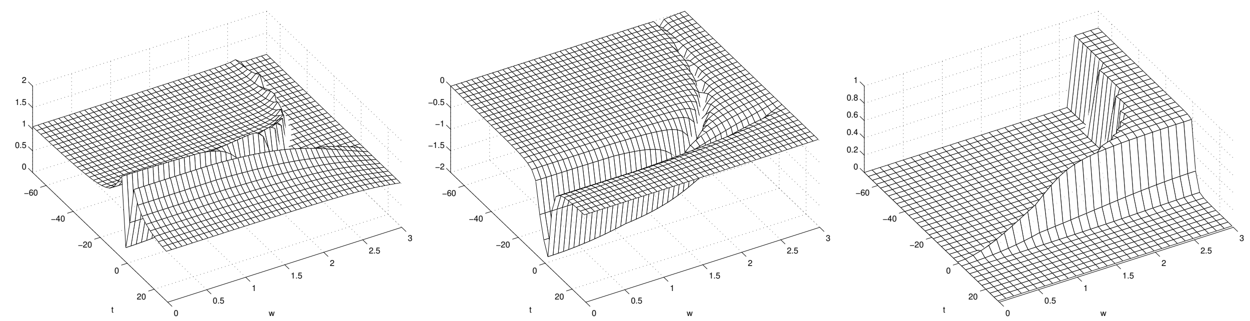

FIG. 1. The exact Xanthopoulos solution for $-70 \le t \le 30$, $0 \le w \le 3$ and $a = 0.5$. Plots are from left to right: $\nu(t,w)$, $\tau(t,w)$ and $\gamma(t,w)$. One clearly sees the incoming pulses in ν and τ. As the pulse hits the axis the string loses energy through gravitational radiation as seen in γ.

The final exact solution we consider is one due to Piran et al. [10] which also has two degrees of freedom representing the two polarization states. As in the case of Xanthopoulos' solution, it represents two incoming pulses that implode on the axis and then move away from it. Piran et al. obtained their solution by starting with the Kerr metric in Boyer-Linquist form, transforming to cylindrical polar coordinates and then swapping the t and z coordinates (and introducing some factors of i to maintain a real Lorentzian metric). See [10] for details. The resulting metric may be written in JEKK form. The solution is rather complicated but may be simplified by introducing the following additional quantities

$$R = b^{-1}[\sqrt{b^2 + (t-\rho)^2} - t + \rho], \tag{70}$$

$$S = b^{-1}[\sqrt{b^2 + (t+\rho)^2} + t + \rho], \tag{71}$$

$$T = 1 + RS + 2a^{-1}\sqrt{(a^2-1)RS} \tag{72}$$

and

$$
\begin{aligned}
X &= (1+R^2)(1+S^2), && (73)\\
Y &= a^2T^2 + (R-S)^2, && (74)\\
Z &= a^2(1-RS)^2 + (R+S)^2, && (75)
\end{aligned}
$$

where $1 \le a < \infty$ and $0 \le b < \infty$. The metric coefficients are then given by

$$
\begin{aligned}
\psi(t,\rho) &= \frac{1}{2}\ln\frac{Z}{Y}, && (76)\\
\omega(t,\rho) &= b\sqrt{a^2-1}\left[2\left(1+\frac{\sqrt{a^2-1}}{a}\right) - \frac{(R+S)^2T}{\sqrt{RSZ}}\right], && (77)\\
\gamma(t,\rho) &= \frac{1}{2}\ln\frac{Z}{X}. && (78)
\end{aligned}
$$

Notice that Minkowski space is obtained in the limit that $a \to 1$, and that we can also consider the case $b \to 0$ in which case the rotation vanishes. This is not true for the Xanthopoulos solution which is only real for a sufficiently large rotation.

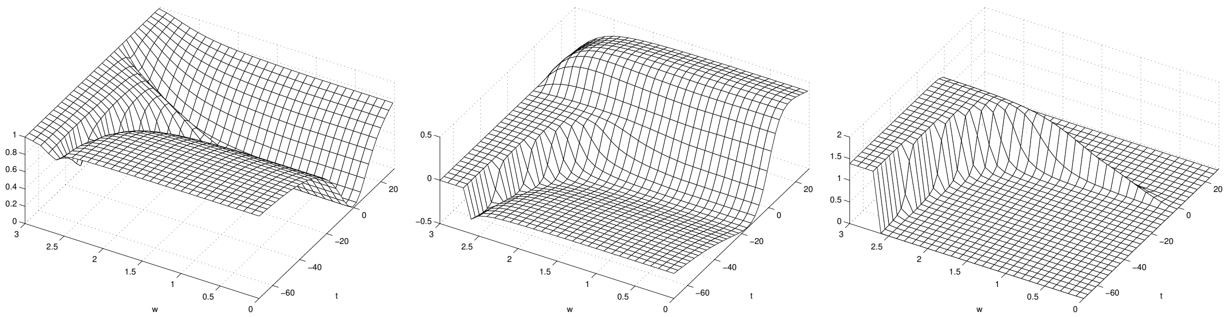

FIG. 2. The exact Piran et al. solution for $-70 \le t \le 30$, $0 \le w \le 3$, $a = 4$ and $b = 2$. Plots are from left to right: $\nu(t,w)$, $\tau(t,w)$ and $\gamma(t,w)$. Notice that ν does not have a double ridge as found in the Xanthopoulos solution.

The solution is regular on the axis, but like the Xanthopoulos solution ω diverges as one approaches null infinity. Again the answer is to transform to the ν, τ variables which are regular both on the axis and at null infinity. Finding ν is straightforward; however solving the differential equations for the Geroch potential τ for such a complicated metric is extremely difficult, but τ may be found by first finding the Geroch potential for the *timelike* Killing vector of the Kerr solution and then making the appropriate transformations. Note that the same process transforms the Killing vector into one along the z-axis. One then finds

$$
\begin{aligned}
\nu(t,\rho) &= \frac{Z}{Y}, && (79)\\
\tau(t,\rho) &= -\frac{4\sqrt{(a^2-1)RS}(R-S)}{[2\sqrt{(a^2-1)RS} + a(1+RS)]^2 + (R-S)^2}. && (80)
\end{aligned}
$$

The corresponding results in the characteristic region are easily found by transforming to (u, y) coordinates. Plots of ν, τ and γ as given by (78)–(80) are shown in Fig. 2. The error in the numerical results as computed using the CCM code are shown in Fig. 5.

We now briefly describe the accuracy and the convergence analysis for the explicit CCM version of our code. The results for the the implicit version are similar and are given in paper II. We define the pointwise error at the ith time slice and jth grid point for some function f by

$$
\xi^i_j(f) = f(t_i, w_j)_{\text{exact}} - f(t_i, w_j)_{\text{computed}}. \tag{81}
$$

The pointwise error for the vacuum solutions is shown in Fig. 3–5 for 600 grid points and 10,000 time steps corresponding to $0 \le t \le 15$. The code is stable and accurate for at least 20,000 time steps with a Courant factor of 0.45. Beyond this point the metric functions have almost decayed to zero and the dynamical behaviour is very slow.

In order to analyze the convergence of the code we define the spacetime ℓ_2-norm for some function f as

$$
\ell_2[f^{N_1}_{N_2}] = \sqrt{\frac{\sum_{i,j}[\xi^i_j(f)]^2}{N_1 \cdot N_2}}, \tag{82}
$$

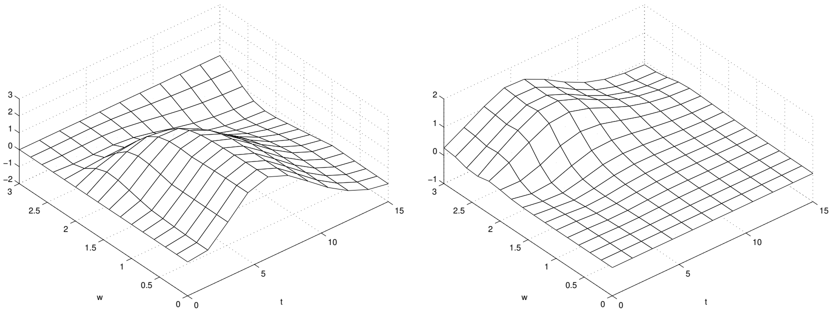

FIG. 3. Pointwise error for the Weber–Wheeler solution. From left to right: $\psi(t,w)\cdot 10^6$ and $\gamma(t,w)\cdot 10^7$.

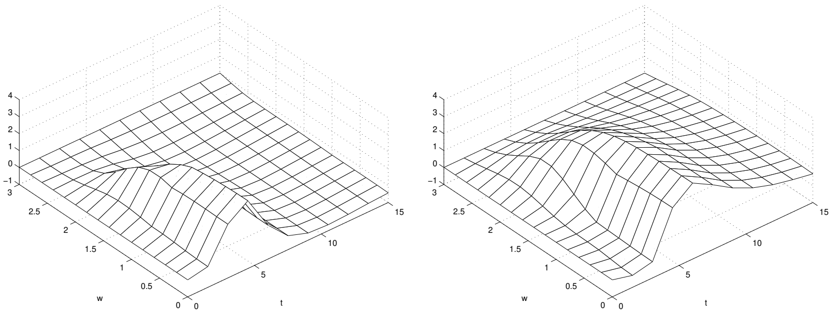

FIG. 4. Pointwise error for the Xanthopoulos solution. From left to right: $\nu(t,w)\cdot 10^5$ and $\tau(t,w)\cdot 10^5$.

where N_1 is the number of time slices and N_2 is the number of grid points on each slice. We also define the relative norm

$$f_r = \sqrt{\frac{\sum_{i,j}[\xi^i_j(f)]^2}{\sum_{i,j}[f^i_{j\ \mathrm{exact}}]^2}}. \tag{83}$$

Convergence testing of the code is done by doubling the grid size keeping the Courant factor constant. We therefore also need to double the number of time steps. We measure the convergence through

$$f_c = \frac{\ell_2[f^{N_1}_{N_2}]}{\ell_2[f^{2N_1}_{2N_2}]}. \tag{84}$$

For a convergent second order code one should have $f_c = 4$, and we can clearly see from Tables I–III that the code has achieved second order convergence in time and space.

Grid pts.	ν_r	γ_r	Time steps	ν_c	γ_c
300	$6.29\cdot 10^{-7}$	$9.38\cdot 10^{-6}$	5,000	–	–
600	$1.11\cdot 10^{-7}$	$1.66\cdot 10^{-6}$	10,000	4.01	4.01
1,200	$1.96\cdot 10^{-8}$	$2.94\cdot 10^{-7}$	20,000	4.00	4.00
2,400	$3.47\cdot 10^{-9}$	$5.19\cdot 10^{-8}$	40,000	4.00	4.00
4,800	$6.14\cdot 10^{-10}$	$9.18\cdot 10^{-9}$	80,000	4.00	4.00

TABLE I. Convergence test: the Weber–Wheeler solution.

Grid pts.	ν_r	τ_r	γ_r	ν_c	τ_c	γ_c
300	$1.70\cdot 10^{-6}$	$5.71\cdot 10^{-6}$	$1.66\cdot 10^{-6}$	–	–	–
600	$2.99\cdot 10^{-7}$	$1.01\cdot 10^{-6}$	$2.93\cdot 10^{-7}$	4.01	4.00	4.01
1,200	$5.28\cdot 10^{-8}$	$1.78\cdot 10^{-7}$	$5.18\cdot 10^{-8}$	4.01	4.00	4.01
2,400	$9.32\cdot 10^{-9}$	$3.15\cdot 10^{-8}$	$9.16\cdot 10^{-9}$	4.00	4.00	4.00
4,800	$1.65\cdot 10^{-9}$	$5.57\cdot 10^{-9}$	$1.62\cdot 10^{-9}$	4.00	4.00	4.00

TABLE II. Convergence test: the Xanthopoulos solution.

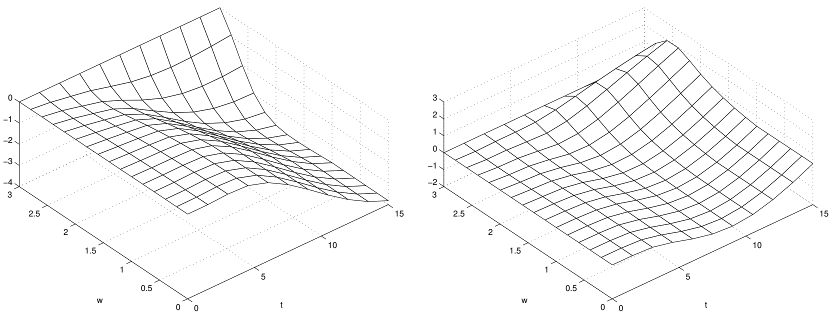

FIG. 5. Pointwise error for the Piran et al. solution. From left to right: $\nu(t,w)\cdot 10^5$ and $\tau(t,w)\cdot 10^5$.

Grid pts.	ν_r	τ_r	γ_r	ν_c	τ_c	γ_c
300	$1.07\cdot 10^{-5}$	$3.94\cdot 10^{-6}$	$2.76\cdot 10^{-6}$	–	–	–
600	$1.89\cdot 10^{-6}$	$6.96\cdot 10^{-7}$	$4.87\cdot 10^{-7}$	4.00	4.00	4.01
1,200	$3.35\cdot 10^{-7}$	$1.23\cdot 10^{-7}$	$8.61\cdot 10^{-8}$	4.00	4.00	4.00
2,400	$5.92\cdot 10^{-8}$	$2.17\cdot 10^{-8}$	$1.52\cdot 10^{-8}$	4.00	4.00	4.00
4,800	$1.05\cdot 10^{-8}$	$3.84\cdot 10^{-9}$	$2.69\cdot 10^{-9}$	4.00	4.00	4.00

TABLE III. Convergence test: the Piran et al. solution.

V. THE COSMIC STRING IN MINKOWSKI SPACETIME

In this section we examine the field equations for the cosmic string. The simplest case is to look at the equations of motion on a fixed Minkowski background. If we do this then the Euler–Lagrange equations for (1) give

$$\Box S = S[4\lambda(S^2-\eta^2) - \rho^{-2}P^2], \tag{85}$$

$$\Box P - 2\rho^{-1}P_{,\rho} = e^2 S^2 P, \tag{86}$$

where $\Box$ is the d'Alembertian in cylindrical polar coordinates given by (29).

It will turn out that the solutions of this simpler set of equations are qualitatively similar to those of the full system of equations for a dynamic cosmic string coupled to Einstein's equations. However the full system enables one to perturb a static string with a pulse of gravitational radiation and in turn look at the effect of the string's oscillations on the gravitational waves.

A special case of (85), (86) is when one looks for a static solution. This has been looked at before by a number of authors, for example Garfinkle [26], Laguna et al. [11], Laguna–Castillo et al. [12] and Dyer et al. [13]. For a static string equations (85) and (86) reduce to

$$\rho\frac{d}{d\rho}\left(\rho\frac{dS}{d\rho}\right) = S[4\lambda\rho^2(S^2-\eta^2)+P^2], \tag{87}$$

$$\rho\frac{d}{d\rho}\left(\rho^{-1}\frac{dP}{d\rho}\right) = e^2S^2P. \tag{88}$$

This pair of coupled second order equations requires four boundary conditions. For the physically relevant finite energy solution these are

$$S(0)=0, \qquad \lim_{\rho\to\infty} S(\rho)=\eta, \tag{89}$$

$$P(0)=1, \qquad \lim_{\rho\to\infty} P(\rho)=0. \tag{90}$$

It is not possible to obtain an exact solution to these equations but one can investigate the asymptotic behavior for large ρ. The solution satisfying the above boundary conditions has asymptotic behaviour given by [27]

$$S(\rho) \sim \eta - K_0(\sqrt{8\lambda}\eta\rho) \sim \eta - \sqrt{\frac{\pi}{\sqrt{32\lambda}\eta}}\rho^{-1/2}e^{-\sqrt{8\lambda}\eta\rho}, \tag{91}$$

$$P(\rho) \sim \rho K_1(e\eta\rho) \sim \sqrt{\frac{\pi}{2e\eta}}\rho^{1/2}e^{-e\eta\rho}. \tag{92}$$

However as well as these physical solutions, the equations admit non-physical solutions which have exponentially divergent behavior as $\rho \to \infty$. It is the existence of these non-physical solutions which make the problem rather delicate from a numerical point of view and makes a method such as shooting hard to apply.

Before proceeding further we follow Garfinkle [26] by introducing rescaled variables and constants which simplify the algebra (and are also important when considering the thin string limit). Provided we rescale the time coordinate this also simplifies the fully coupled system. Let us introduce

$$X = \frac{S}{\eta}, \tag{93}$$

$$r = \sqrt{\lambda}\eta\rho, \tag{94}$$

$$\tilde{t} = \sqrt{\lambda}\eta t, \tag{95}$$

$$\alpha = \frac{e^2}{\lambda}. \tag{96}$$

Thus α represents the relative strength of the coupling between the scalar and vector field given by e compared to the self-coupling of the scalar field given by λ. Critical coupling, when the masses of the scalar and vector fields are equal, is given by $\alpha = 8$. In the theory of superconductivity, $\alpha = 8$ corresponds to the interface between type I and type II behaviour [27].

With the above rescaling equations (87) and (88) become

$$r\frac{d}{dr}\left(r\frac{dX}{dr}\right) = X[4r^2(X^2-1)+P^2], \tag{97}$$

$$r\frac{d}{dr}\left(r^{-1}\frac{dP}{dr}\right) = \alpha X^2 P. \tag{98}$$

Note the rescaled version of (26)–(33) may be found simply by making the replacements

$$S \to X, \tag{99}$$

$$\rho \to r, \tag{100}$$

$$t \to \tilde{t}, \tag{101}$$

$$e^2 \to \alpha, \tag{102}$$

$$\lambda \to 1. \tag{103}$$

The boundary conditions of the cosmic string are given by

$$X(0) = 0, \qquad \lim_{r\to\infty} X(r) = 1, \tag{104}$$

$$P(0) = 1, \qquad \lim_{r\to\infty} P(r) = 0. \tag{105}$$

Because of the boundary conditions at infinity it is desirable to introduce a new coordinate which brings in infinity to a finite coordinate value. For this purpose we again introduce an inner region ($r \leq 1$) and an outer region ($r \geq 1$) where we use the radial coordinate y given by

$$y = \frac{1}{\sqrt{r}}. \tag{106}$$

In order to solve equations (97), (98) numerically we introduce a spatial grid consisting of n_1 points in the inner region and n_2 points in the outer region. The points $r_1, \ldots, r_{n_1}$ cover the range $0 \leq r \leq 1$, and $y_{n_1+1}, \ldots, y_{n_1+n_2}$ cover the range $1 \geq y \geq 0$. Thus, $r = 1 = y$ is represented by two points. These two points form the interface of the code where r-derivatives are transformed into y-derivatives. The static equations in the outer region are

$$y\frac{d}{dy}\left(y\frac{dX}{dy}\right) = 4X\left[4\frac{(X^2-1)}{y^4} + P^2\right], \tag{107}$$

$$y\frac{d}{dy}\left(y^5\frac{dP}{dy}\right) = 4\alpha X^2 P. \tag{108}$$

In order to apply boundary conditions at both $r = 0$ and $y = 0$ the equations were solved numerically using a relaxation scheme (as described in [28] for example). For this purpose we wrote the equations as a first order system in both regions (see paper II) and used second order centered finite differencing. Solutions were computed in this way for different resolutions $N = n_1 + n_2$ to check the convergence of the code. Since there is no exact solution available the convergence was checked by calculating the ℓ_2-norm with respect to a high resolution result for $N = 2400$ (1200 points in each region)

$$\ell_2[\Delta f^N] = \sqrt{\frac{\sum_{i=1}^{N}(f_i^N - f_i^{2400})^2}{N}}, \tag{109}$$

where f stands for either X or P. For a second order convergent code one would expect the ℓ_2-norm to decrease by a factor of 4 if the number of grid points is doubled. We find that the code shows clear second order convergence. Since the convergence in this case is very similar to that of a static string coupled to gravity considered in the next section, we only give the results for the latter case in Table IV.

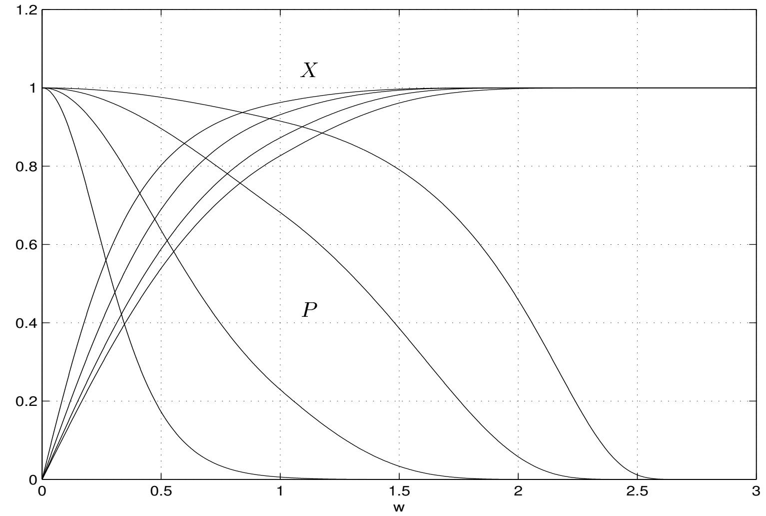

FIG. 6. Cosmic string in Minkowski spacetime. X and P for α: 0.125, 1, 8, 64. As α increases, both X and P become more concentrated towards the origin.

In Fig. 6. we plot X and P for $\alpha = 0.125$, 1, 8, 64. The results show that for fixed values of the self-coupling λ and vacuum expectation value η of the scalar field, both the vector and scalar fields become more concentrated towards the origin as the coupling between the fields e increases. One also finds that for a fixed ratio of the coupling constants α the fields become more concentrated towards the origin as either the self-coupling λ or the vacuum expectation value η of the scalar field increases.

VI. THE STATIC COSMIC STRING COUPLED TO GRAVITY

The next class of solutions we wish to consider are static solutions of the fully coupled equations. In the case of no t dependence equations (26)–(33) reduce to

$$\frac{1}{r}(r\nu_{,r})_{,r} = -\nu_{,r}\mu_{,r} + \frac{\nu_{,r}^2 - \tau_{,r}^2}{\nu} + 8\pi\eta^2\left[\nu^2 e^{-2\mu}\frac{P_{,r}^2}{\alpha r^2} - 2e^{2(\gamma+\mu)}(X^2-1)^2\right], \tag{110}$$

$$\frac{1}{r}(r\tau_{,r})_{,r} = \tau_{,r}\left(2\frac{\nu_{,r}}{\nu} - \mu_{,r}\right), \tag{111}$$

$$\frac{1}{r^2}(r^2\mu_{,r})_{,r} = -\mu_{,r}^2 - 8\pi\eta^2\left[e^{2\gamma}\frac{X^2P^2}{r^2} + 2\frac{e^{2(\gamma+\mu)}}{\nu}(X^2-1)^2\right], \tag{112}$$

$$\gamma_{,r} = \frac{r}{1+r\mu_{,r}}\left\{\frac{1}{4\nu^2}\left(\tau_{,r}^2 + \nu_{,r}^2\right) + 4\pi\eta^2\left[X_{,r}^2 + \nu e^{-2\mu}\frac{P_{,r}^2}{\alpha r^2} - e^{2\gamma}\frac{X^2P^2}{r^2} - 2\frac{e^{2(\gamma+\mu)}}{\nu}(X^2-1)^2\right]\right\} - \mu_{,r}, \tag{113}$$

$$\frac{1}{r}(rX_{,r})_{,r} = -X_{,r}\mu_{,r} + X\left[4\frac{e^{2(\gamma+\mu)}}{\nu}(X^2-1) + e^{2\gamma}\frac{P^2}{r^2}\right], \tag{114}$$

$$r\left(\frac{1}{r}P_{,r}\right)_{,r} = P_{,r}\left(\mu_{,r} - \frac{\nu_{,r}}{\nu}\right) + \alpha\frac{e^{2(\gamma+\mu)}}{\nu}PX^2. \tag{115}$$

Notice that (111) with the corresponding boundary conditions is satisfied by the trivial solution $\tau = 0$. One also has from the field equations

$$(r\gamma_{,r})_{,r} = -r\gamma_{,r}\mu_{,r} + \mu_{,r} + 8\pi\eta^2\left(e^{2\gamma}\frac{X^2P^2}{r} + e^{-2\mu}\nu\frac{P_{,r}^2}{\alpha r}\right). \tag{116}$$

This equation is a direct consequence of the other equations and will not be used in the calculations but is instead used as a check for the code. The equations in the outer region as well as the resulting first order system, the interface equations and boundary conditions are given in paper II. We use the same grid and numerical method as in the Minkowskian case. In order to check the code for convergence we again compute the ℓ_2-norm with respect to a high resolution calculation. The results are shown in Table IV for $\alpha = 1$ and the large value $\eta = 0.1$ and clearly indicate second order convergence. Small deviations from a convergence factor of 4 are expected since we compare against a high resolution reference solution rather than an exact solution. The same result has been obtained for other choices of α and η.

	ν	μ	γ	X	P
$\ell_2(f^{1200})$	$1.28 \cdot 10^{-7}$	$2.51 \cdot 10^{-6}$	$2.39 \cdot 10^{-6}$	$4.16 \cdot 10^{-7}$	$5.95 \cdot 10^{-7}$
$\ell_2(f^{150})/\ell_2(f^{300})$	3.56	3.59	3.58	3.37	4.04
$\ell_2(f^{300})/\ell_2(f^{600})$	3.76	3.79	3.78	3.60	4.19
$\ell_2(f^{600})/\ell_2(f^{1200})$	4.58	4.61	4.60	4.44	4.98

TABLE IV. Convergence test: static cosmic string coupled to gravity.

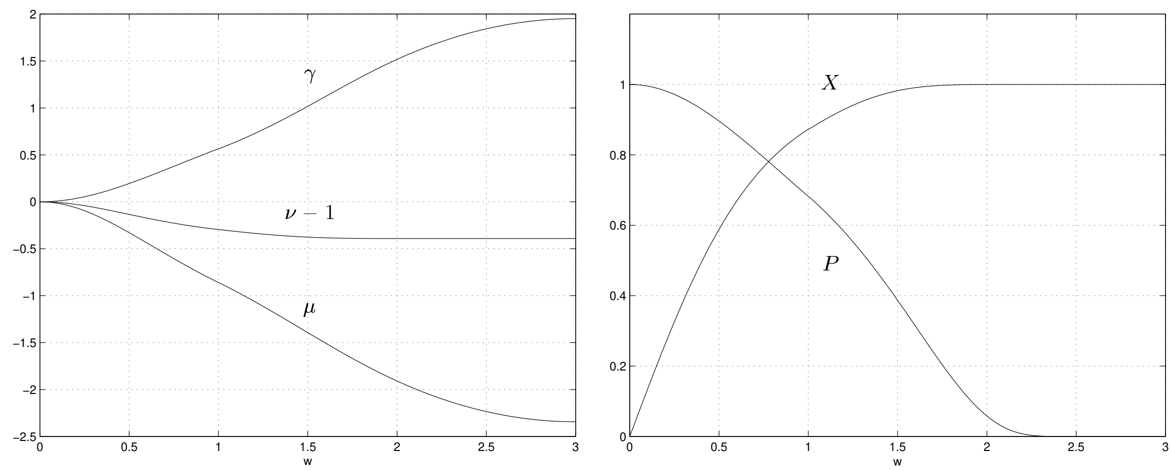

FIG. 7. Cosmic string coupled to gravity. $\alpha = 1$ and $\eta = 10^{-3}$. On the left the metric variables: γ, $\nu - 1$ and μ are plotted multiplied by 10^5. On the right X and P are plotted.

The metric and matter variables for $\alpha = 1$ and $\eta = 10^{-3}$ are shown in Fig. 7. We find that the behaviour of X and P is very close to that obtained for a static cosmic string in Minkowski spacetime shown in Fig. 6. For physically realistic values of η the metric variables at infinity are close to their Minkowskian values although the non-zero value of $\gamma_0 = \lim_{r\to\infty}\gamma(r)$ indicates that the spacetime is asymptotically conical, that is Minkowski spacetime minus a wedge with deficit angle $\Delta\phi = 2\pi(1 - e^{-\gamma_0})$. Thus a string with $\alpha = 1$ and $\eta = 10^{-3}$ has an angular deficit of about 2×10^{-5} which corresponds to a grand unified symmetry breaking scale of about 10^{16} GeV [27]. For larger values of η, however, the deviation from the Minkowskian case becomes substantial. Close to critical coupling the deficit angle exceeds 2π for values of η greater than about 0.28 which explains why the code converges well for $\eta \lesssim 0.28$.

VII. THE DYNAMIC COSMIC STRING COUPLED TO GRAVITY

In this section we discuss the interaction between the cosmic string and the gravitational field. Here we will simply outline the numerical scheme, the full details are given in paper II. The field equations in the (t, ρ) coordinates are given by equations (26)–(33) while those in (u, y) coordinates are given by equations (43)–(49). In both cases there are two additional Einstein equations which are consequences of the others and are used to check the code. In fact two numerical schemes were employed. The first was an explicit CCM scheme similar to that employed by Dubal et al. [19] and d'Inverno et al. [20]. However the use of the geometrical variables ν and τ in both the interior Cauchy region and the exterior characteristic region significantly improves the interface and results in a genuinely second order scheme with good accuracy even with both polarizations present. For the vacuum equations the code also exhibits long term stability. However when the matter variables are included this code performs less satisfactorily. This is because of the existence of exponentially growing non-physical solutions. It is possible to control these diverging solutions by multiplying the u-derivatives of the matter variables by a smooth 'bump function' which vanishes at $y = 0$ but is equal to 1 for $y > c$ (where c is a parameter). This produces satisfactory results but the bump function introduces some noise into the scheme which eventually gives rise to instabilities.

A much better solution is to control the asymptotic behavior at infinity by using an implicit scheme. The main problem with the system of differential equations is the irregularity of the equations at both the origin and null infinity. Therefore just as in the static case considered in the previous section the scheme employed divides the spacetime into two regions, an inner region $r \leq 1$ where coordinates (u, r) are used and an exterior region $r \geq 1$ where the (u, y) coordinates are used. The equations in both regions are written as a first order system connected by an interface and the evolution is accomplished using a code based on the implicit Crank–Nicholson scheme. This implicit scheme provided a simple way of implementing the boundary conditions and thus circumventing all problems with the irregularities. The outer boundary conditions as well as the equations in the outer region, the first order system used for the numerics and the interface are given in paper II. The implicit code showed very good agreement with both the exact (vacuum) and previously obtained (static) numerical solutions. It also showed clear second order convergence and very long term stability. For convenience characteristic coordinates were also used in the inner region but we do not think that their use was responsible for the good features of this code and believe that an implicit CCM scheme would have produced similar accuracy, convergence and long term stability.

The code is able to consider the interaction of the cosmic string with a gravitational field with two degrees of freedom, however here we simply describe the interaction with a Weber–Wheeler wave which has just one degree of freedom. We consider a pulse which comes in from past null infinity and interacts with a cosmic string in its static equilibrium configuration. This interaction causes the string to oscillate which in turn affects the gravitational field as measured by ν and τ. The oscillations in both X and P decay as one approaches null infinity (i.e. as $r \to \infty$ for fixed u) and also for fixed r as $u \to \infty$. After the oscillation has died away the string variables X and P return to their static values. Note however that this decay is rather slow and being able to show this effect depends upon the long term stability of the code. Although oscillations are observed in both X and P the character and frequency of these oscillations is rather different.

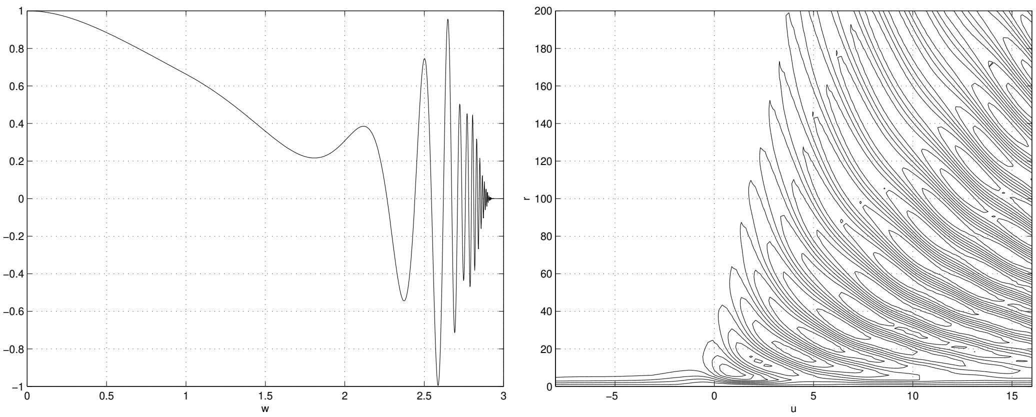

FIG. 8. On the left: $P(u, w)$ for $\alpha = 1$, $\eta = 10^{-3}$ and $u = 8.5$. On the right: The corresponding contour plot of $P(u, r)$ for $0 \leq r \leq 200$ and $-8 \leq u \leq 16$. Note the use of r for values greater than one in this plot.

In Fig. 8. we plot P for $\alpha = 1$ and $\eta = 10^{-3}$ at a time $u = 8.5$ (left panel). The oscillations out to large radii can be clearly seen . The contour plot on the right shows the ringing behaviour of the string and the slow decay of the oscillations in P. In contrast the oscillations in X displayed in Fig. 9. are restricted to small radii and decay on a shorter timescale.

An investigation of the frequencies $\tilde{f}_X$ and $\tilde{f}_P$ of the oscillations of the scalar field X and the vector field P (in the rescaled unphysical variables) indicates that they are relatively insensitive to the value of η and the Weber–Wheeler pulse which excites the string. They are also largely independent of the radius at which they are measured. However,

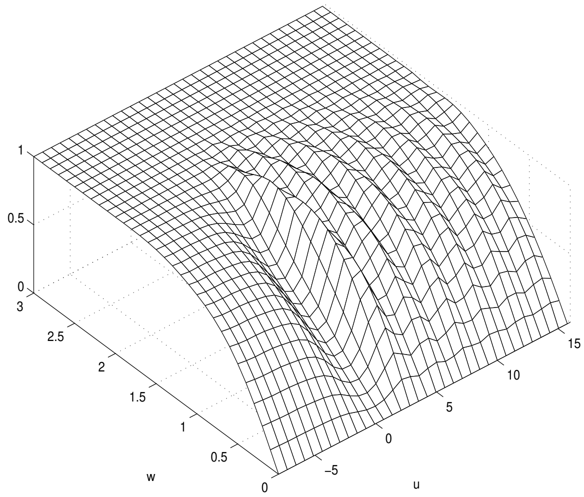

FIG. 9. $X(u, w)$ for $\alpha = 1$, $\eta = 10^{-3}$.

although $\tilde{f}_X$ is also independent of α, we find that $\tilde{f}_P$ is proportional to $\sqrt{\alpha}$. When we convert to the physical fields S and P and use the physical coordinates (t, ρ) we find that the frequencies in natural units are given by

$$f_S \sim \sqrt{\lambda}\eta \tag{117}$$

$$f_P \sim e\eta. \tag{118}$$

If we now use the fact that the masses of the scalar and vector fields are given by $m_S^2 \sim \lambda\eta^2$ and $m_P^2 \sim e^2\eta^2$ [27] this gives

$$f_S \sim m_S \tag{119}$$

$$f_P \sim m_P. \tag{120}$$

In fact these frequencies are also obtained by considering the simpler model of a dynamic cosmic string in Minkowski space with equations of motion (76)–(77) provided one gives S and P similar initial conditions to that produced by the interaction with the pulse of gravitational radiation. Thus the main role of the gravitational field as far as the string is concerned is to provide a mechanism for perturbing the string. This is discussed more fully in paper II.

VIII. CONCLUSION

In this paper we have shown how the method of Geroch decomposition may be used to recast the field equations for a time dependent cylindrical cosmic string in four dimensions in terms of fields on a reduced 3-dimensional spacetime. This has the advantage that it has a well defined notion of null infinity. It is therefore possible to conformally compactify the 3-dimensional spacetime and avoid the need for artificial outgoing radiation conditions. An additional feature of this approach is that it naturally introduces two geometrically defined variables ν and τ which encode the two gravitational degrees of freedom in the original spacetime.

We have described how the field equations for a cosmic string coupled to gravity may be solved numerically by dividing the 3-dimensional spacetime into two regions; an interior region $r \leq 1$, and an exterior region $r \geq 1$ in which the coordinate y is used. The use of the geometric variables ν and τ greatly simplifies the transmission of information at the interface $r = 1$. Although an explicit CCM code worked very effectively in the vacuum case, the asymptotic behavior of the matter fields S and P made it less effective when the string is coupled to the gravitational field. An alternative implicit fully characteristic scheme however exhibited good accuracy and long term stability.

In this paper we have demonstrated the effectiveness of the CCM codes in reproducing the results of the Weber–Wheeler solution and two vacuum spacetimes with two degrees of freedom due to Xanthopoulos and Piran et al. This involved calculating the Geroch potential for these solutions and using it to compare with the numerically computed values. We have also described a code for a static cosmic string which uses a relaxation scheme and provides initial data for the dynamic code.

In the final section of the paper we briefly described the interaction of the cosmic string with a Weber–Wheeler gravitational wave. The pulse of gravitational radiation excites the string and causes the fields S and P to oscillate with frequencies proportional to their respective masses. The full details of the code, the convergence testing and the interaction between the string and the gravitational field are described in paper II.

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