A new numerical approach to radial oscillations of neutron stars

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Keywords

– Neutron stars

– Fully non-linear oscillations

– Perturbations

Overview

(1) Introduction
(2) The static case (TOV)
(3) The dynamic case
(4) The non-linear perturbation approach
(5) Comparison

Introduction

- Oscillations of neutron stars $\Rightarrow$ grav. waves (?)
- Numerical treatment usually in simplified cases:
  - Linearised equations,
  - (Post-) Newtonian case,
  - Cowling Approximation (frozen metric),...
- But eventually need the fully relativistic scenario!

Problems:

- Relativistic, non-linear 3-dim case complicated,
- Computational power $\Rightarrow$ low resolution,
- Stability issues.

So use systems with higher symmetry to

- Investigate non-linear features,
- Study numerical problems.

Simplest case: Spherical symmetry

The static (TOV) case

Line element in spherical symmetry

(radial gauge, polar slicing)

$$ds^2 = -\lambda^2 dt^2 + \mu^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Four velocity and energy momentum tensor

$$u_a = (\lambda, 0, 0, 0)$$

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab}$$

Equations:

$$\frac{\lambda_{,r}}{\lambda} = \frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 P$$

$$\frac{\mu_{,r}}{\mu} = -\frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 \rho$$

$$P_{,r} = -(\rho + P)\frac{\lambda_r}{\lambda}$$

$$r_{,y} = C, \quad \text{where} \quad C^2 := \frac{\partial P}{\partial \rho}$$

Equation of state (polytrope):

$$P = K\rho^\gamma$$

$$\gamma = 1 + \frac{1}{n}$$

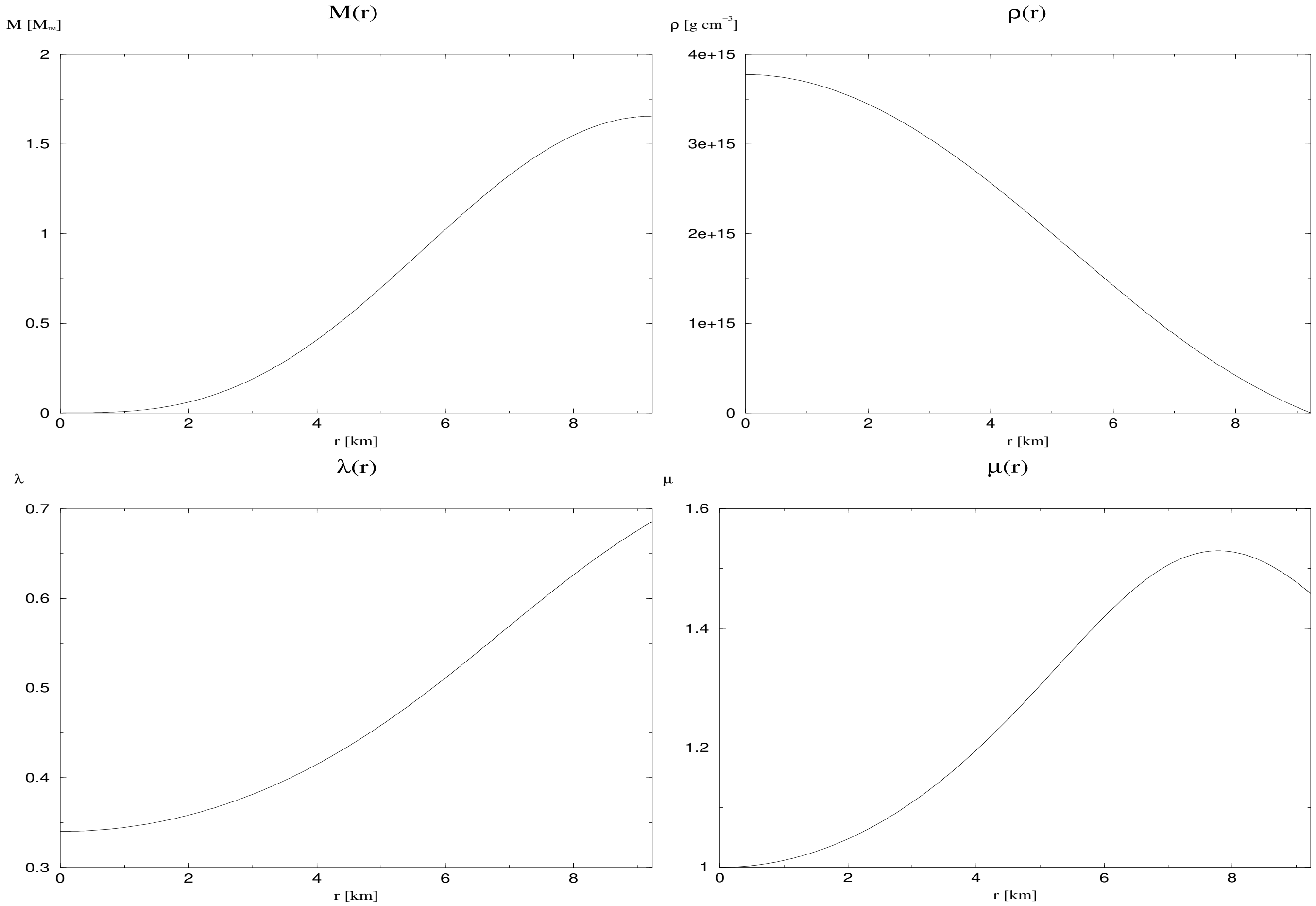
Boundary conditions:

$$\mu(0) = 1$$

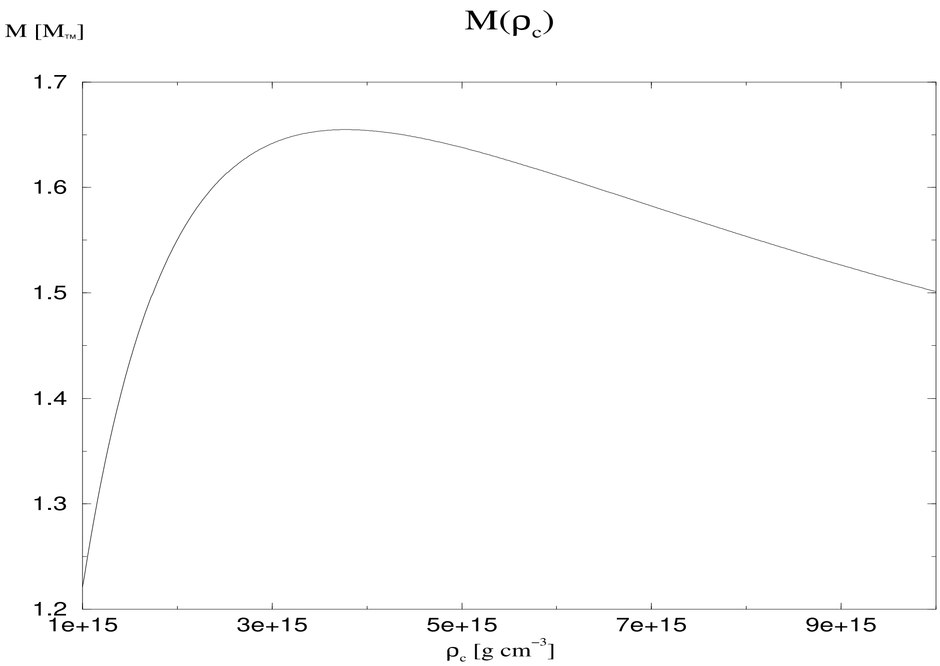
$$P(r_{\mathrm{s}}) = 0$$

Solutions: One parameter families (ρ_c)

$$\gamma = 2, \quad \rho_c = 3.8 \cdot 10^{15}\ g\,cm^{-3}$$



Stability curve



The dynamic case

- λ, μ, ρ, P are functions of (r,t) now
- Four velocity: $u_a = (v(r,t), w(r,t), 0, 0)$

Field Equations:

$$\frac{\hat{\mu}_{,r}}{\hat{\mu}} = -\frac{\hat{\mu}^2 - 1}{2r} + 4\pi r\hat{\mu}^2\left[\hat{\rho} + (\hat{\rho} + \hat{P})\hat{\mu}^2 w^2\right]$$

$$\frac{\hat{\lambda}_{,r}}{\hat{\lambda}} = \frac{\hat{\mu}^2 - 1}{2r} + 4\pi r\hat{\mu}^2\left[\hat{P} + (\hat{\rho} + \hat{P})\,\hat{\mu}^2 w^2\right]$$

Hydro-Equations:

$$\hat{\rho}_{,t} = \alpha_{11}\hat{\rho}_{,r} + \alpha_{12}w_{,r} + b_1,$$

$$w_{,t} = \alpha_{21}\hat{\rho}_{,r} + \alpha_{11}w_{,r} + b_2,$$

with

$$D = -v\left(1 - \hat{C}^2\frac{\hat{\mu}^2 w^2}{1 + \hat{\mu}^2 w^2}\right)$$

$$\alpha_{11} = \frac{w(1 - \hat{C}^2)}{D}$$

$$\alpha_{12} = \frac{\hat{\rho} + \hat{P}}{(1 + \hat{\mu}^2 w^2)D}$$

$$\alpha_{21} = \frac{\hat{C}^2}{(\hat{\rho} + \hat{P})\hat{\mu}^2 D}$$

$$b_1 = \frac{1}{D}(\hat{\rho} + \hat{P})\left(\frac{w\hat{\mu}_{,r}/\hat{\mu} + v\hat{\mu}_{,t}/\hat{\mu}}{1 + \hat{\mu}^2 w^2} + 2\frac{w}{r}\right)$$

$$b_2 = \frac{1}{D}\left[w^2\left(\frac{\hat{\mu}_{,r}}{\hat{\mu}} + \frac{\hat{\lambda}_{,r}}{\hat{\lambda}} - \frac{2}{r}\hat{C}^2 + 2\frac{v}{w}\frac{\hat{\mu}_{,t}}{\hat{\mu}}\right) + \frac{\hat{\lambda}_{,r}}{\hat{\lambda}\hat{\mu}^2}\right]$$

A non-linear perturbation approach

$$\hat{\lambda} = \lambda + \delta\lambda$$

$$\hat{\mu} = \mu + \delta\mu$$

$$\hat{\rho} = \rho + \delta\rho$$

$$\hat{P} = P + \delta P$$

Note: The perturbations (δ's) are NOT infinitesimal!

Field equations:

$$\begin{aligned}
\delta\mu_{,r} = & -\mu\frac{2\mu\delta\mu + \delta\mu^2}{2r} - \delta\mu\frac{\hat{\mu}^2 - 1}{2r} + 4\pi r\mu^3\left[\delta\rho + (\rho + \delta\rho + \hat{P})\hat{\mu}^2 w^2\right] \\
& + 4\pi r(3\mu^2\delta\mu + 3\mu\delta\mu^2 + \delta\mu^3)\left[\hat{\rho} + (\rho + \delta\rho + \hat{P})\hat{\mu}^2 w^2\right]
\end{aligned}$$

$$\begin{aligned}
\delta\lambda_{,r} = & \lambda\frac{2\mu\delta\mu + \delta\mu^2}{2r} + \delta\lambda\frac{\hat{\mu}^2 - 1}{2r} + 4\pi r\lambda\mu^2\left[\delta P + (\rho + \delta\rho + \hat{P})\hat{\mu}^2 w^2\right] \\
& + 4\pi r\left[\lambda(2\mu\delta\mu + \delta\mu^2) + \delta\lambda\hat{\mu}^2\right]\left[\hat{P} + (\rho + \delta\rho + \hat{P})\hat{\mu}^2 w^2\right]
\end{aligned}$$

Hydro-equations:

$$\delta\rho_{,t} = \alpha_{11}\delta\rho_{,r} + \alpha_{12}w_{,r} + \hat{b}_1$$

$$w_{,t} = \alpha_{21}\delta\rho_{,r} + \alpha_{11}w_{,r} + \hat{b}_2$$

Comparison

Stellar model:

- $\gamma = 2$ polytrope
- $\rho_c = 1.68 \cdot 10^{15}\ g\,cm^{-3}$
- $M = 1.48\ M_\odot$
- $r = 11.3\ km$

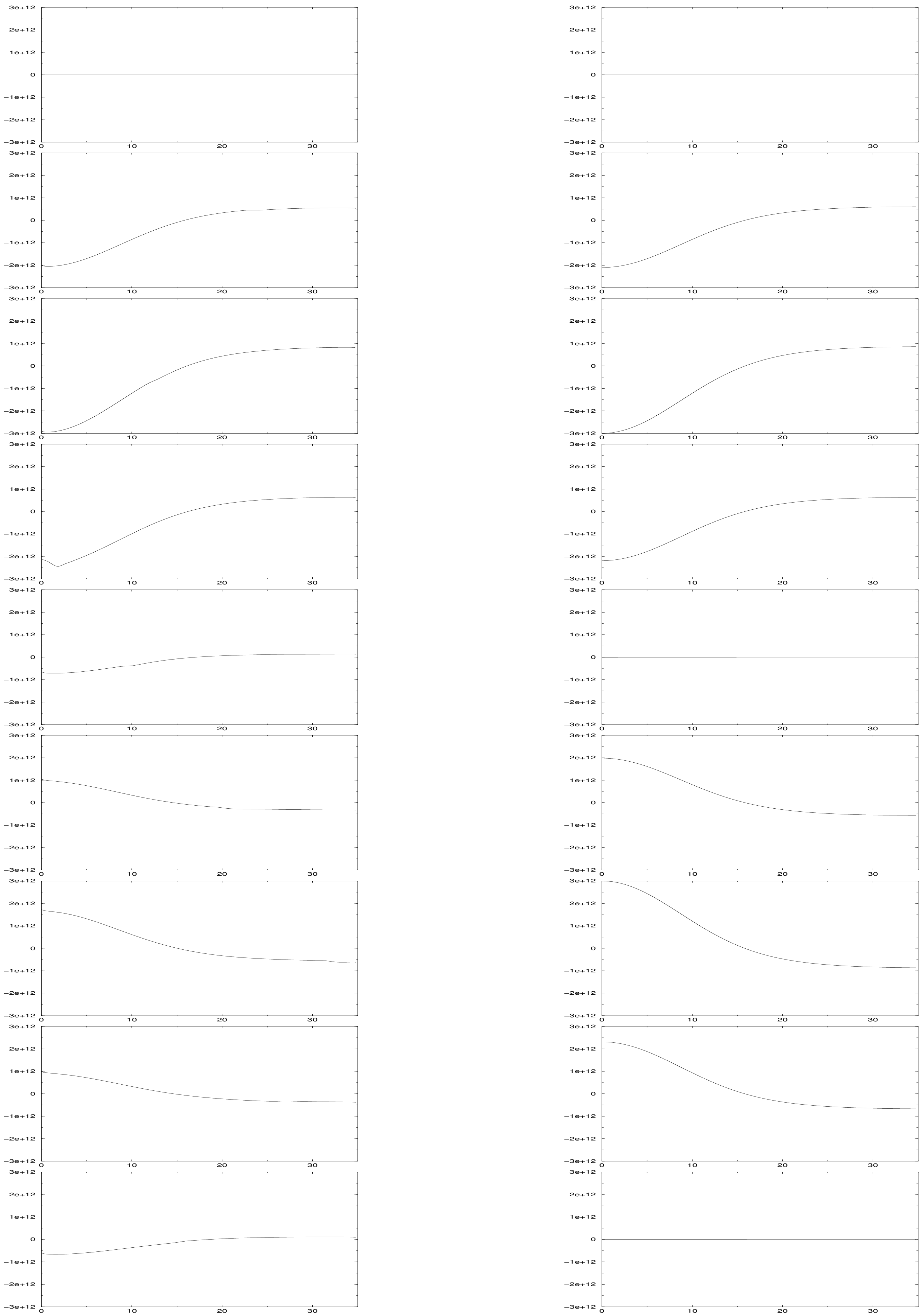
Perturbation:

- Fundamental eigenmode
- Amplitude: $3 \cdot 10^{12}\ g\,cm^{-3}$ (several m)
- linear regime

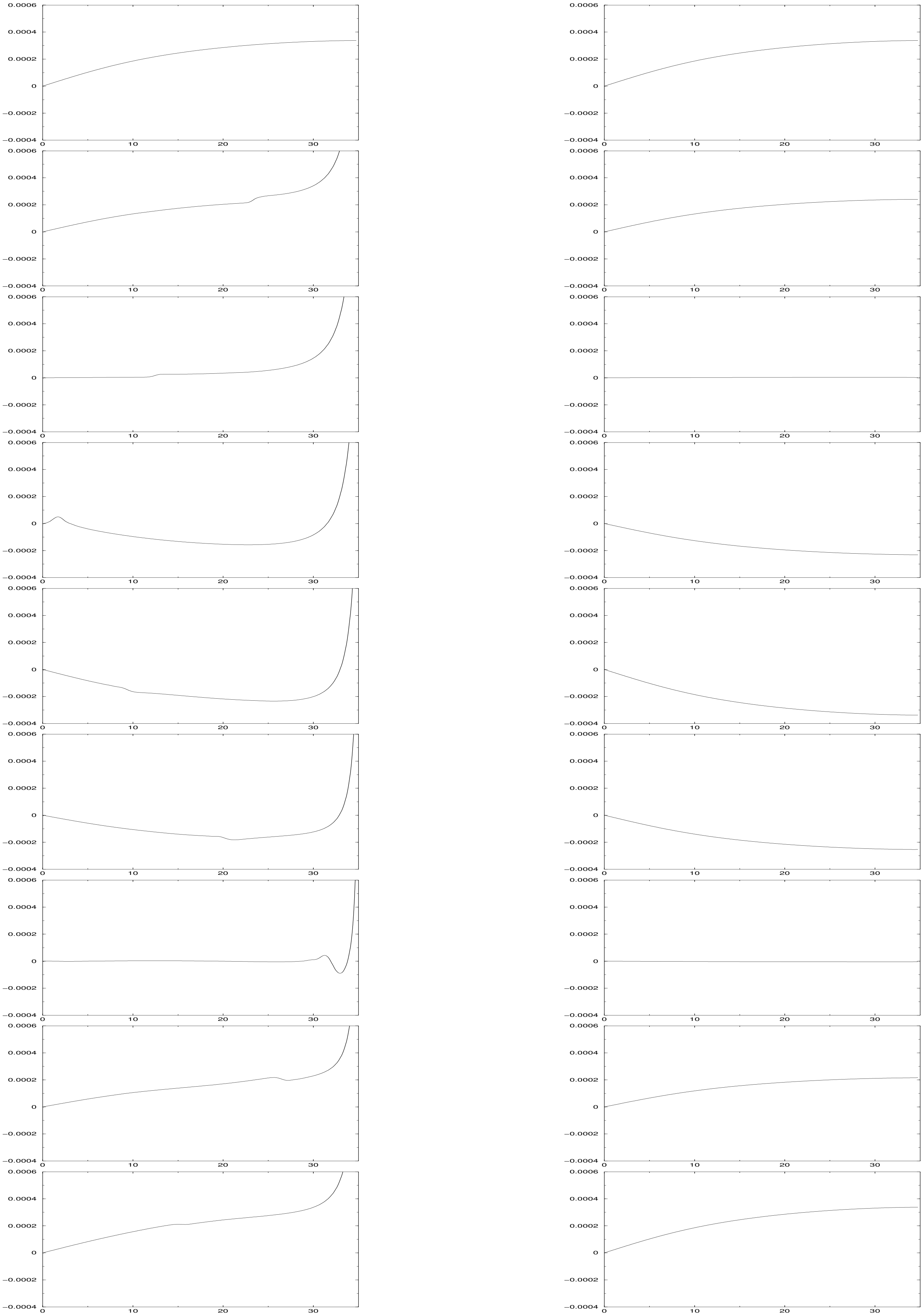
Numerics:

- Mc Cormack (2^{nd} order)
- ≈ 400 gridpoints

$\delta\rho(r,t)$



$w(r,t)$



Conclusions

Using full variables:

- Background contaminations,
- Problems at boundaries,
- better performance for larger amplitudes...
  but: non-linear effects,
- better performance for higher resolution...
  but: works in 1-dim only.

Using perturbational approach:

- Much better performance at low resolution,
- Equations NOT much more complicated,
- ONE code for linear and non-linear regime.

Applications

- Nonlinear stability,
- Shocks, Modecoupling,
- Generic numerical problems (surface).