Harald Dimmelmeier

“MARIAGE DES MAILLAGE”:
Combining Finite Difference Schemes and Spectral Methods in Relativistic Hydrodynamics

Work with

José A. Font and José M. Ibáñez (Universidad de Valencia)
Jérôme Novak (LUTH, Meudon)
Ewald Müller (MPA Garching)
Nick Stergioulas (Aristotle University, Thessaloniki)


http://www.mpa-garching.mpg.de/relhydro/

Seminar, Aristotle University, Thessaloniki, Greece, 2004
Outline of the Talk
Outline of the Talk

- Astrophysical motivation.
- Gravitational waves.
- Various approaches for numerical simulations.
- Results from our previous simulations.
- Spectral methods.
- Our new 3d code.
- Tests: The ladder of credibility.
- First applications:
  - Neutron star oscillations, supernova core collapse, bar mode instabilities.
Outline of the Talk

- Astrophysical motivation.
- Gravitational waves.
- Various approaches for numerical simulations.
- Results from our previous simulations.
- Spectral methods.
- Our new 3d code.
- Tests: The ladder of credibility.
- First applications: Neutron star oscillations, supernova core collapse, bar mode instabilities.

And particularly we demonstrate:

**Spectral methods can be used to efficiently solve elliptic equations in 3d!**

This is crucial for **new schemes in numerical relativity!**
Supernova Core Collapse
Supernova Core Collapse

Our code was originally tailored for supernova core collapse.

Current standard model for core collapse supernova:

- Subsequent nuclear burning in massive star yield shell structure.
- Iron core with $M \sim 1.4 M_\odot$ and $R \sim 1000$ km develops in center.
- Equation of state: Relativistic degenerate fermion gas, $\gamma = 4/3$.
- Instability due to photo-disintegration and electron capture.
Supernova Core Collapse

Our code was originally tailored for supernova core collapse.

Current standard model for core collapse supernova:

- Subsequent nuclear burning in massive star yield shell structure.
- Iron core with $M \sim 1.4 M_\odot$ and $R \sim 1000$ km develops in center.
- Equation of state: Relativistic degenerate fermion gas, $\gamma = 4/3$.
- Instability due to photo-disintegration and electron capture.

- Collapse to nuclear matter densities in $T \sim 100$ ms.
- Stiffening of EoS, bounce, and formation of prompt shock.
- Stalled shock revived by neutrinos energy deposition.
- Delayed shock propagates out and disrupts envelope of star.
- Proto-neutron star cools and shrinks to neutron star.
Gravitational Waves

Shock breaks through stellar surface: See supernova explosion as EM waves (light curve).

This happens hours after core collapse!

⇒ Light curve is only “echo” of supernova driving engine.
Gravitational Waves

Shock breaks through stellar surface: See supernova explosion as EM waves (light curve).

This happens hours after core collapse!

$\implies$ Light curve is only “echo” of supernova driving engine.

But: Can use two complementary methods to look at “heart” of supernova:

- **Neutrinos**: Store main (gravitational) energy of collapse and drive supernova shock.
Gravitational Waves

Shock breaks through stellar surface: See supernova explosion as EM waves (light curve).

This happens hours after core collapse!

⇒ Light curve is only “echo” of supernova driving engine.

But: Can use two complementary methods to look at “heart” of supernova:

- **Neutrinos**: Store main (gravitational) energy of collapse and drive supernova shock.
- **Gravitational waves**: Dynamically unimportant, very weak effect ($h \sim 10^{-20}$).
Various Approaches to Simulations of Core Collapse

Historically, supernova core collapse simulations focus on correct treatment of microphysics
\[\Rightarrow\text{ Neglect general relativity, use Newtonian gravity (possibly with corrections).}\]
Various Approaches to Simulations of Core Collapse

Historically, supernova core collapse simulations focus on correct treatment of microphysics \(\implies\) Neglect general relativity, use Newtonian gravity (possibly with corrections).

Since few years: Multidimensional general relativistic hydrodynamic simulations feasible (typically derived from Cartesian black hole vacuum codes, with simple hydro added).
Various Approaches to Simulations of Core Collapse

Historically, supernova core collapse simulations focus on correct treatment of microphysics

\[ \implies \text{Neglect general relativity, use Newtonian gravity (possibly with corrections)} \]

Since few years: Multidimensional general relativistic hydrodynamic simulations feasible (typically derived from Cartesian black hole vacuum codes, with simple hydro added).

Advantages:

- Consistent relativistic formulation.
- Gravitational wave emission arises naturally.
Various Approaches to Simulations of Core Collapse

Historically, supernova core collapse simulations focus on correct treatment of microphysics \( \rightarrow \) Neglect general relativity, use Newtonian gravity (possibly with corrections).

Since few years: Multidimensional general relativistic hydrodynamic simulations feasible (typically derived from Cartesian black hole vacuum codes, with simple hydro added).

Advantages:

- Consistent relativistic formulation.
- Gravitational wave emission arises naturally.

Difficulties:

- Coordinates usually not adopted to geometry of core collapse.
- Mesh refinement (in core collapse: radial contraction scale \( \sim 100! \)).
- Issue of gauge freedom in relativity (separate invariants from “coordinate” effects).
- Nonuniqueness of formulating metric equations (crucial issue!).
Reformulations of the Einstein metric equations

ADM equations in standard form: Split into 10 evolution and 4 constraint equations.

Free evolution: Constraint violating mode grows exponentially $\implies$ Unphysical results!
Reformulations of the Einstein metric equations

ADM equations in standard form: Split into 10 evolution and 4 constraint equations.
Free evolution: Constraint violating mode grows exponentially $\implies$ Unphysical results!

Two approaches for reformulating Einstein equations:

- Emphasize hyperbolicity, evolve all degrees of freedom:
  Bona–Massó, ADM-BSSN (NOK), Z4.
  $\implies$ Long-term numerical stability improves, but ultimately constraints are violated.
Reformulations of the Einstein metric equations

ADM equations in standard form: Split into 10 evolution and 4 constraint equations.
Free evolution: Constraint violating mode grows exponentially \( \implies \) Unphysical results!

Two approaches for reformulating Einstein equations:

- Emphasize hyperbolicity, evolve all degrees of freedom:
  Bona–Massó, ADM-BSSN (NOK), Z4.
  \( \implies \) Long-term numerical stability improves, but ultimately constraints are violated.

- Can physical system fit into time window of stability? \( \implies \) Unsatisfactory approach!
  - Possible way out: Reprojection onto constraint hypersurface (Holst et al., 2004).

Seminar, Aristotle University, Thessaloniki, Greece, 2004
Reformulations of the Einstein metric equations

ADM equations in standard form: Split into 10 evolution and 4 constraint equations.
Free evolution: Constraint violating mode grows exponentially $\implies$ Unphysical results!

Two approaches for reformulating Einstein equations:

- Emphasize hyperbolicity, evolve all degrees of freedom:
  Bona–Massó, ADM-BSSN (NOK), Z4.
  $\implies$ Long-term numerical stability improves, but ultimately constraints are violated.
  - Can physical system fit into time window of stability? $\implies$ Unsatisfactory approach!
  - Possible way out: Reprojection onto constraint hypersurface (Holst et al., 2004).

- Emphasize constraints, evolve fewest possible degrees of freedom (minimally two):
  ADM constrained evolution, maximally constrained evolution scheme (Meudon).
  $\implies$ Constraint violation impossible (by definition).
Reformulations of the Einstein metric equations

ADM equations in standard form: Split into 10 evolution and 4 constraint equations.
Free evolution: Constraint violating mode grows exponentially $\Rightarrow$ Unphysical results!

Two approaches for reformulating Einstein equations:

- **Emphasize hyperbolicity, evolve all degrees of freedom:**
  Bona–Massó, ADM-BSSN (NOK), Z4.
  $\Rightarrow$ Long-term numerical stability improves, but ultimately constraints are violated.
  - Can physical system fit into time window of stability? $\Rightarrow$ Unsatisfactory approach!
  - Possible way out: Reprojection onto constraint hypersurface (Holst et al., 2004).

- **Emphasize constraints, evolve fewest possible degrees of freedom (minimally two):**
  ADM constrained evolution, maximally constrained evolution scheme (Meudon).
  $\Rightarrow$ Constraint violation impossible (by definition).
  - Experience (so far limited) shows: Much more stable!
  - Numerically expensive (solve many elliptic equations during evolution).
  - Big issue at GR 17 in Dublin (formulations and numerical solvers become available).
Our Approach

Try to bridge gap between numerical relativity and astrophysical simulations.
Our Approach

Try to bridge gap between numerical relativity and astrophysical simulations.

Started some years ago with general relativistic simulations of supernova core collapse:

- Spherical polar coordinates restricted to axisymmetry.
- Simple matter model with hybrid ideal gas EoS.
- Approximation of ADM equations by assuming conformal flatness condition (CFC).
  \[ \implies \text{Metric equations reduce to 5 elliptic constraint equations.} \]
  - No problems with numerical stability!
  - Tradeoff: Solution computationally very expensive…
Our Approach

Try to bridge gap between numerical relativity and astrophysical simulations.

Started some years ago with general relativistic simulations of supernova core collapse:

- Spherical polar coordinates restricted to axisymmetry.
- Simple matter model with hybrid ideal gas EoS.
- Approximation of ADM equations by assuming conformal flatness condition (CFC).
  \[ \Rightarrow \text{Metric equations reduce to 5 elliptic constraint equations.}\]
  - No problems with numerical stability!
  - Tradeoff: Solution computationally very expensive...
- Restriction of initial models to polytropes in equilibrium
  (various rotation rates and profiles; like in \texttt{rnsid} by Nick Stergioulas).
- Wave extraction with Newtonian quadrupole formula.
- Extensive parameter study of models and comparison with Newtonian results.
- Very versatile code: Simulations of critical collapse or rapidly rotating neutron stars in full nonlinear evolution possible.
Results from our Previous Simulations

Inclusion of relativistic effects result primarily in deeper effective potential
\[ \implies \text{Higher densities during bounce, proto-neutron star more compact.} \]
Results from our Previous Simulations

Inclusion of relativistic effects result primarily in deeper effective potential \[ \Rightarrow \text{Higher densities during bounce, proto-neutron star more compact.} \]

Collapse type can change

- from standard single bounce with instantaneous formation of proto-neutron star
- to multiple centrifugal bounce and re-expansion (possibly at subnuclear matter density).

(Dimmelmeier, Font, and Müller, 2002)
Waveforms

With simple EoS and parametrized rotation:

Multiple bounces suppressed!
Waveforms

With simple EoS and parametrized rotation:

**Multiple bounces suppressed!**

Also get shift to higher frequencies in signal.
Waveforms

With simple EoS and parametrized rotation: **Multiple bounces suppressed!**

Also get shift to higher frequencies in signal.

![Graph showing energy spectrum and frequency shift](image)

No change in bulk of models due to relativity.  
**⇒ In principle Galactic supernova detectable!**

(Dimmelmeier, Font, and Müller, 2002)
The Extension to Three Dimensions

Reasons for removing restriction to axisymmetry:

• This is state of the art for relativistic hydro codes.  
  \[ \implies \text{Full comparison with other codes possible!} \]

For tests, axisymmetry is unwanted restriction.
The Extension to Three Dimensions

Reasons for removing restriction to axisymmetry:

- This is state of the art for relativistic hydro codes.  
  \[ \Rightarrow \text{Full comparison with other codes possible!} \]
  For tests, axisymmetry is unwanted restriction.

- Newtonian simulations show: Convection essential for explosion.  
  \[ \Rightarrow \text{Third dimension may be decisive for reviving shock!} \]

(Keil, Janka, and Müller, 1996)  
(Janka and Müller, 1996)  
(Rampp and Janka, 2000)
The Extension to Three Dimensions

Reasons for removing restriction to axisymmetry:

- This is state of the art for relativistic hydro codes.  
  \[\Rightarrow\] Full comparison with other codes possible!  
  For tests, axisymmetry is unwanted restriction.

- Newtonian simulations show: Convection essential for explosion.  
  \[\Rightarrow\] Third dimension may be decisive for reviving shock!

- In axisymmetry: Triaxial instabilities cannot be investigated.  
  \[\Rightarrow\] Need 3d code for simulations of bar modes in rapidly rotating neutron stars.

(Keil, Janka, and Müller, 1996)  
(Janka and Müller, 1996)  
(Rampp and Janka, 2000)
The Extension to Three Dimensions

Reasons for removing restriction to axisymmetry:

- This is state of the art for relativistic hydro codes.  
  \[ \rightarrow \] Full comparison with other codes possible!
  For tests, axisymmetry is unwanted restriction.

- Newtonian simulations show: Convection essential for explosion.
  \[ \rightarrow \] Third dimension may be decisive for reviving shock!

- In axisymmetry: Triaxial instabilities cannot be investigated.
  \[ \rightarrow \] Need 3d code for simulations of bar modes in rapidly rotating neutron stars.

- In 3d: One more polarization of gravitational radiation.

(Keil, Janka, and Müller, 1996)  
(Janka and Müller, 1996)  
(Rampp and Janka, 2000)
High-Resolution Shock-Capturing Methods

For solving hydroequations, we exploit their hyperbolic and conservative form:

\[
\frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} F^0}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = S,
\]

with vectors of conserved quantities $F^0$, fluxes $F^i$ and sources $S$. 

Seminar, Aristotle University, Thessaloniki, Greece, 2004
High-Resolution Shock-Capturing Methods

For solving hydro equations, we exploit their hyperbolic and conservative form:

$$\frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} F^0}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = S,$$

with vectors of conserved quantities $F^0$, fluxes $F^i$ and sources $S$.

Modern recipe for such equations: High-resolution shock-capturing (HRSC) methods. Use analytic solution of (approximate) Riemann problems.
High-Resolution Shock-Capturing Methods

For solving hydro equations, we exploit their hyperbolic and conservative form:

\[
\frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} F^0}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = S,
\]

with vectors of conserved quantities \( F^0 \), fluxes \( F^i \) and sources \( S \).

Modern recipe for such equations: High-resolution shock-capturing (HRSC) methods. Use analytic solution of (approximate) Riemann problems.

This method guarantees

- convergence to physical solution of the problem,
- correct propagation velocities of discontinuities, and
- sharp resolution of discontinuities.

**HRSC methods are particularly well suited for situations with shocks!**

No problem with extending HRSC methods from 2d to 3d!
The CFC Metric Equations

In CFC approximation:

\[
\begin{align*}
\text{ADM equations for exact metric} \\
\downarrow \\
\text{System of five coupled elliptic equations for CFC metric}
\end{align*}
\]
The CFC Metric Equations

In CFC approximation:

\[
\Delta \phi = -2\pi \phi^5 \left( \rho W^2 - P + \frac{K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta \alpha \phi = 2\pi \alpha \phi^5 \left( \rho h (3W^2 - 2) + 5P + \frac{7K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta \beta^i = 16\pi \alpha \phi^4 S^i + 2K^{ij} \nabla_j \left( \frac{\alpha}{\phi^6} \right) - \frac{1}{3} \nabla^i \nabla_k \beta^k,
\]

where \( \nabla \) and \( \Delta \) are Nabla and Laplace operator, respectively.
The CFC Metric Equations

In CFC approximation:

\[
\Delta \phi = -2\pi \phi^5 \left( \rho W^2 - P + \frac{K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta \alpha \phi = 2\pi \alpha \phi^5 \left( \rho \left( 3W^2 - 2 \right) + 5P + \frac{7K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta \beta^i = 16\pi \alpha \phi^4 S^i + 2K^{ij} \nabla_j \left( \frac{\alpha}{\phi^6} \right) - \frac{1}{3} \nabla^i \nabla_k \beta^k,
\]

where \( \nabla \) and \( \Delta \) are Nabla and Laplace operator, respectively.

**Task:** Solve these equations efficiently (particularly in 3d)!

(This is nontrivial and interesting for anyone who needs to solve elliptic equations!)
Old Metric Solvers

- **Newton–Raphson iteration:**

  Discretize equations and define root-finding problem.
  
  $$\Rightarrow$$ Use multi-dimensional Newton–Raphson solver.
Old Metric Solvers

- **Newton–Raphson iteration:**

  Discretize equations and define root-finding problem.
  \[ \Rightarrow \text{Use multi-dimensional Newton–Raphson solver.} \]

Disadvantages:
- Needs **explicit boundary conditions** at outer boundary (generally unknown).
- Associated **linear problem is huge** (already in 2d).
Old Metric Solvers

- **Newton–Raphson iteration:**

  Discretize equations and define root-finding problem.
  \[ \implies \text{Use multi-dimensional Newton–Raphson solver.} \]

  **Disadvantages:**
  - Needs explicit boundary conditions at outer boundary (generally unknown).
  - Associated linear problem is huge (already in 2d).

- **Conventional integral Poisson iteration:**

  **Exploit Poisson-like structure** of metric equations, \( \Delta u = S(u) \).
  \[ \implies \text{Keep r.h.s. fixed, solve linear Poisson equations, iterate until convergence.} \]
Old Metric Solvers

- **Newton–Raphson iteration:**
  
  Discretize equations and define root-finding problem.
  
  ➞ Use multi-dimensional Newton–Raphson solver.

  Disadvantages:
  - Needs explicit boundary conditions at outer boundary (generally unknown).
  - Associated linear problem is huge (already in 2d).

- **Conventional integral Poisson iteration:**
  
  Exploit Poisson-like structure of metric equations, \( \Delta u = S(u) \).
  
  ➞ Keep r.h.s. fixed, solve linear Poisson equations, iterate until convergence.

  Disadvantages:
  - Low relaxation factor due to choice of coordinates.
  - No convergence for high meridional resolution or in 3d.
Old Metric Solvers

- Newton–Raphson iteration:
  Discretize equations and define root-finding problem.
  \[ \Rightarrow \text{Use multi-dimensional Newton–Raphson solver.} \]

  Disadvantages:
  - Needs explicit boundary conditions at outer boundary (generally unknown).
  - Associated linear problem is huge (already in 2d).

- Conventional integral Poisson iteration:
  Exploit Poisson-like structure of metric equations, \( \Delta u = S(u) \).
  \[ \Rightarrow \text{Keep r.h.s. fixed, solve linear Poisson equations, iterate until convergence.} \]

  Disadvantages:
  - Low relaxation factor due to choice of coordinates.
  - No convergence for high meridional resolution or in 3d.

Both solvers feasible in axisymmetry, but no extension to 3d possible!

Alternative: Try integral Poisson solver based on spectral methods!
Introduction to Spectral Methods

Finite difference methods:
Approximate analytic function by overlapping *local polynomials* of low order.

Spectral methods:
Approximate analytic function by *global smooth functions*.
Introduction to Spectral Methods

Finite difference methods:
Approximate analytic function by overlapping *local polynomials* of low order.

Spectral methods:
Approximate analytic function by *global smooth functions*.

Functions are complete basis of orthogonal

- Legendre or Chebyshev polynomials in radial direction, and
- spherical harmonics in angular directions.
Introduction to Spectral Methods

Finite difference methods:
Approximate analytic function by overlapping \textit{local polynomials} of low order.

Spectral methods:
Approximate analytic function by \textit{global smooth functions}.

Functions are complete basis of orthogonal

- Legendre or Chebyshev polynomials in radial direction, and
- spherical harmonics in angular directions.

Thus: Arbitrary function in 3d can be approximated by series of trial functions.

$\implies$ All information stored in coefficients of this series.

\textbf{Applying linear (differential) operators reduces to simple operations on coefficients!}
Sources of Errors in Spectral Methods

There are two main sources of errors:

- **Cut series** at some expansion number $\hat{n}$ (depending on coordinate direction).
  $\Rightarrow$ Source of **truncation error**.

Still: Exact for e.g. polynomial of degree $\leq 2\hat{n} + 1$. 
Sources of Errors in Spectral Methods

There are two main sources of errors:

- **Cut series** at some expansion number \( \hat{n} \) (depending on coordinate direction).
  \[ \Rightarrow \text{Source of truncation error}. \]
  Still: Exact for e.g. polynomial of degree \( \leq 2\hat{n} + 1 \).

- **Calculation of expansion coefficients from original function requires solution of integral** (can only be performed numerically).
  \[ \Rightarrow \text{Source of interpolation error}. \]
  There are various interpolation methods available to minimize that error (and also to suppress contamination by high frequency noise).
Sources of Errors in Spectral Methods

There are two main sources of errors:

- **Cut series** at some expansion number $\hat{n}$ (depending on coordinate direction).
  $\implies$ Source of truncation error.

  Still: Exact for e.g. polynomial of degree $\leq 2\hat{n} + 1$.

- Calculation of expansion coefficients from original function requires solution of integral (can only be performed numerically).
  $\implies$ Source of interpolation error.

There are various interpolation methods available to minimize that error (and also to suppress contamination by high frequency noise).

For $C^\infty$ function:

- Representation in spectral expansion with errors decreasing exponentially with $\hat{n}$!
  (Compare: Decrease only with $n^k$ for finite difference method of order $k$!)
Comparison of Spectral Methods with Finite Difference Methods

For typical analytical test functions: Need much less points for accurate representation!
Comparison of Spectral Methods with Finite Difference Methods

For typical analytical test functions: Need much less points for accurate representation!

But discontinuities cannot be well represented by expansion methods (think of Fourier expansion):

- Low number collocation points: No sharp resolution of discontinuity.
- High number of collocation points: Gibbs phenomenon.

⇒ Not well suited for hydrodynamics (supernova shocks, thin discs).

In such cases: HRSC schemes superior.
Comparison of Spectral Methods with Finite Difference Methods

For typical analytical test functions: Need **much less points** for accurate representation!

But **discontinuities** cannot be well represented by expansion methods (think of Fourier expansion):

- Low number collocation points: **No sharp resolution** of discontinuity.
- High number of collocation points: **Gibbs phenomenon**.

⇒ **Not well suited for hydrodynamics** (supernova shocks, thin discs).

In such cases: HRSC schemes superior.

But metric is smooth even in presence of matter discontinuities!

**Breakthrough concept** by Valencia/Meudon groups:
Use HRSC methods for hydrodynamics and spectral methods for metric!

**Known as “Mariage des Maillages” (grid wedding) approach.**

(Combine best of both worlds!)

Seminar, Aristotle University, Thessaloniki, Greece, 2004
Incorporation of Spectral Methods in Our 3d Code

Now Garching group jumps onto spectral methods bandwagon!

\[ \implies \text{Use publicly available package in C++ from Meudon group: Lorene} \]

(widely used and tested in various astrophysical applications).
Incorporation of Spectral Methods in Our 3d Code

Now Garching group jumps onto spectral methods bandwagon!

Use publicly available package in C++ from Meudon group: Lorene
(widely used and tested in various astrophysical applications).

Based on ideas by Marck and Ibáñez and previous work by Novak:
Build metric solver using spectral methods into 3d HRSC relativistic hydrodynamics code.
Incorporation of Spectral Methods in Our 3d Code

Now Garching group jumps onto spectral methods bandwagon!

⇒ Use publicly available package in C++ from Meudon group: Lorene
(widely used and tested in various astrophysical applications).

Based on ideas by Marck and Ibáñez and previous work by Novak:
Build metric solver using spectral methods into 3d HRSC relativistic hydrodynamics code.

Result: First successful grid wedding in 3d!
New Metric Solver: Integral Poisson Iteration using Spectral Methods

Solver works similar to conventional integral Poisson iterative solver. But: Based on spectral methods.

- Matter lives happily on finite difference grid.
- Spacetime metric spends its life on spectral grid.
New Metric Solver: Integral Poisson Iteration using Spectral Methods

Solver works similar to conventional integral Poisson iterative solver. But: Based on spectral methods.

- Matter lives happily on finite difference grid.
- Spacetime metric spends its life on spectral grid.

As in any marriage: Communication necessary (here between grids):

- Interpolation from finite difference grid to spectral grid before metric calculation (interpolation accuracy tested by analytic test functions).
- Reconstruction from spectral grid to finite difference grid after metric calculation.
New Metric Solver: Integral Poisson Iteration using Spectral Methods

Solver works similar to conventional integral Poisson iterative solver. But: Based on spectral methods.

- Matter lives happily on finite difference grid.
- Spacetime metric spends its life on spectral grid.

As in any marriage: Communication necessary (here between grids):

- Interpolation from finite difference grid to spectral grid before metric calculation (interpolation accuracy tested by analytic test functions).
- Reconstruction from spectral grid to finite difference grid after metric calculation.

Now summarize important features and tests of new spectral solver...
Grid Setup

Spectral solver uses several (typically 3 – 6) radial domains (easy with Lorene package):

- **Nucleus** limited by $r_d$ roughly at largest density gradient.
- Several **shells** up to $r_{fd}$.
- Compactified radial **vacuum domain** out to $r = \infty$. 
Grid Setup

Spectral solver uses several (typically 3 – 6) radial domains (easy with LORENE package):

- **Nucleus** limited by $r_d$ roughly at largest density gradient.
- Several **shells** up to $r_{fd}$.
- Compactified radial **vacuum domain** out to $r = \infty$.

In contraction phase of core collapse, inner domain boundaries are **allowed to move**. (In core collapse relevant **radial scale contracts** by factor 100!)
Grid Setup

Spectral solver uses several (typically 3 – 6) radial domains (easy with Lorene package):

- **Nucleus** limited by \( r_d \) roughly at largest density gradient.
- Several **shells** up to \( r_{fd} \).
- Compactified radial **vacuum domain** out to \( r = \infty \).

In contraction phase of core collapse, inner domain boundaries are **allowed to move**. (In core collapse relevant radial scale contracts by factor 100!)

Example: Influence of **bad spectral grid setup** on collapse dynamics.

---

Seminar, Aristotle University, Thessaloniki, Greece, 2004
Convergence Properties

Use axisymmetry to compare convergence rate (not speed!) of metric solvers 1/2/3:
Convergence Properties

Use axisymmetry to compare convergence rate (not speed!) of metric solvers 1/2/3:

- Newton–Raphson solver exhibits quadrativ convergence (as expected).
- Conventional Poisson solver has poor convergence rate (low relaxation factor).
- Spectral Poisson solver can use relaxation factor of 1. \(\Rightarrow\) Fair convergence.

It doesn’t suffer from convergence problems either (bad initial guess, 3d)
Preservation of Symmetry

With spherical polar coordinates:
Reduction of dimensions for symmetric configurations trivial.

⇒ Crucial test for code:
Can lower-dimensional symmetry be maintained against small perturbations.
Preservation of Symmetry

With spherical polar coordinates:
Reduction of dimensions for symmetric configurations trivial.

⇒ Crucial test for code:
Can lower-dimensional symmetry be maintained against small perturbations.

Example: Nonaxisymmetric perturbation of rotating neutron star in axisymmetry.

⇒ Perturbations do not grow and are only slowly damped (low numerical viscosity of HRSC codes!).
Oscillations of Rotating Neutron Stars

Another stringent test: Can code keep rotating neutron star in equilibrium.

Test criterion: Preservation of rotation velocity profile.
**Oscillations of Rotating Neutron Stars**

Another stringent test: Can code keep *rotating neutron star in equilibrium*.

**Test criterion**: Preservation of rotation velocity profile.

Axisymmetric oscillations in rotating neutron stars can be evolved as in other codes. **No crucial differences** between 2d and 3d version of code.

→ Use code with spectral metric solver for *axisymmetric simulations of rotating stars* (project with Nick Stergioulas).

---

*Seminar, Aristotle University, Thessaloniki, Greece, 2004*
Oscillations of Rotating Neutron Stars

Another stringent test: Can code keep rotating neutron star in equilibrium.

Test criterion: Preservation of rotation velocity profile.

Axisymmetric oscillations in rotating neutron stars can be evolved as in other codes.

No crucial differences between 2d and 3d version of code.

⇒ Use code with spectral metric solver for axisymmetric simulations of rotating stars (project with Nick Stergioulas).

Proof of principle: Code is ready for simulations of dynamical triaxial instabilities!
The Ladder of Credibility

New code has passed all tests in axisymmetry (against previous axisymmetric code and other codes).

Accuracy of the “Mariage des Maillage” has been checked in various situations.
The Ladder of Credibility

New code has passed all tests in axisymmetry
(against previous axisymmetric code and other codes).

Accuracy of the “Mariage des Maillage” has been checked in various situations.
Generic Nonaxisymmetric Configurations

Now finally explore nonaxisymmetric configurations in 3d.

Extension from axisymmetry to nonaxisymmetry trivial with LORENE!
Generic Nonaxisymmetric Configurations

Now finally explore nonaxisymmetric configurations in 3d.

Extension from axisymmetry to nonaxisymmetry trivial with Lorene!

Setup: Rotating neutron star with strong “bar” perturbation (unphysical!).

Rotation generates spiral arms!
Three-Dimensional Wave Extraction

Test shows: Code can handle nonaxisymmetric matter distributions in strong gravity!
Three-Dimensional Wave Extraction

Test shows: Code can handle nonaxisymmetric matter distributions in strong gravity!

Matter in bars is partially shed into spiral arms and partially accreted onto neutron star. Ring-down-like oscillation of neutron star visible in central density.

\[ \begin{align*}
\rho_c &\mid [10^{14} \text{ g cm}^{-3}] \\
\rho_{\text{nuc}} &
\end{align*} \]

Seminar, Aristotle University, Thessaloniki, Greece, 2004
Applications

Three-Dimensional Wave Extraction

Test shows: Code can handle nonaxisymmetric matter distributions in strong gravity!

Matter in bars is partially shed into spiral arms and partially accreted onto neutron star. $\Rightarrow$ Ring-down-like oscillation of neutron star visible in central density.

Also: Wave extraction based on 3d Newtonian quadrupole formula implemented.

Contrary to axisymmetry: Both polarizations of gravitational radiation excited!

Comparison with 3d Cartesian code Cactus/Whisky is underway (also in axisymmetry).
Rotating Neutron Stars in Axisymmetry

We are currently performing parameter study of rotating neutron stars (extension of ToniK simulation series to full metric evolution).
Rotating Neutron Stars in Axisymmetry

We are currently performing parameter study of rotating neutron stars (extension of ToniK simulation series to full metric evolution).

Model summary:

- Different rotation rates and profiles:
  - Spherical symmetry to mass shedding limit.
  - Uniform rotation to extremely differential rotation.

- Oscillations excited by various artificial and recycled eigenfunctions.
Rotating Neutron Stars in Axisymmetry

We are currently performing parameter study of rotating neutron stars (extension of ToniK simulation series to full metric evolution).

Model summary:

- Different rotation rates and profiles:
  - Spherical symmetry to mass shedding limit.
  - Uniform rotation to extremely differential rotation.

- Oscillations excited by various artificial and recycled eigenfunctions.

With excitation by eigenfunction, specific eigenmodes can be selected. Compare these results to perturbation codes!
Rotating Neutron Stars in Axisymmetry

For extremely rapidly rotating neutron stars: Observe persistent mass shedding. This is damping mechanism for oscillations.
Rotating Neutron Stars in Axisymmetry

For extremely rapidly rotating neutron stars: Observe persistent mass shedding. 
→ This is damping mechanism for oscillations.

Damping time scale is rather long. Nevertheless: Can use dynamic simulations for this (only two dimensions in axisymmetry).
Rotating Neutron Stars in Axisymmetry

For extremely rapidly rotating neutron stars: Observe persistent mass shedding. 
\[ \Rightarrow \text{This is damping mechanism for oscillations.} \]

Damping time scale is rather long. Nevertheless: Can use dynamic simulations for this (only two dimensions in axisymmetry).

We want to investigate role of gravity and EoS in mass shedding (now polytope versus ideal gas).
Future Projects

Bonazzola et al. have presented new formulation of Einstein metric equations: Maximally constrained evolution.
**Future Projects**

Bonazzola et al. have presented new formulation of Einstein metric equations: **Maximally constrained evolution**.

Idea (contrary to previous approaches):

Satisfy all constraints and evolve only two physical degrees of freedom!

⇒ No constraint-violating modes present in evolution.
Future Projects

Bonazzola et al. have presented new formulation of Einstein metric equations: Maximally constrained evolution.

Idea (contrary to previous approaches):

Satisfy all constraints and evolve only two physical degrees of freedom!

⇒ No constraint-violating modes present in evolution.

- Hope: Numerically stable formulation.
- Spectral methods are ideally suited for this formulation.
  ⇒ With these methods, solution of constraints during evolution feasible!
- With "Mariage des Maillage":
  Can implement these equations into relativistic hydrodynamic code.
Future Projects

Bonazzola et al. have presented new formulation of Einstein metric equations: Maximally constrained evolution.

Idea (contrary to previous approaches):

Satisfy all constraints and evolve only two physical degrees of freedom!

⇒ No constraint-violating modes present in evolution.

- Hope: Numerically stable formulation.
- Spectral methods are ideally suited for this formulation.
  ⇒ With these methods, solution of constraints during evolution feasible!
- With “Mariage des Maillage”:
  Can implement these equations into relativistic hydrodynamic code.

This will probably be future for our code (full relativity in 3d)!
Advantages of a Compactified Radial Grid

We use quadrupole formula for wave extraction. Cannot make use of possibility to extend metric (without matter) into wave zone...

Major benefit from compactified grid:

- All metric equation terms (also with noncompact support) are taken into account.
- No need for explicit boundary conditions at $r_{fd}$ (simply assume asymptotic flatness).

Example:
Rotating neutron star with $r_{fd}$ close to $r_{se}$.

Artificial boundary condition $\beta_\varphi = 0$ bad!

Solver 3 yields accurate result (difference due to CFC and different grid).