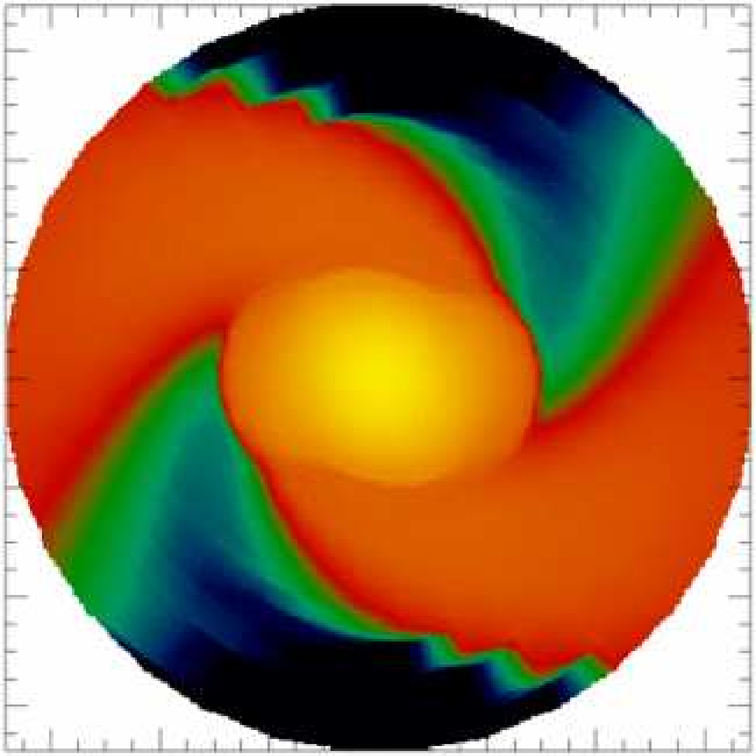
Max Planck Institute
for Astrophysics,
Garching, Germany



Harald Dimmelmeier

"Mariage des Maillage": Combining Finite Difference Schemes and Spectral Methods in Relativistic Hydrodynamics

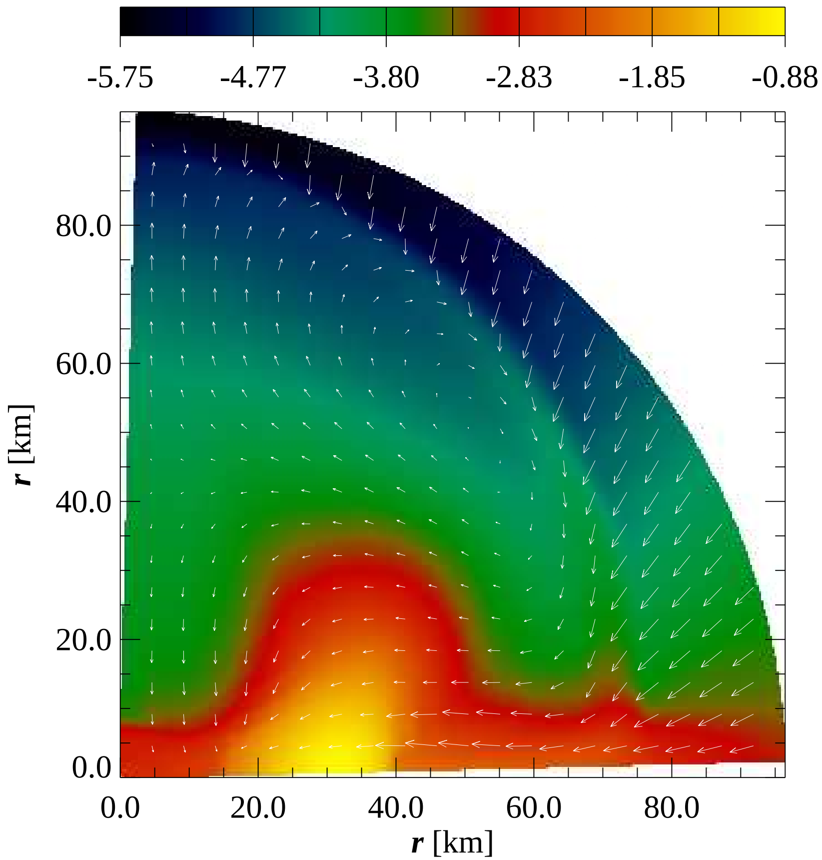
Work with

José A. Font and José M. Ibáñez (Universidad de Valencia)
Jérôme Novak (LUTH, Meudon)
Ewald Müller (MPA Garching)
Nick Stergioulas (Aristotle University, Thessaloniki)

Dimmelmeier, Font, Müller, *Astron. Astrophys.*, 388, 917–935, (2002), astro-ph/0103088
Dimmelmeier, Font, Müller, *Astron. Astrophys.*, 393, 523–542, (2002), astro-ph/0103089
Dimmelmeier, Novak, Font, Ibáñez, Müller, submitted to *Phys. Rev. D*, (2004), astro-ph/0407174
Stergioulas, Dimmelmeier, Font, in preparation, (2004)

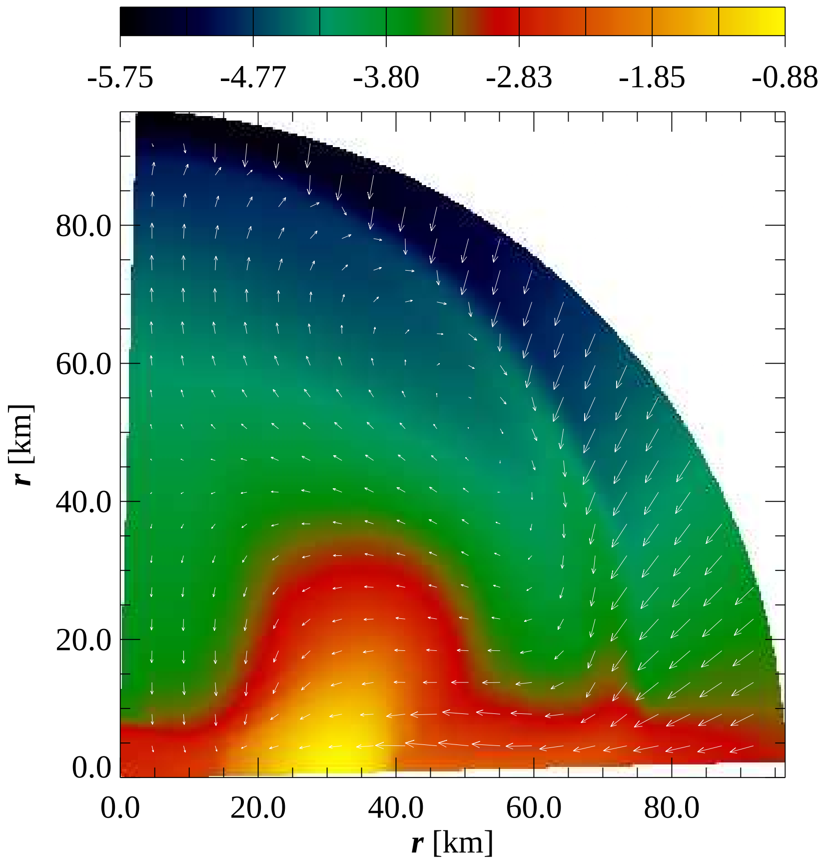
`http://www.mpa-garching.mpg.de/relhydro/`

Outline of the Talk



Outline of the Talk

- Astrophysical motivation.
- Gravitational waves.
- Various approaches for numerical simulations.
- Results from our previous simulations.
- Spectral methods.
- Our new 3d code.
- Tests: The ladder of credibility.
- First applications:
  Neutron star oscillations, supernova core collapse, bar mode instabilities.



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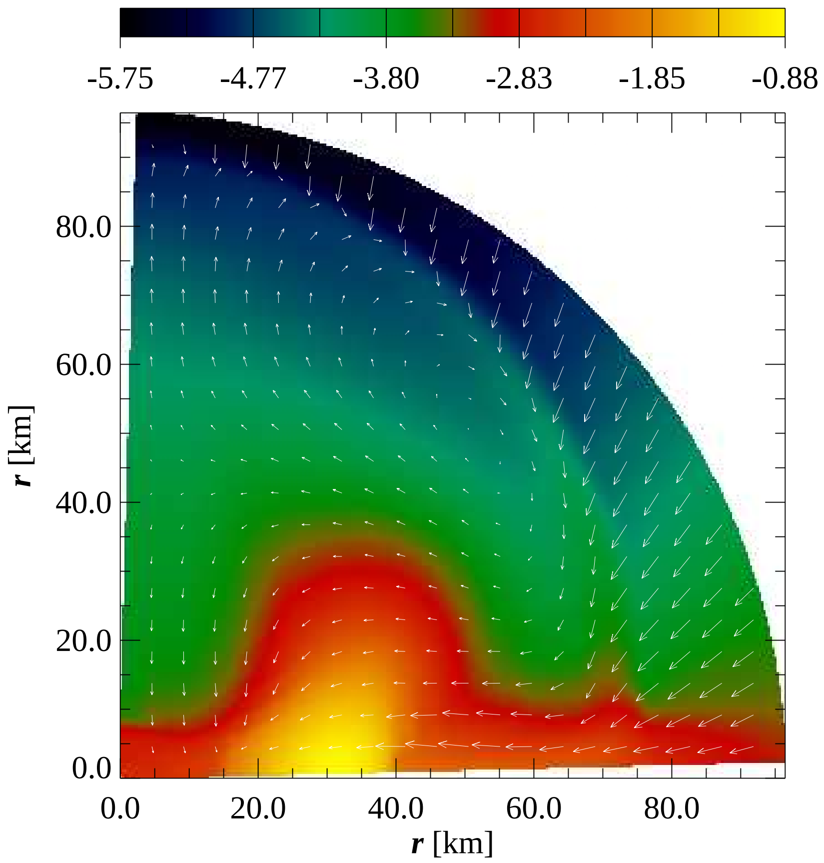
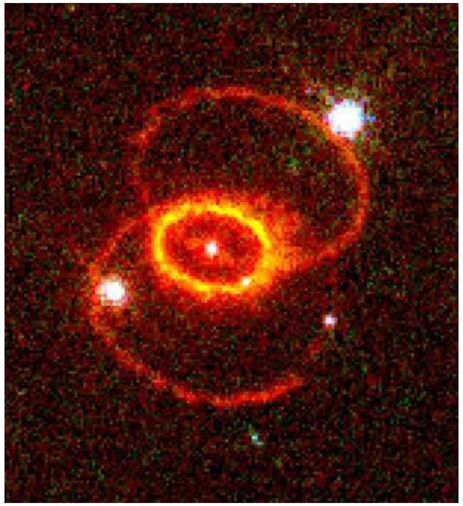


And particularly we demonstrate:

Spectral methods can be used to efficiently solve elliptic equations in 3d!

This is crucial for new schemes in numerical relativity!

Supernova Core Collapse



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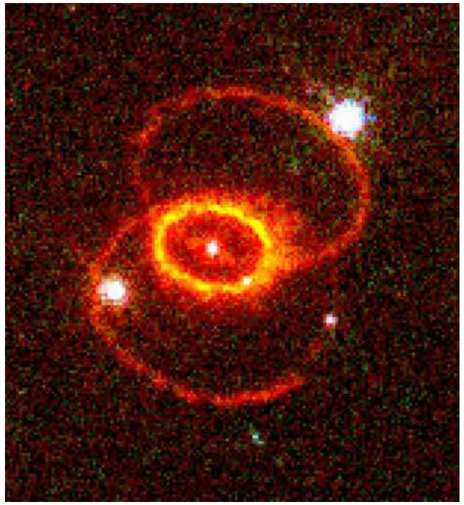

Our code was originally tailored for supernova core collapse.

Current standard model for core collapse supernova:

- Subsequent nuclear burning in massive star yield shell structure.
- Iron core with $M \sim 1.4\, M_\odot$ and $R \sim 1000$ km develops in center.
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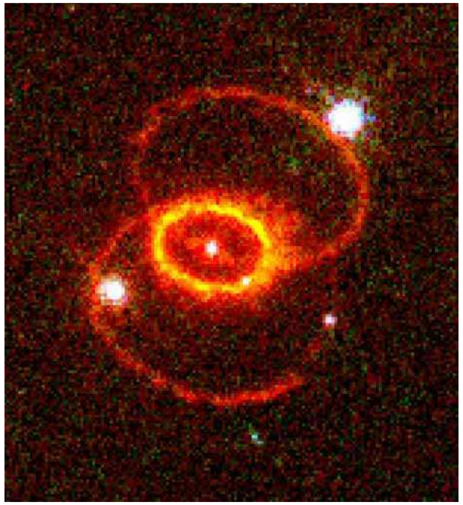

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- Collapse to nuclear matter densities in $T \sim 100$ ms.
- Stiffening of EoS, bounce, and formation of prompt shock.
- Stalled shock revived by neutrinos energy deposition.
- Delayed shock propagates out and disrupts envelope of star.
- Proto-neutron star cools and shrinks to neutron star.

Gravitational Waves

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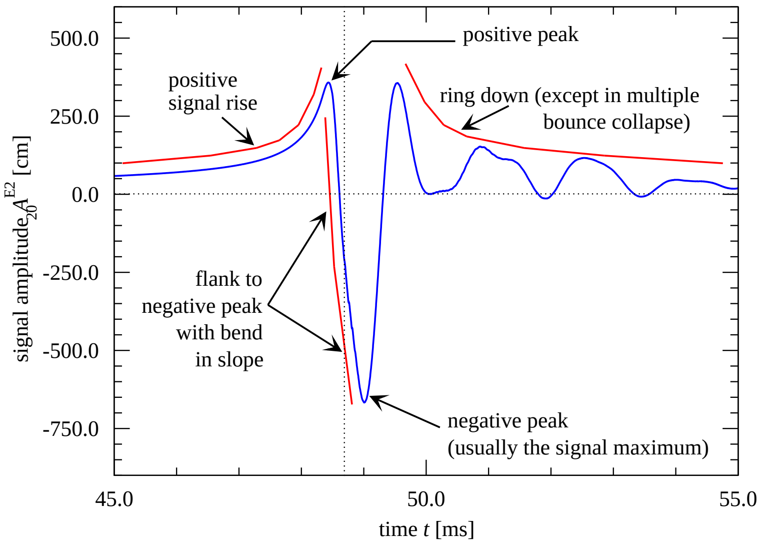

Various Approaches to Simulations of Core Collapse

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Advantages:

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Difficulties:

- Coordinates usually not adopted to geometry of core collapse.
- Mesh refinement (in core collapse: radial contraction scale $\sim 100!$).
- Issue of gauge freedom in relativity (separate invariants from "coordinate" effects).
- Nonuniqueness of formulating metric equations (crucial issue!).

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ADM equations in standard form: Split into 10 evolution and 4 constraint equations.

Free evolution: Constraint violating mode grows exponentially $\Longrightarrow$ Unphysical results!

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  - Experience (so far limited) shows: Much more stable!
  - Numerically expensive (solve many elliptic equations during evolution).
  - Big issue at GR 17 in Dublin (formulations and numerical solvers become available).

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- Spherical polar coordinates restricted to axisymmetry.
- Simple matter model with hybrid ideal gas EoS.
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- Restriction of initial models to polytropes in equilibrium
  (various rotation rates and profiles; like in `rnsid` by Nick Stergioulas).
- Wave extraction with Newtonian quadrupole formula.
- Extensive parameter study of models and comparison with Newtonian results.
- Very versatile code: Simulations of critical collapse or rapidly rotating neutron stars in full nonlinear evolution possible.

Results from our Previous Simulations

Inclusion of relativistic effects result primarily in deeper effective potential
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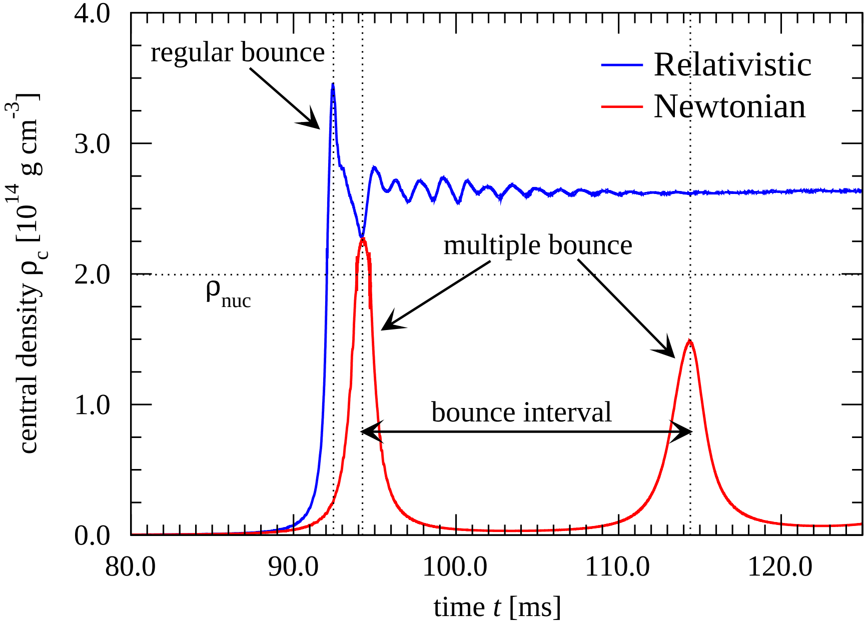
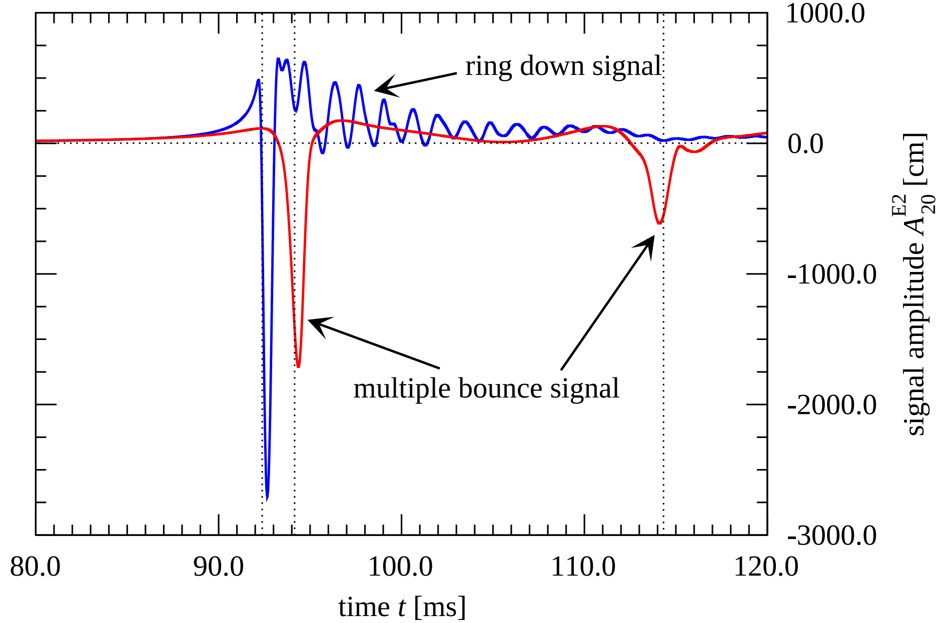
Collapse type can change

- **from standard single bounce with instantaneous formation of proto-neutron star**
- **to multiple centrifucal bounce and re-expansion (possibly at subnuclear matter density).**





(Dimmelmeier, Font, and Müller, 2002)

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Multiple bounces suppressed!

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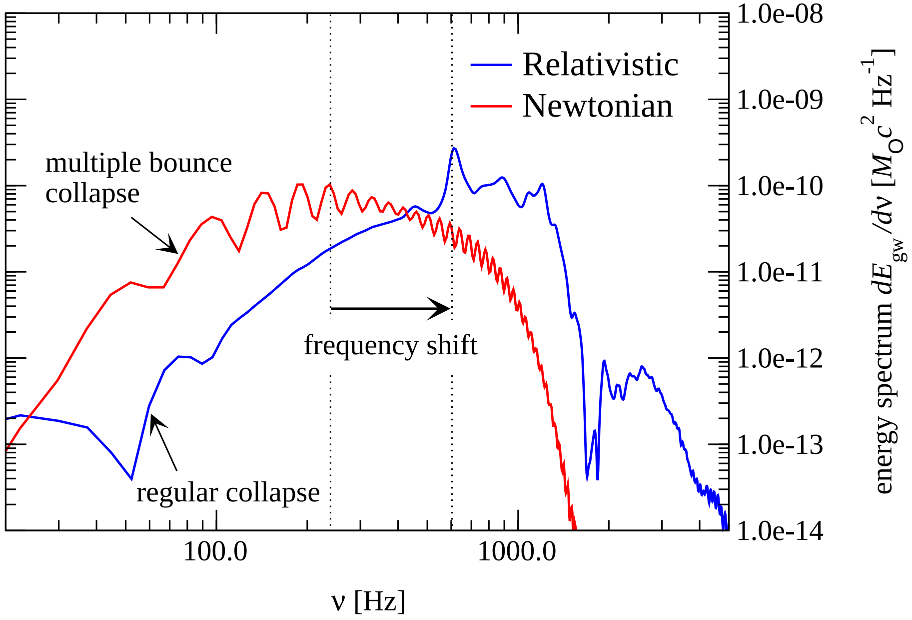
Also get shift to higher frequencies in signal.



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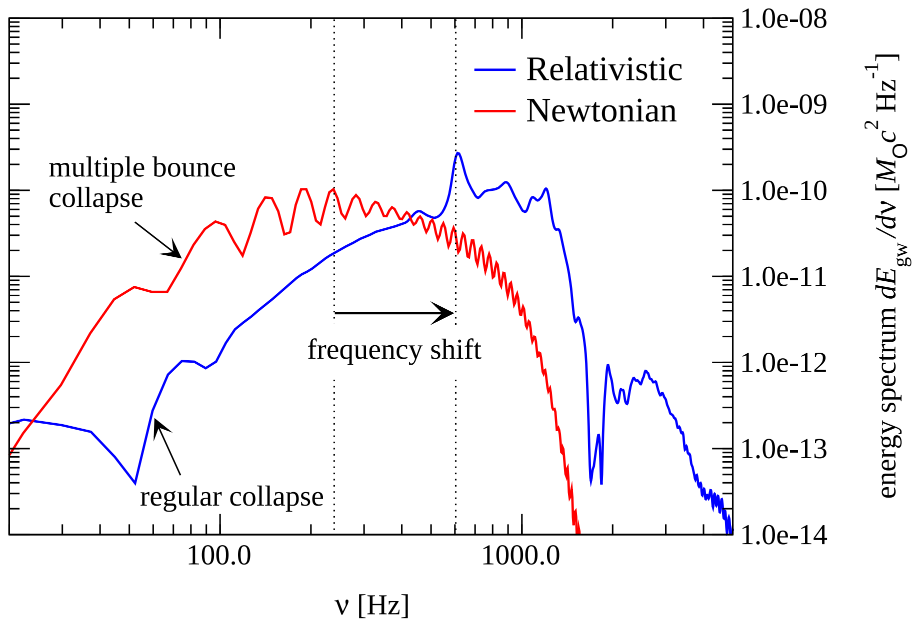

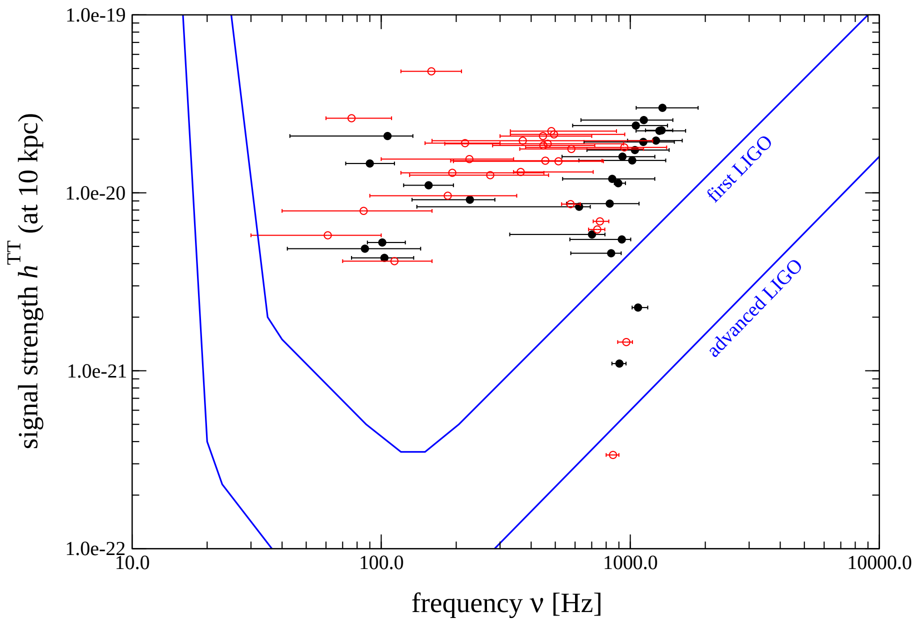

(Dimmelmeier, Font, and Müller, 2002)

No change in bulk of models due to relativity.
$\Longrightarrow$ In principle Galactic supernova detectable!

The Extension to Three Dimensions

Reasons for removing restriction to axisymmetry:

- **This is state of the art for relativistic hydro codes.**
  $\Longrightarrow$ Full comparison with other codes possible!

  For tests, axisymmetry is unwanted restriction.

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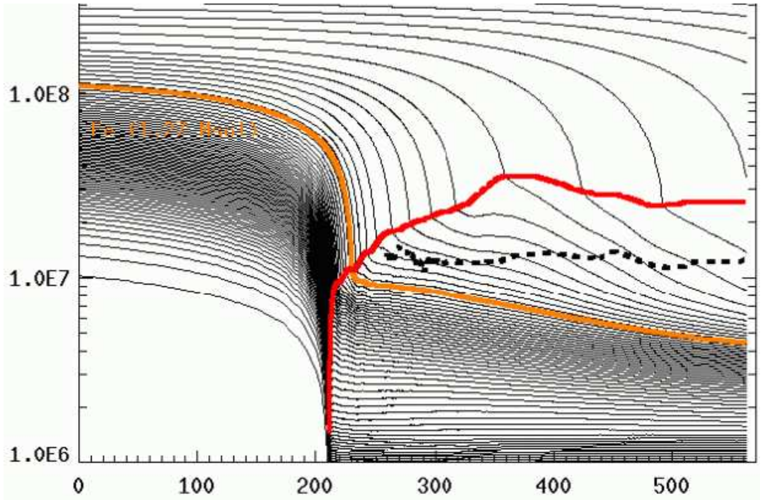

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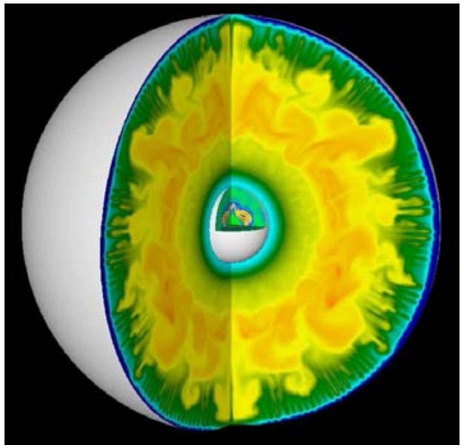

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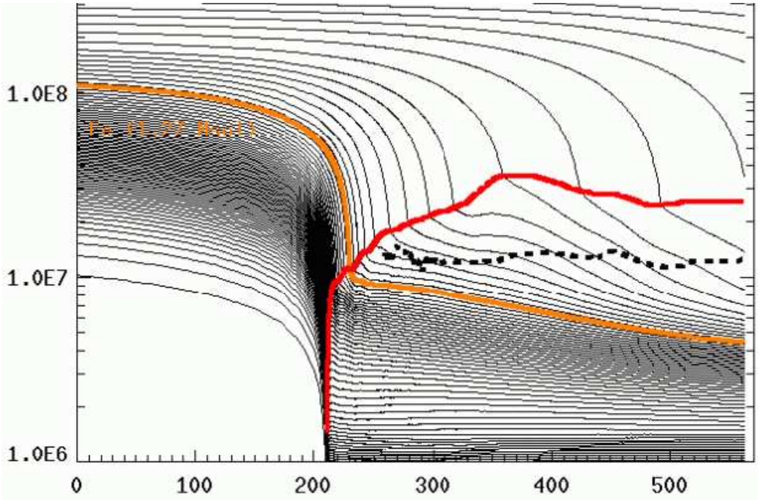

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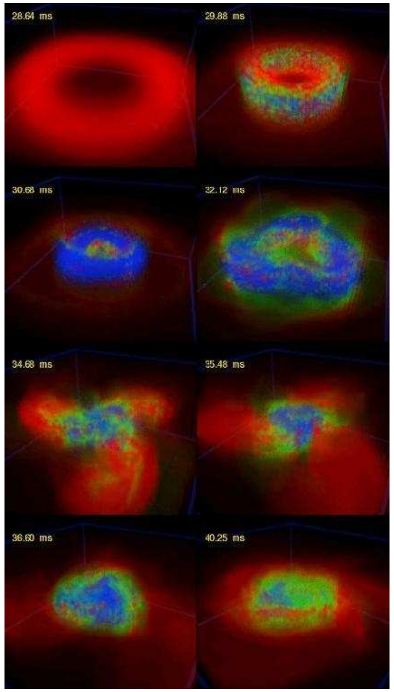

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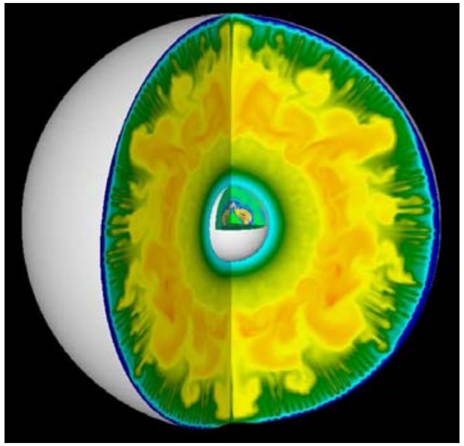

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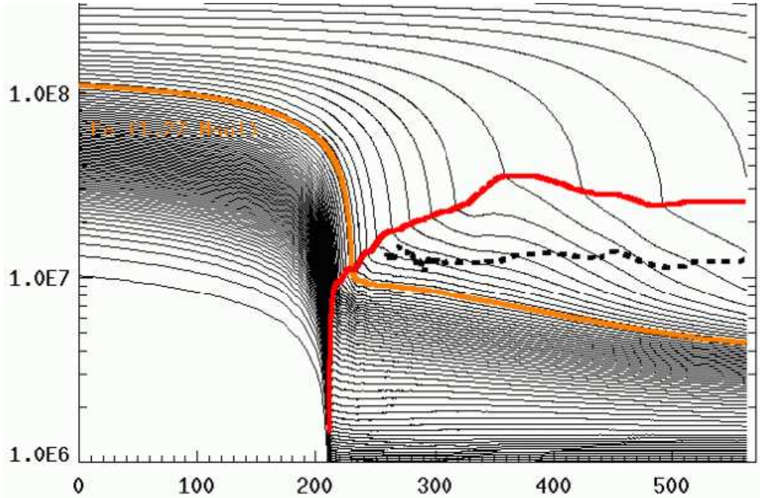

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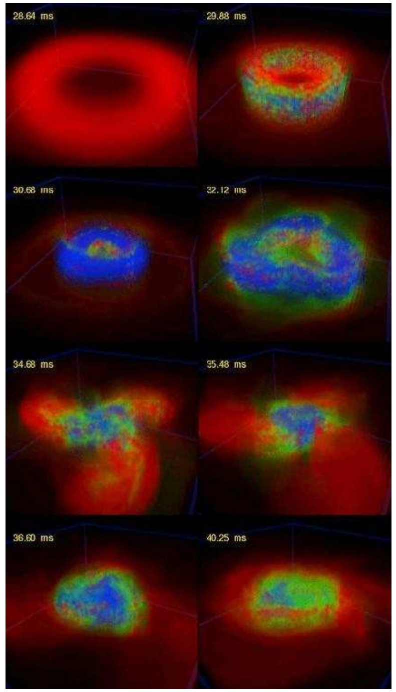

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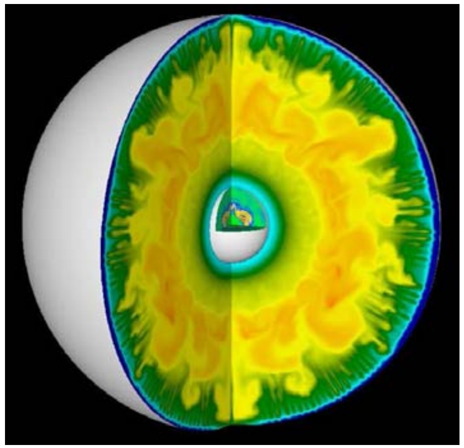

- In 3d: One more polarization of gravitational radiation.

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High-Resolution Shock-Capturing Methods

For solving hydro equations, we exploit their hyperbolic and conservative form:

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\boldsymbol{F}^0}{\partial t}+\frac{\partial\sqrt{-g}\boldsymbol{F}^i}{\partial x^i}\right]=\boldsymbol{S},$$

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$\Longrightarrow$ Use analytic solution of (approximate) Riemann problems.

This method guarantees

- convergence to physical solution of the problem,
- correct propagation velocities of discontinuities, and
- sharp resolution of discontinuities.

HRSC methods are particularly well suited for situations with shocks!

No problem with extending HRSC methods from 2d to 3d!

The CFC Metric Equations

In CFC approximation:

$$\boxed{\begin{array}{c}\text{ADM equations for exact metric}\\ \downarrow \\ \text{System of five coupled elliptic equations for CFC metric}\end{array}}$$

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Task: Solve these equations efficiently (particularly in 3d)!

(This is nontrivial and interesting for anyone who needs to solve elliptic equations!)

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  Discretize equations and define root-finding problem.
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Both solvers feasible in axisymmetry, but no extension to 3d possible!

Alternative: Try integral Poisson solver based on spectral methods!

Introduction to Spectral Methods

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- **spherical harmonics in angular directions.**

Thus: Arbitrary function in 3d can be approximated by series of trial functions.

$\Longrightarrow$ All information stored in coefficients of this series.

Applying linear (differential) operators reduces to simple operations on coefficients!

Sources of Errors in Spectral Methods

There are two main sources of errors:

- Cut series at some expansion number $\hat{n}$ (depending on coordinate direction).
  $\Longrightarrow$ Source of truncation error.

  Still: Exact for e.g. polynomial of degree $\leq 2\hat{n} + 1$.

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For C^∞ function:

$\boxed{\text{Representation in spectral expansion with errors decreasing exponentially with } \hat{n}!}$

(Compare: Decrease only with n^k for finite difference method of order k!)

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But discontinuities cannot be well represented by expansion methods
(think of Fourier expansion):

- Low number collocation points: No sharp resolution of discontinuity.
- High number of collocation points: Gibbs phenomenon.

$\Longrightarrow$ Not well suited for hydrodynamics (supernova shocks, thin discs).

In such cases: HRSC schemes superior.

Comparison of Spectral Methods with Finite Difference Methods

For typical analytical test functions: Need much less points for accurate representation!

But discontinuities cannot be well represented by expansion methods (think of Fourier expansion):

- **Low number collocation points: No sharp resolution of discontinuity.**
- **High number of collocation points: Gibbs phenomenon.**

$\Longrightarrow$ Not well suited for hydrodynamics (supernova shocks, thin discs).

In such cases: HRSC schemes superior.

But metric is smooth even in presence of matter discontinuities!

Breakthrough concept by Valencia/Meudon groups:
Use HRSC methods for hydrodynamics and spectral methods for metric!

Known as "MARIAGE DES MAILLAGES" (grid wedding) approach.

(Combine best of both worlds!)

Incorporation of Spectral Methods in Our 3d Code

Now Garching group jumps onto spectral methods bandwagon!
$\Longrightarrow$ Use publicly available package in C++ from Meudon group: LORENE (widely used and tested in various astrophysical applications).

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Result: First successful grid wedding in 3d!

New Metric Solver: Integral Poisson Iteration using Spectral Methods

Solver works similar to conventional integral Poisson iterative solver.
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- **Interpolation from finite difference grid to spectral grid before metric calculation (interpolation accuracy tested by analytic test functions).**
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Now summarize important features and tests of new spectral solver...

Grid Setup

Spectral solver uses several (typically 3 – 6) radial domains (easy with LORENE package):

- Nucleus limited by r_{d} roughly at largest density gradient.
- Several shells up to r_{fd}.
- Compactified radial vacuum domain out to $r = \infty$.

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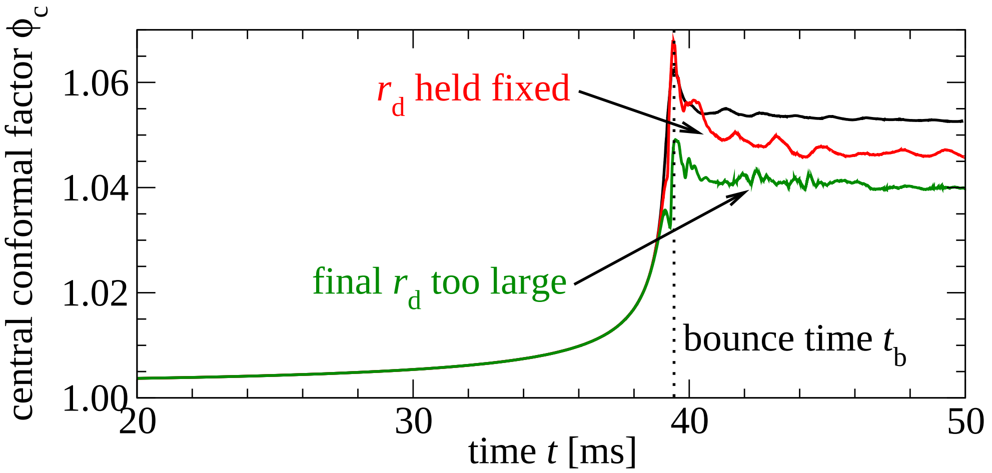
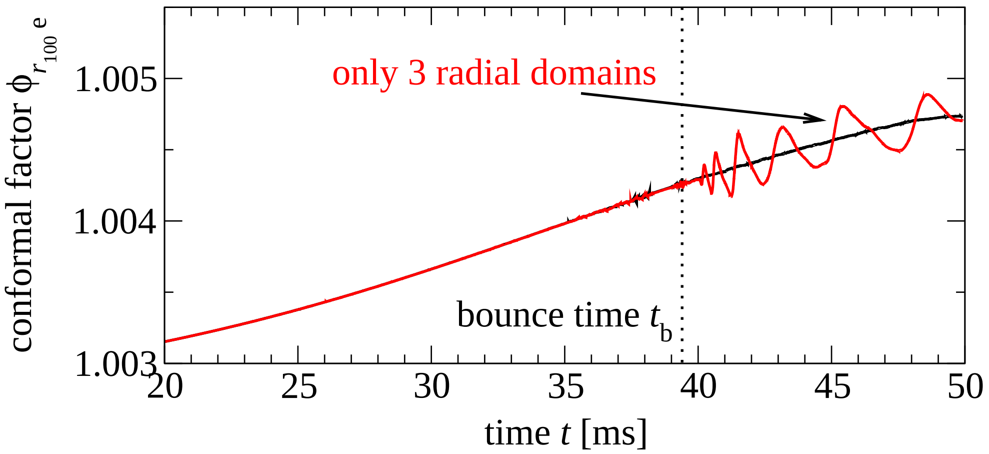
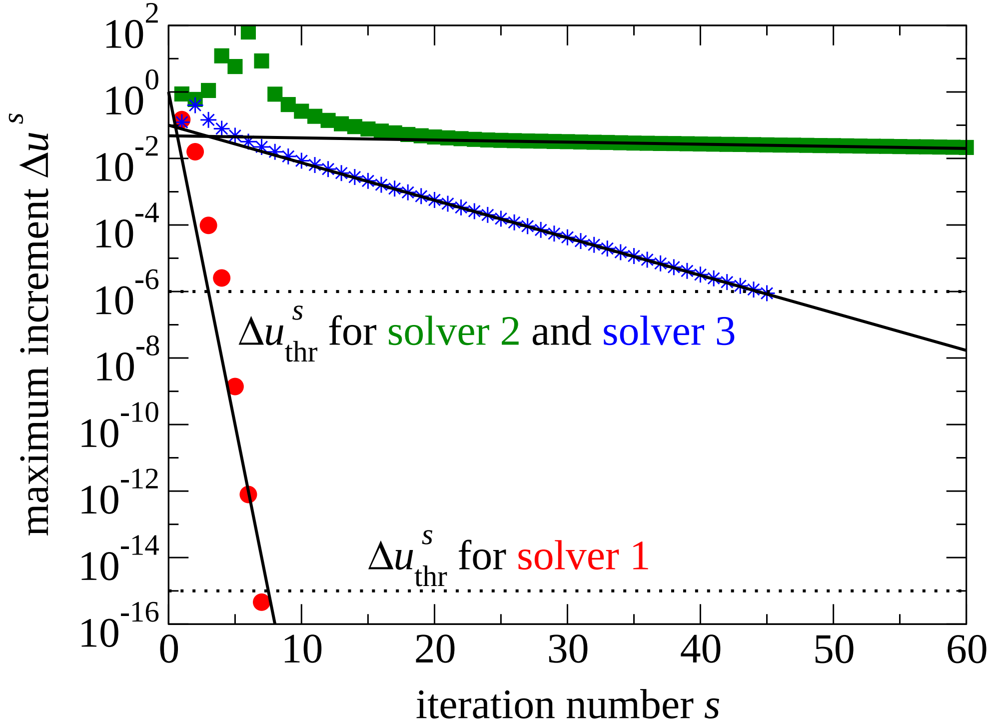
Example: Influence of bad spectral grid setup on collapse dynamics.





Convergence Properties

Use axisymmetry to compare convergence rate (not speed!) of metric solvers 1/2/3:



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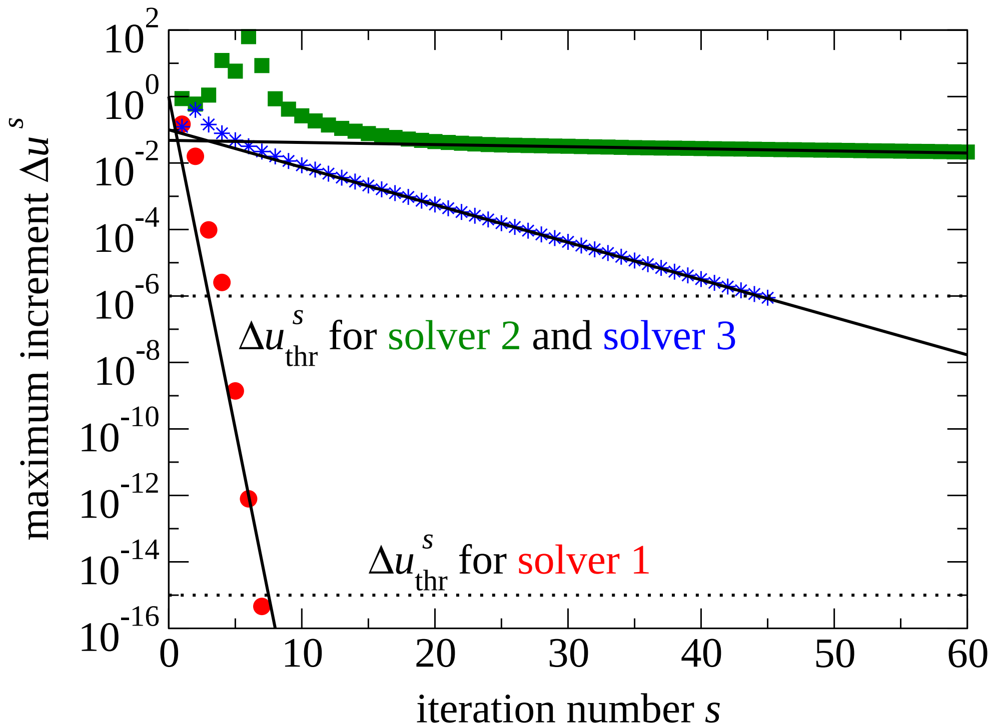

- **Newton–Raphson solver exhibits quadrativ convergence (as expected).**
- **Conventional Poisson solver has poor convergence rate (low relaxation factor).**
- **Spectral Poisson solver can use relaxation factor of 1. $\Longrightarrow$ Fair convergence.**
  It doesn't suffer from convergence problems either (bad initial guess, 3d)

Preservation of Symmetry

With spherical polar coordinates:
Reduction of dimensions for symmetric configurations trivial.

$\Longrightarrow$ Crucial test for code:
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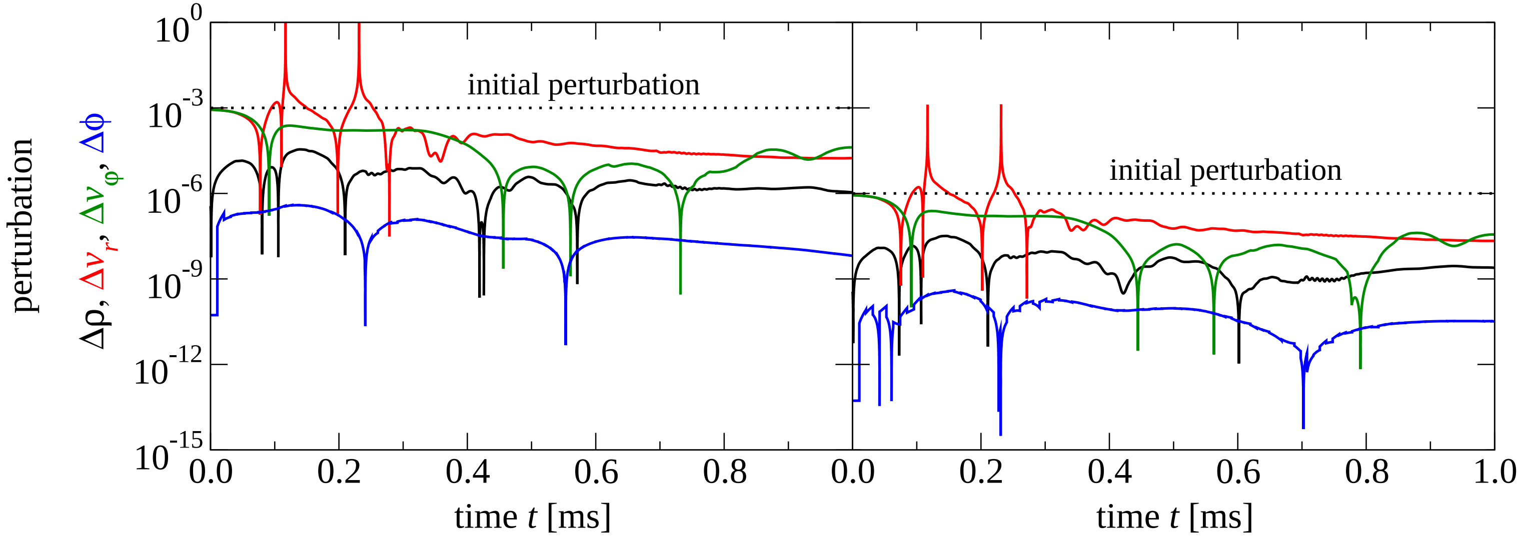
Example: Nonaxisymmetric perturbation of rotating neutron star in axisymmetry.

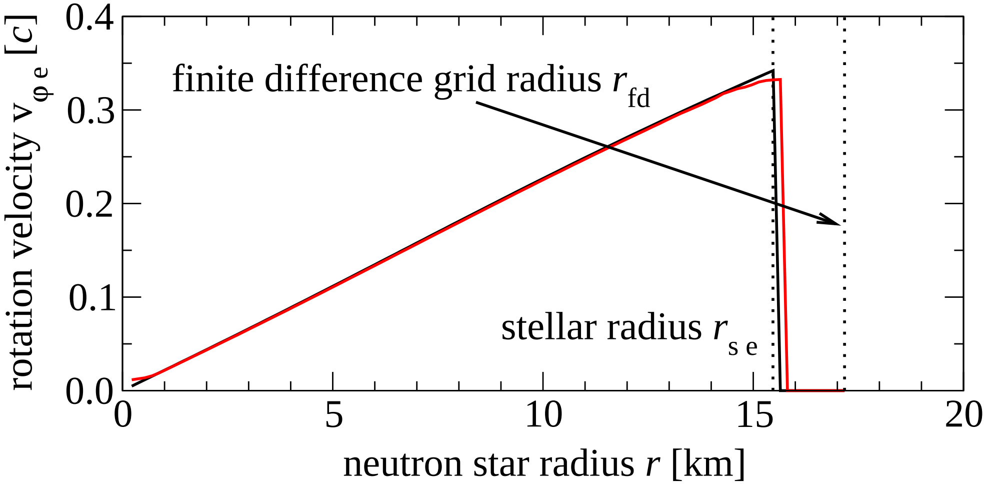


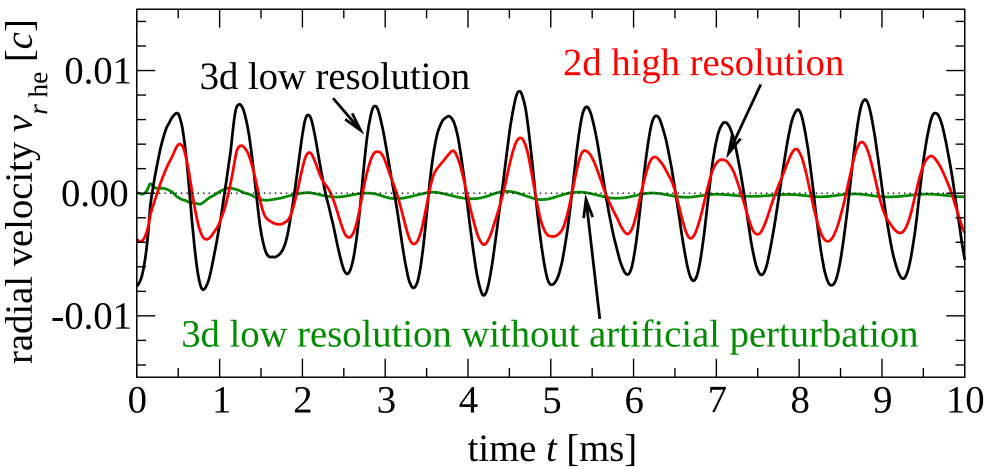
⟹ Perturbations do not grow and are only slowly damped
(low numerical viscosity of HRSC codes!).

Oscillations of Rotating Neutron Stars

Another stringent test: Can code keep rotating neutron star in equilibrium.

Test criterion: Preservation of rotation velocity profile.





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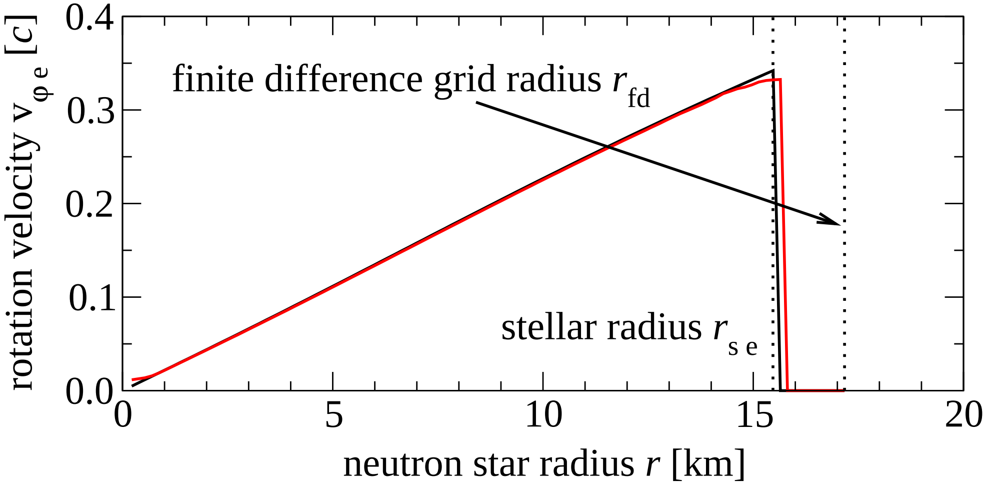

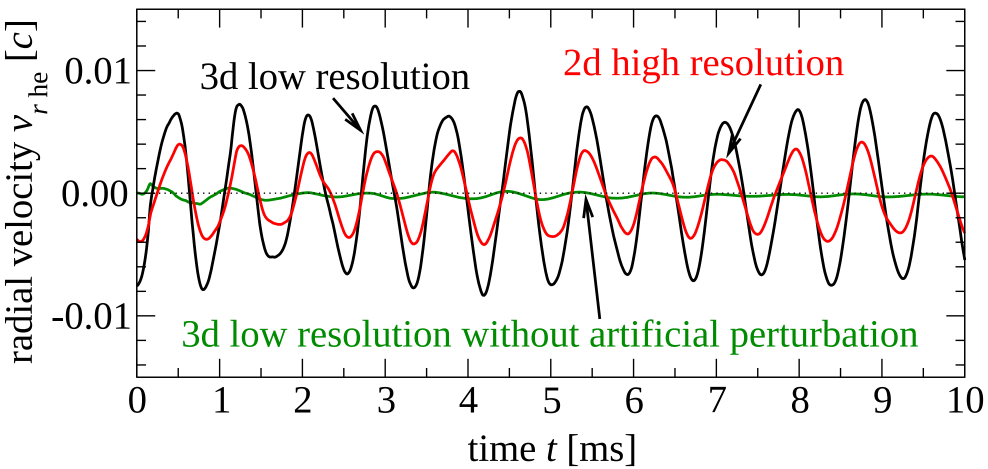

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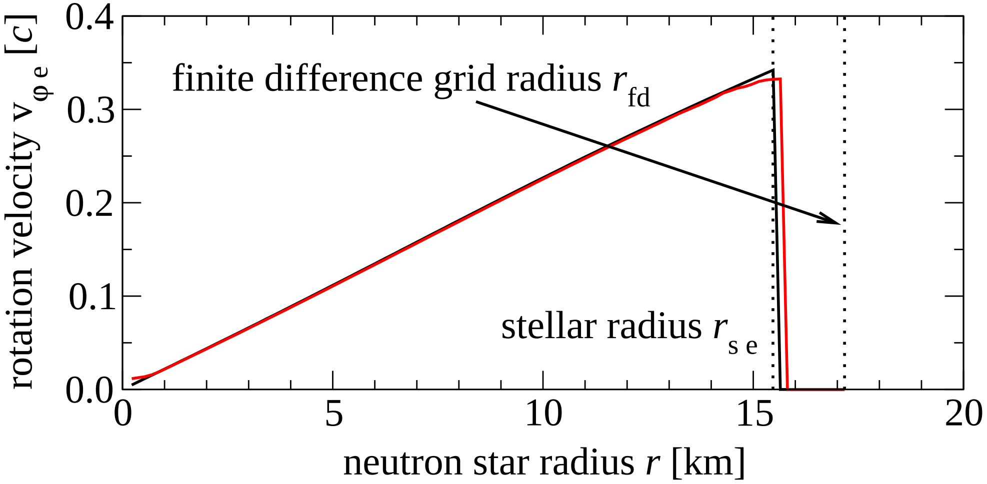

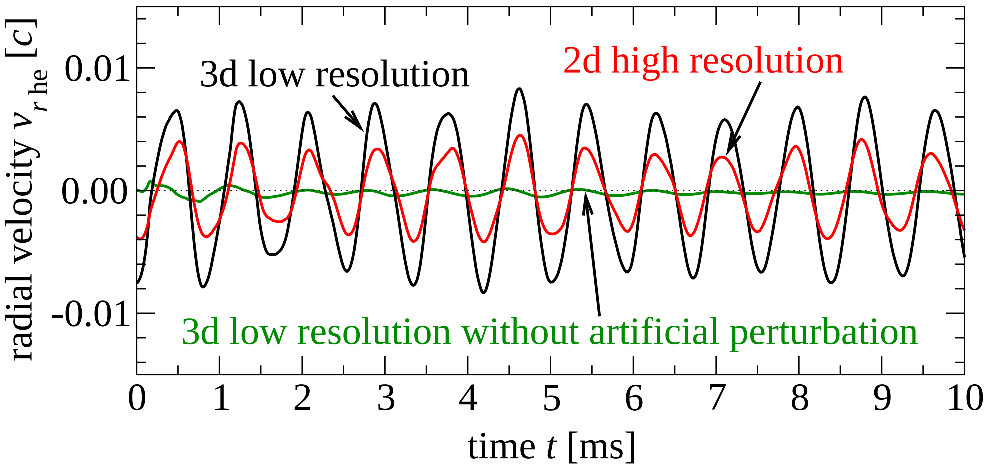

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Proof of principle: Code is ready for simulations of dynamical triaxial instabilities!

The Ladder of Credibility

New code has passed all tests in axisymmetry (against previous axisymmetric code and other codes).

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Code has climbed "Ladder of Credibility"!

Generic Nonaxisymmetric Configurations

Now finally explore nonaxisymmetric configurations in 3d.

Extension from axisymmetry to nonaxisymmetry trivial with LORENE!

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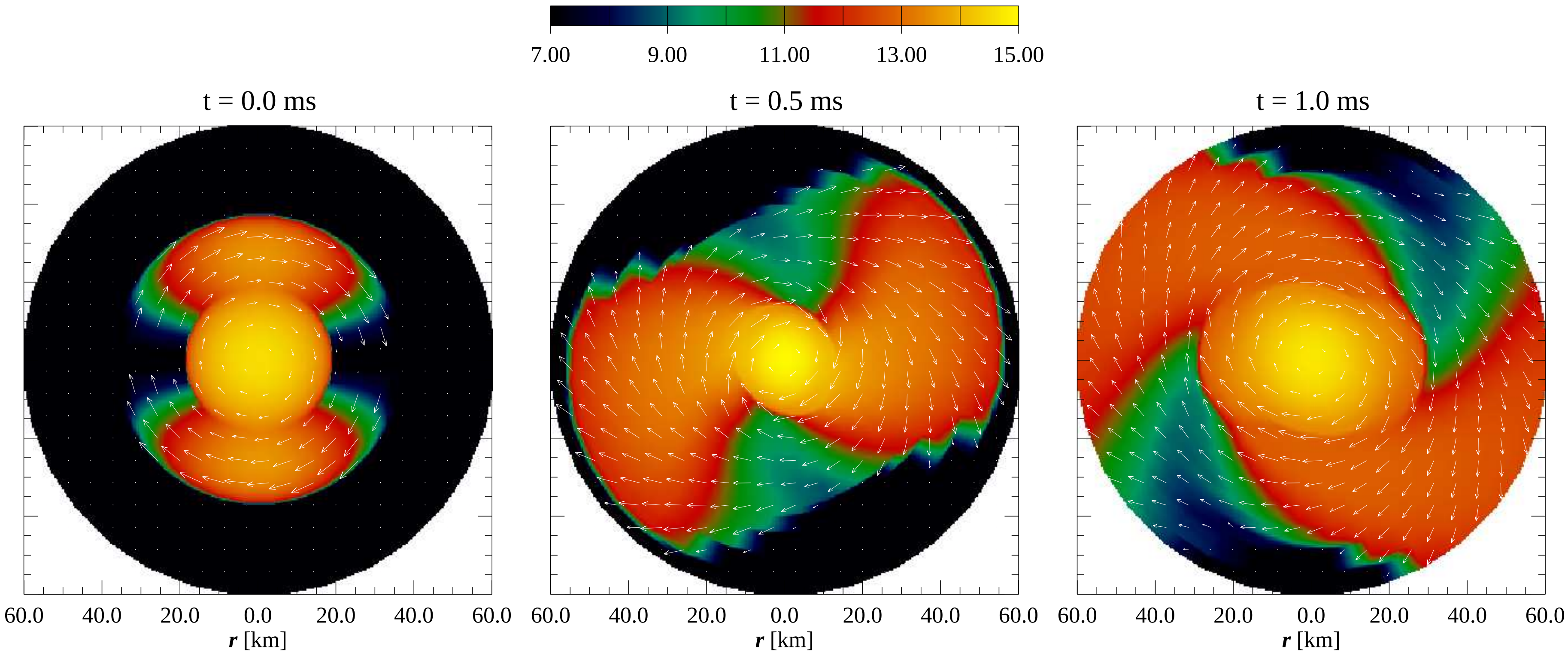
Setup: Rotating neutron star with strong “bar” perturbation (unphysical!).



Rotation generates spiral arms!

Three-Dimensional Wave Extraction

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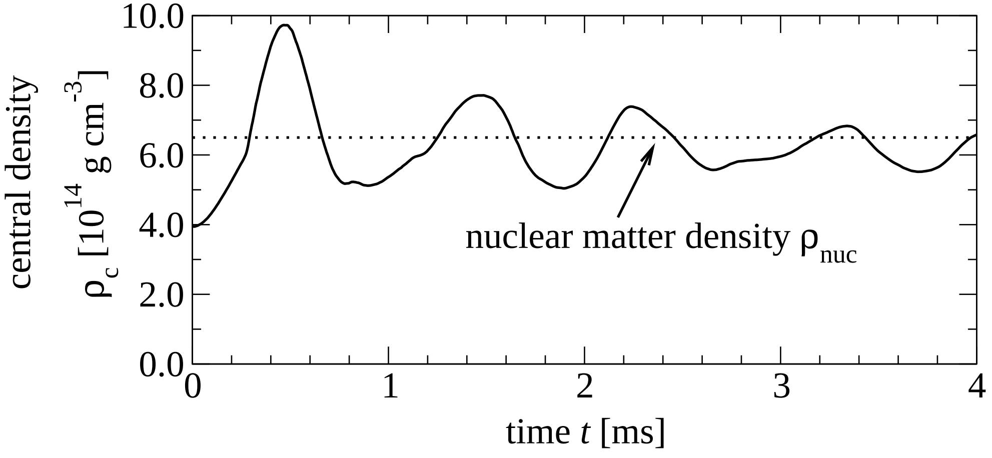

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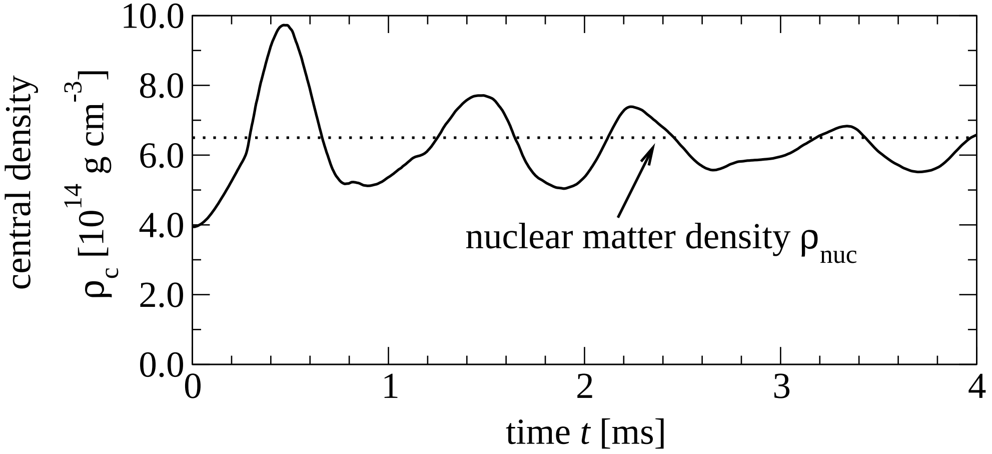

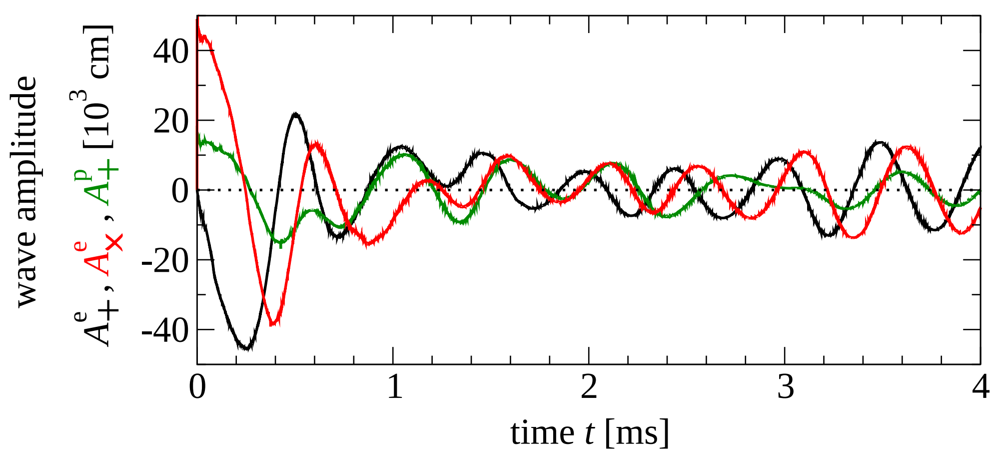

Also: Wave extraction based on 3d Newtonian quadrupole formula implemented.

Contrary to axisymmetry: Both polarizations of gravitational radiation excited!

Comparison with 3d Cartesian code `Cactus/Whisky` is underway (also in axisymmetry).

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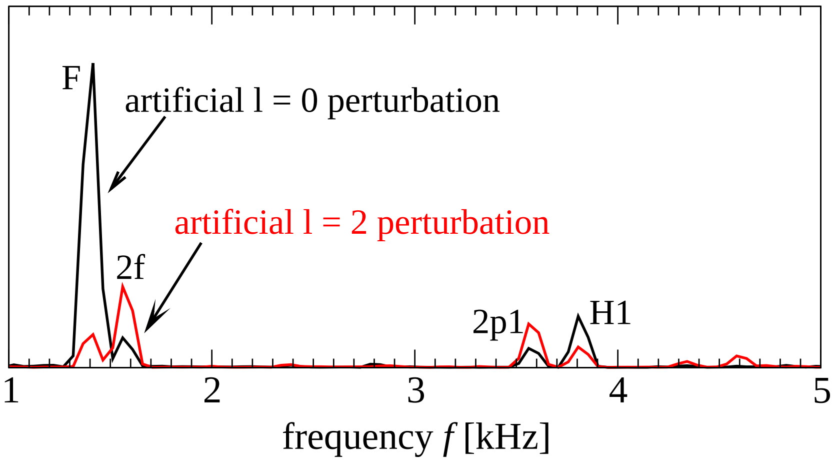
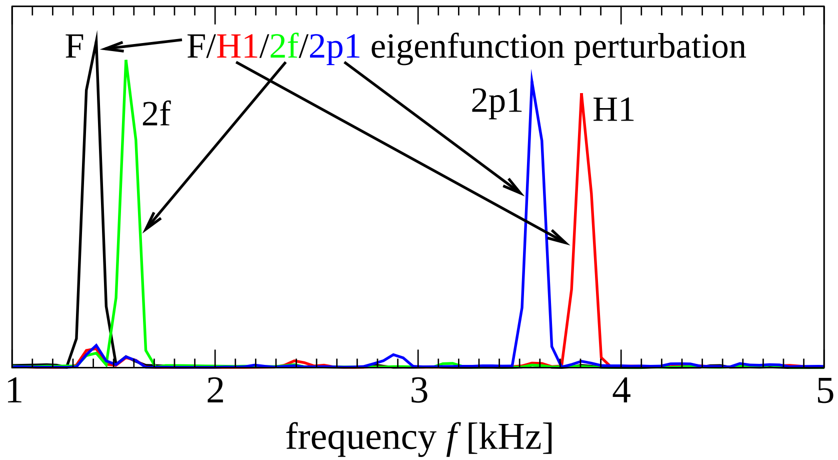
With excitation by eigenfunction, specific eigenmodes can be selected.
$\Longrightarrow$ Compare these results to perturbation codes!





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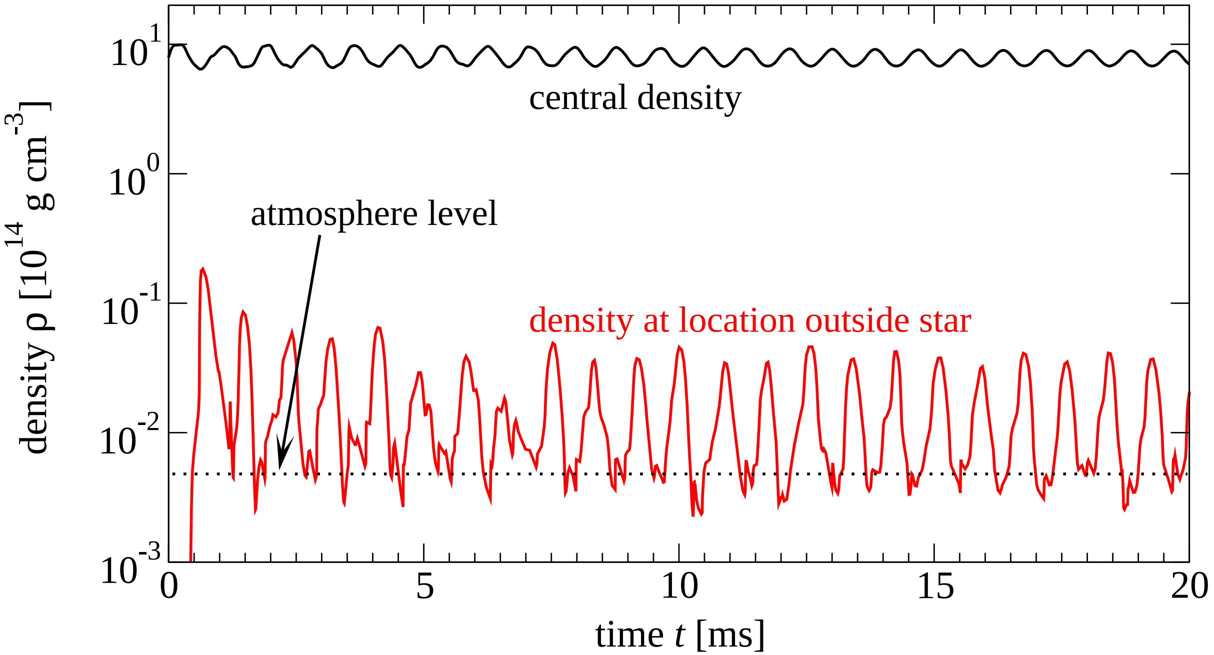

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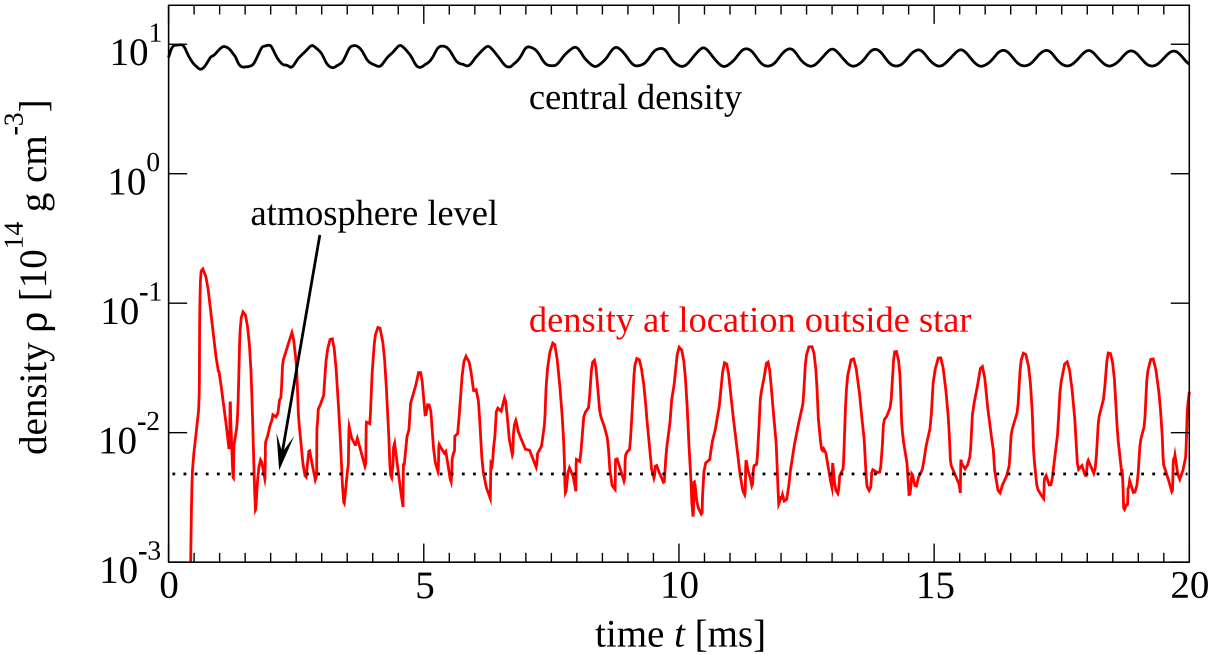

We want to investigate role of gravity and EoS in mass shedding
(now polytope versus ideal gas).

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This will probably be future for our code (full relativity in 3d)!

Advantages of a Compactified Radial Grid

We use quadrupole formula for wave extraction.
$\Longrightarrow$ Cannot make use of possibility to extend metric (without matter) into wave zone...

Major benefit from compactified grid:

- **All metric equation terms (also with noncompact support) are taken into account.**
- **No need for explicit boundary conditions at r_{fd} (simply assume asymptotic flatness).**

Example:

Rotating neutron star with r_{fd} close to $r_{\mathrm{s\,e}}$.

Artificial boundary condition $\beta_\varphi = 0$ bad!

Solver 3 yields accurate result
(difference due to CFC and different grid).