Propagation of Gravitational Waves in a FRW Universe

in other words…

What a Cosmological Gravitational Wave may look like

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INTRODUCTION & MOTIVATION

What are we trying to do?

We look for exact gravitational wave solutions in a curved cosmological background!

What for?

To investigate the effect of the background curvature and the spacetime dynamics on the propagation (and the expected characteristics) of a RELIC gravitational wave!

Give me a good reason why they should have any effect.

Due to the non-linearity of the gravitational field equations even a weak gravitational wave may interact with the background gravitational field (the gravity gravitates)!

This interaction could alter the dynamical characteristics of the wave, resulting in its dispersion, its amplification etc.

Why should we bother?

The detection of relic gravitational waves is probably the only way to obtain information about the very early stages of the Universe evolution!

- In order to detect them, we need to specify what do we expect to see, i.e. to determine their characteristics!

- The safest way to do so, is to evaluate SOME exact solutions!

Some exact solutions?

Yes! The extreme physical conditions holding at the early stages of the Universe may have resulted in many different states of evolution!

A gravitational wave solution should be compatible with the spacetime evolution during the time period corresponding to each of these states!

COSMOLOGICAL GRAVITATIONAL WAVES

What are they?

The so-called cosmological gravitational waves ($h_{\mu\nu}$) represent small corrections to the Universal metric tensor!

The far-field propagation (away from the source) of a weak gravitational wave in a curved and non-vacuum spacetime, is governed by the differential equations:

$$h_{\mu\nu;\alpha}^{\;;\alpha} - 2\Re_{\alpha\mu\nu\beta} h^{\alpha\beta} = 0$$

$$\left(h^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} h \right)_{;\beta} = 0$$

where $g_{\alpha\beta}$ is the background metric.

SETTING THE PROBLEM

We are interested in studying the evolution of a cosmological gravitational wave in the transition of the Universe:

(i) From the inflationary epoch to the radiation dominated era (at $t = t_{GUT}$)

(ii) From the radiation dominated epoch to the matter dominated era (at $t = t_{REC}$)

Each of these transitions is assumed to be instantaneous! (Very restrictive! To be relaxed!)

The changes taking place within these three epochs are two-fold:

(i) Each transition modifies the background dynamics!

(ii) In accordance, the wave propagation equation changes its differential type!

THE BIG PICTURE

A gravitational wave is created during the inflationary epoch (probably due to true-vacuum bubble collisions) and propagates!

In the meantime...

The Universe experiences a number of phase-transitions, mostly due to non-gravitational Physics:

Inflation era - Radiation era - Matter era

We try to explore how the gravitational wave responds to all these modifications of the spacetime dynamics.

To do so, we consider a linearly polarized, plane gravitational wave propagating in a spatially flat FRW cosmological model!

THE BACKGROUND DYNAMICS

The cosmological model: A spatially flat FRW model

$$ds^2 = c^2 dt^2 - R^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

Is a solution of: The Friedmann equations for k=0

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho(t)$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 = -\frac{8\pi G}{c^2}p(t)$$

With matter-content: In the form of a perfect-fluid

$$T_{\mu\nu} = (\rho c^2 + p)u^{\mu}u^{\nu} - g_{\mu\nu}p$$

Obeying:

- The conservation law:

$$T^{\mu\nu}_{;\nu} = 0 \Rightarrow \dot{\rho} + 3(\rho + \frac{p}{c^2})\frac{\dot{R}}{R} = 0$$

- The equation of state:

$$p = \left(\frac{m}{3} - 1\right)\rho c^2$$

Which is decomposed to:

- Quantum Vacuum: m = 0
- Gas of Strings: m = 2
- Dust: m = 3
- Radiation: m = 4
- Stiff Matter: m = 6

Determining the behaviour of the matter content:

The conservation law implies:

$$\rho(t) = \frac{const}{R^m} \qquad \text{where,} \qquad const = \sqrt{\frac{8\pi G}{3}\rho_0 R_0^m}$$

and ρ_0, R_0 are the typical energy density and scale factor corresponding to the m-epoch!

Determining the spacetime evolution:

The Friedmann equations imply:

$$R^{\frac{m}{2}-1}\dot{R}=C$$

We consider the following cases:

(i) Quantum vacuum: m = 0

$$R(t)=R_0e^{Ct}$$

i.e. an inflationary model!

(ii) Gas of strings: m = 2

$$R(t)=R_2t$$

i.e. a Milne model!

(iii) Other matter-contents: m ≠ 0, 2

$$R(t)=R_mt^{\frac{2}{m}}$$

i.e. the standard model scenario, since, for m = 3 it results in the E-DS model and for m = 4 in the Friedmann radiation model!

RESOLVING THE PROBLEM

In a FRW background the cosmological wave perturbations are defined by the expression:

$$ds^2 = c^2 dt^2 - R^2(t)(\delta_{ij} + h_{ij}) dx^i dx^j$$

We introduce the conformal-time coordinate

$$\tau = \int \frac{cdt}{R(t)}$$

in terms of which the scale factor is written in the form:

- For m = 0 (at the inflationary era):

$$S(\tau) = \frac{S_0}{\tau}$$

- For m = 2 (at the string regime):

$$S(\tau) = S_2 e^{C\tau}$$

- For m ≠ 0, 2 (at the standard model scenario):

$$S(\tau) = S_m \tau^{\frac{2}{m-2}}$$

Then, the gravitational wave equation of propagation reduces to:

$$h_{ik}'' + 2\frac{S'}{S}h_{ik}' + \delta^{lm}h_{ik,lm} = 0$$

To decompose it, we represent the metric corrections in the form:

$$h_{ik}(\tau, x^j) = \frac{h(\tau)}{S(\tau)}G_{ik}(x^j)$$

where, h(τ) is the time-dependent part of the modes and

$$G_{ik}(x^j) = \alpha\varepsilon_{ik}e^{ik_j x^j}$$

is a tensor eigenfunction of the wave-number k attributed to the Laplace operator of the flat space.

Accordingly, we end up with a differential equation for the time-dependent part of the modes:

$$h'' + (k^2 - \frac{S''}{S})h = 0$$

EXACT SOLUTIONS

For m = 0: At the inflationary phase:

$$h'' + (k^2 - \frac{2}{\tau^2})h = 0 \Rightarrow h_0(\tau) = \sqrt{\tau} H_{3/2}^{(1,2)}(k\tau)$$

For m ≠ 0, 2: Within the standard model scenario:

$$h'' + \left[k^2 - 2\frac{4-m}{(m-2)^2}\frac{1}{\tau^2} \right] h = 0 \Rightarrow h_m(\tau) = \sqrt{\tau} H_\nu^{(1,2)}(k\tau)$$

where, the Hankel functions' order is

$$\nu = \frac{1}{2}\left(\frac{m-6}{m-2}\right)$$

thus, resulting in:

For m=4 (at the radiation epoch): $\nu = 1/2$

For m=3 (at the matter epoch): $\nu = 3/2$ (again!)

Accordingly, we obtain:

At the inflationary era:

$$h_{\inf l}(\tau)=\sqrt{\frac{2}{\pi k}}C_{\inf l}\left(\frac{1}{k\tau}+i\right)e^{-ik\tau}\Rightarrow$$

$$h_{\inf l}(\tau)=\sqrt{\frac{2}{\pi k}}C_{\inf l}\sqrt{1+\frac{1}{k^2\tau^2}}e^{-i[k-\frac{1}{\tau}\tan^{-1}(k\tau)]\tau}$$

At the radiation-dominated epoch:

$$h_{rad}(\tau)=\sqrt{\frac{2}{\pi k}}C_{rad}e^{-ik\tau}$$

At the matter-dominated epoch:

$$h_{mat}(\tau)=\sqrt{\frac{2}{\pi k}}C_{mat}\left(\frac{1}{k\tau}+i\right)e^{-ik\tau}\Rightarrow$$

$$h_{mat}(\tau)=\sqrt{\frac{2}{\pi k}}C_{mat}\sqrt{1+\frac{1}{k^2\tau^2}}e^{-i[k-\frac{1}{\tau}\tan^{-1}(k\tau)]\tau}$$

Notice that:

$$\omega_{mat}=k-\frac{1}{\tau}\tan^{-1}(k\tau)=\omega_{\inf l}$$

The propagation eqs at the inflationary era and the matter-dominated epoch are "dynamically equivalent", i.e. their spaces of solutions are isomorphic!

Furthermore…

In the string regime (m = 2):

Around Planck-time, the wave propagation problem gets even better!

The corresponding wave equation reduce to

$$h'' + (k^2 c^2 - C^2) h = 0$$

admitting formal plane-wave solutions of the form:

$$h(\tau) = \sqrt{\tau} H_{1/2}^{(1,2)}(\omega\tau) = \sqrt{\frac{2}{\pi k}} C_{str} e^{-i\omega\tau}$$

i.e. a problem dynamically equivalent to the corresponding one of the radiation era. However, now the frequency is given by

$$\omega^2 = k^2 c^2 - \frac{8\pi G}{3} \rho_0 R_0^2$$

easily recognizable as a dispersion relation which implies $\upsilon_{ph} < c$ for the gravitational wave!