MAGNETIZED RELATIVISTIC STARS AND THE STATUS OF NUMERICAL RELATIVISTIC MHD

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The Path to GR-MHD

1. Hydrodynamical Evolution
2. Spacetime Evolution
3. Magnetic Field Evolution

Hydrodynamical Evolution (I)

General-Relativistic Hydrodynamics

$$T^{ab}{}_{;b} = 0 \qquad \text{Energy and momentum conservation}$$

$$(\rho u^a)_{;a} = 0 \qquad \text{Baryon number conservation}$$

In a coordinate basis:

$$\partial_t(\sqrt{-g}\rho u^t) + \partial_i(\sqrt{-g}\rho u^i) = 0 \qquad \text{(baryons)}$$

$$\partial_t(\sqrt{-g}T^t{}_j) + \partial_i(\sqrt{-g}T^i{}_j) = \sqrt{-g}T^k{}_\lambda \Gamma^\lambda{}_{jk} \qquad \text{(momentum)}$$

$$\partial_t(\sqrt{-g}T^t{}_t) + \partial_i(\sqrt{-g}T^i{}_t) = \sqrt{-g}T^k{}_\lambda \Gamma^\lambda{}_{tk} \qquad \text{(energy)}$$

The above system is in a 1st-order flux conservative hyperbolic form:

$$\partial_t \vec{U} + \partial_i \vec{F}^i = \vec{S}$$

Only in this form can one compute shock waves correctly.

Hydrodynamical Evolution (II)

3+1 spacetime split:

Choose an Eulerian observer, with unit vector

$$n^\alpha = \frac{1}{\alpha}(1, \beta^i)$$

orthogonal to a spacelike hypersurface foliation. Then, the metric is written as

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j$$

Define:

$$\gamma = det(\gamma_{ij})$$

$$W = \alpha u^t = \frac{1}{\sqrt{1 - v^2}}$$ (Lorentz factor)

$$v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$ (3-velocity w.r.t. Eulerian observer)

ρ (rest-mass density)

ϵ (specific internal energy)

$e = \rho(1 + \epsilon)$ (energy density)

$$h = 1 + \epsilon + \frac{P}{\rho}$$ (specific enthalpy)

$$P = (\gamma - 1)\rho\epsilon$$ (ideal fluid EOS)

Hydrodynamical Evolution (III)

Then, instead of the primitive variables ρ, v^i, ϵ

one uses the conserved variables

$$
\begin{aligned}
D &= \sqrt{\gamma} W \rho \\
S_i &= \sqrt{\gamma} \rho h W^2 v_i \\
\tau &= \sqrt{\gamma} (\rho h W^2 - P - W \rho)
\end{aligned}
$$

and the 1st-order flux-conservative hyperbolic system becomes

$$
\vec{U} = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix}
\qquad
\vec{F}^i = \begin{bmatrix} \alpha \left(v^i - \beta^i/\alpha \right) D \\ \alpha \left[\left(v^i - \beta^i/\alpha \right) S_j + \sqrt{\gamma} P \delta^i_j \right] \\ \alpha \left[\left(v^i - \beta^i/\alpha \right) \tau + \sqrt{\gamma} v^i P \right] \end{bmatrix}
$$

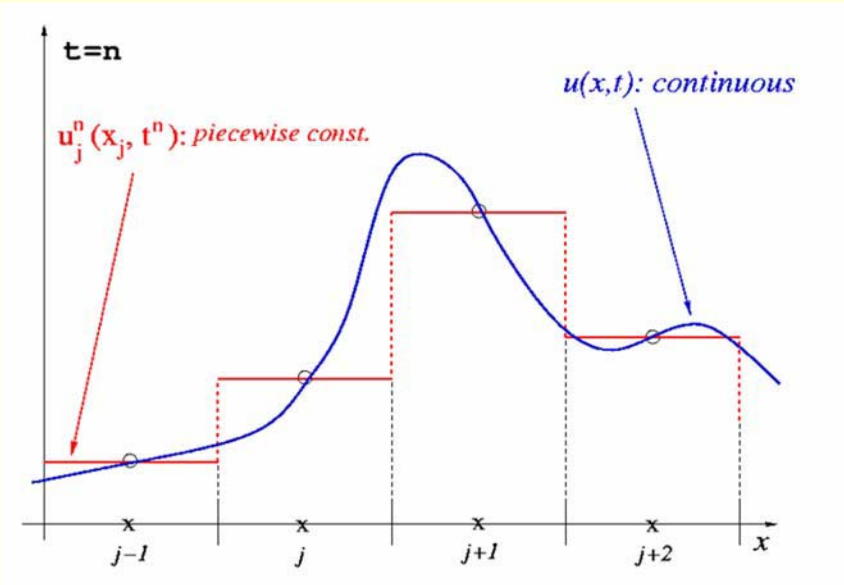
$$
\vec{S} = \begin{bmatrix} 0 \\ \alpha \sqrt{\gamma} T^{\mu\nu} g_{\nu\sigma} \Gamma^{\sigma}{}_{\mu j} \\ \alpha \sqrt{\gamma} (T^{\mu t} \partial_\mu \alpha - \alpha T^{\mu\nu} \Gamma^{t}{}_{\mu\nu}) \end{bmatrix}
$$

Numerical Scheme

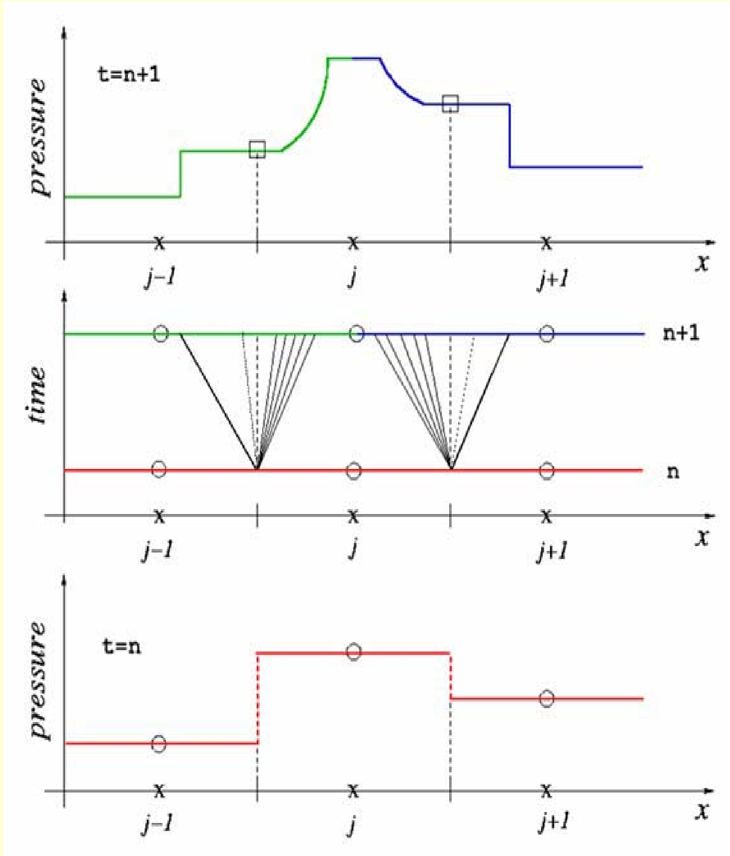
High Resolution Shock Capturing (HRSC) finite-volume schemes

- Relativistic implementations: *Marti, Ibanez, Miralles, PRD, 1991*
- Oscillation-free at discontinuities + Low numerical dissipation

Main idea: Solution of local Riemann problem in each cell



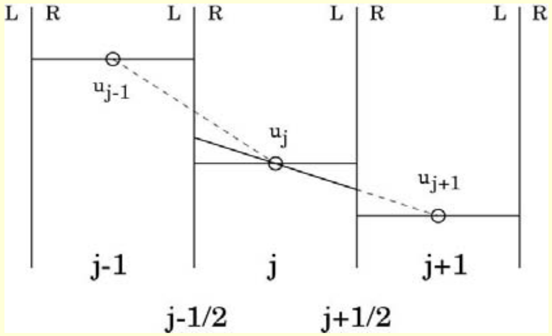
$$\hat{f}_{i+1/2} = \frac{1}{2}\left(f(u^L) + f(u^R) - Q\right)$$



Typical Implementation HRSC Schemes

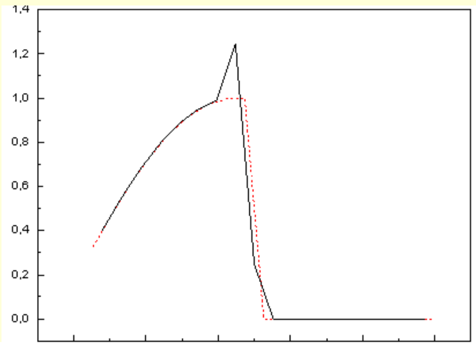
1. Cell reconstruction:

1st, 2nd or 3rd order (PPM) interpolation of variables from cell centers to cell interfaces.



2. Apply slope limiters:

Near discontinuities correct for abrupt changes (MC).



3. Numerical fluxes:

Approximate Riemann solvers (Roe, Marquina). Explicit use of the spectral information of the system. Or, e.g. HLLE (no need for spectral information).

$$\hat{\vec{f}}_i = \frac{1}{2}\left[\vec{f}_i(w_R) + \vec{f}_i(w_L) - \sum_{n=1}^{5} \left|\tilde{\lambda}_n\right| \Delta\tilde{\omega}_n \tilde{R}_n\right]$$

4. Time update:

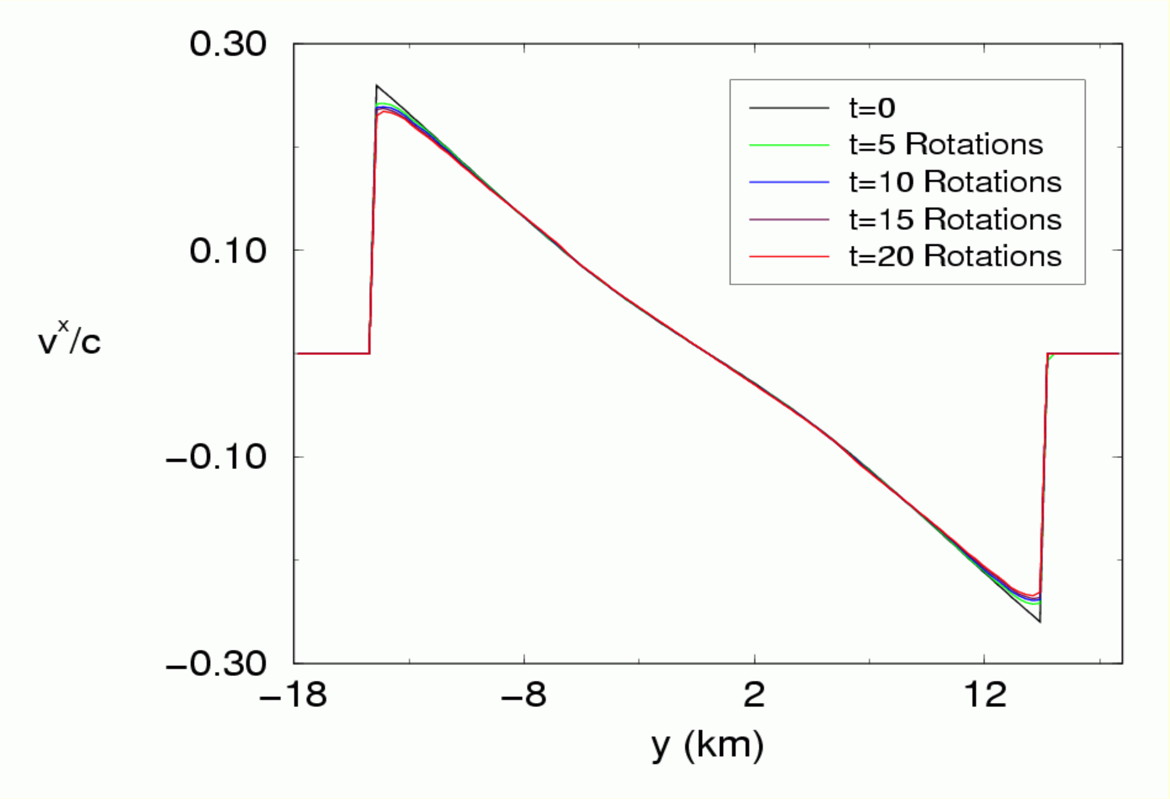
2nd or 3rd order conservative Runge-Kutta schemes

$$\vec{u}_j^{\,n+1} = \vec{u}_j^{\,n} - \frac{\Delta t}{\Delta x}\left(\hat{\vec{f}}^{\,n}_{j+1/2} - \hat{\vec{f}}^{\,n}_{j-1/2}\right)$$

Example: Evolution of Equilibrium Models

Font, N.S. & Kokkotas (2000)

2-D nonlinear evolutions with 3rd order PPM method in Cowling approximation



Spacetime Evolution

Conformal-traceless formulation

Kojima – Oohara - Nakamura - Shibata – Baumgarte - Shapiro

Definitions

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$$

$$e^{4\phi} = \gamma^{1/3} \equiv \det(\gamma_{ij})^{1/3}$$

$$\tilde{A}_{ij} = e^{-4\phi}A_{ij} \qquad A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K,$$

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}{}_{,j}$$

"1+log" slicing condition

$$\partial_t \alpha = -2\alpha A$$

$$\partial_t A = \partial_t K$$

Gamma-driver shift condition

$$\partial_t \beta^i = B^i$$

$$\partial_t B^i = \frac{3}{4}\alpha\partial_t\tilde{\Gamma}^i - e^{-4\phi}\beta^i$$

Alcubierre et al. (2002)

Evolution equations

$$\frac{d}{dt} = \partial_t - \mathcal{L}_\beta$$

$$\frac{d}{dt}\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij},$$

$$\frac{d}{dt}\phi = -\frac{1}{6}\alpha K.$$

$$\frac{d}{dt}K = -\gamma^{ij}D_iD_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}(\rho+S)\right],$$

$$\frac{d}{dt}\tilde{A}_{ij} = e^{-4\phi}[-D_iD_j\alpha + \alpha(R_{ij} - S_{ij})]^{TF} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j),$$

$$\frac{\partial}{\partial t}\tilde{\Gamma}^i = -2\tilde{A}^{ij}\alpha_{,j} + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\phi_{,j}\right) - \frac{\partial}{\partial x^j}\left(\beta^l\tilde{\gamma}^{ij}{}_{,l} - 2\tilde{\gamma}^{m(j}\beta^{i)}_{,m} + \frac{2}{3}\tilde{\gamma}^{ij}\beta^l_{,l}\right).$$

3D Nonlinear Simulations in Full GR

(Baiotti, Hawke, Montero, Loeffler, Rezzolla, N.S., Font & Seidel, 2004)

Initial Data

Rapidly rotating relativistic stars in uniform rotation

(N.S. & Friedman, 1995)

Nonlinear Evolution

Conformal-traceless formulation

(Kojima, Oohara & Nakamura, 1987; Shibata & Nakamura, 1995; Baumgarte & Shapiro, 1999)

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$$

$$e^{4\phi} = \gamma^{1/3} \equiv \det(\gamma_{ij})^{1/3}$$

$$\tilde{A}_{ij} = e^{-4\phi} A_{ij} \qquad A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij} K$$

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}{}_{,j}$$

Hydrodynamics

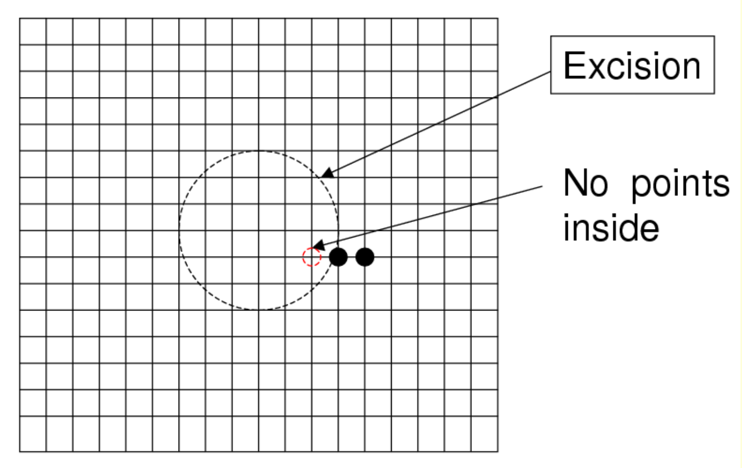
Approximate Riemann solvers with *3^rd-order Piecewise-Parabolic* method

(Font, N.S. & Kokkotas, 2000)

Excision inside Horizon

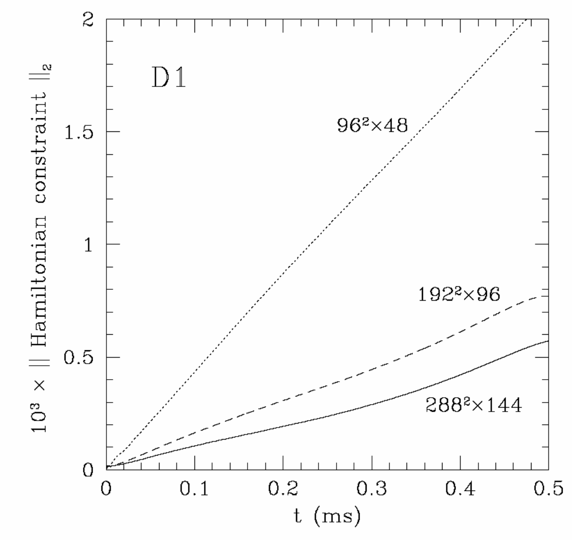
Cubical excision method

(Hawke, Loeffler & Nerozzi, 2004)

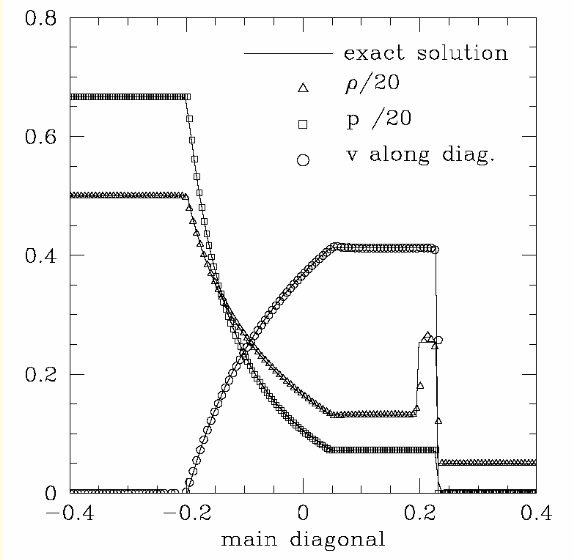


How can we trust our numerical solutions?

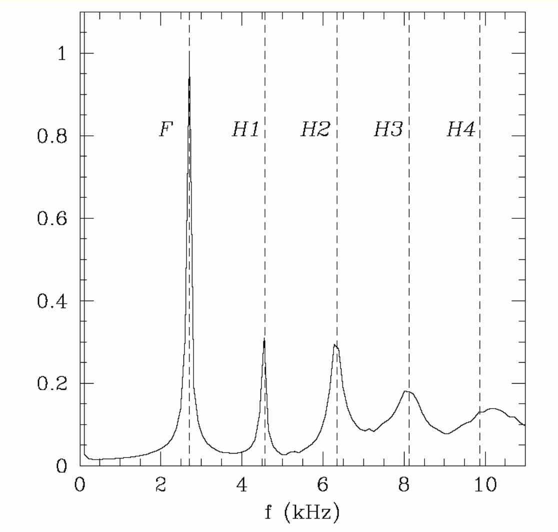
Convergence of Constraints



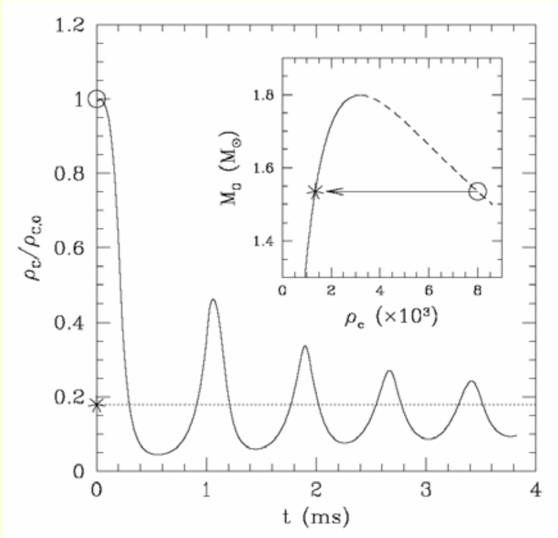
Shock tube



Comparison to linear oscillation modes



NS Migration



Axisymmetric Instability to Collapse

Rotating stars are subject to a secular axisymmetric instability, if:

$$\left(\frac{\partial M}{\partial \epsilon_c}\right)_J < 0$$

(Friedman, Ipser & Sorkin, 1988).

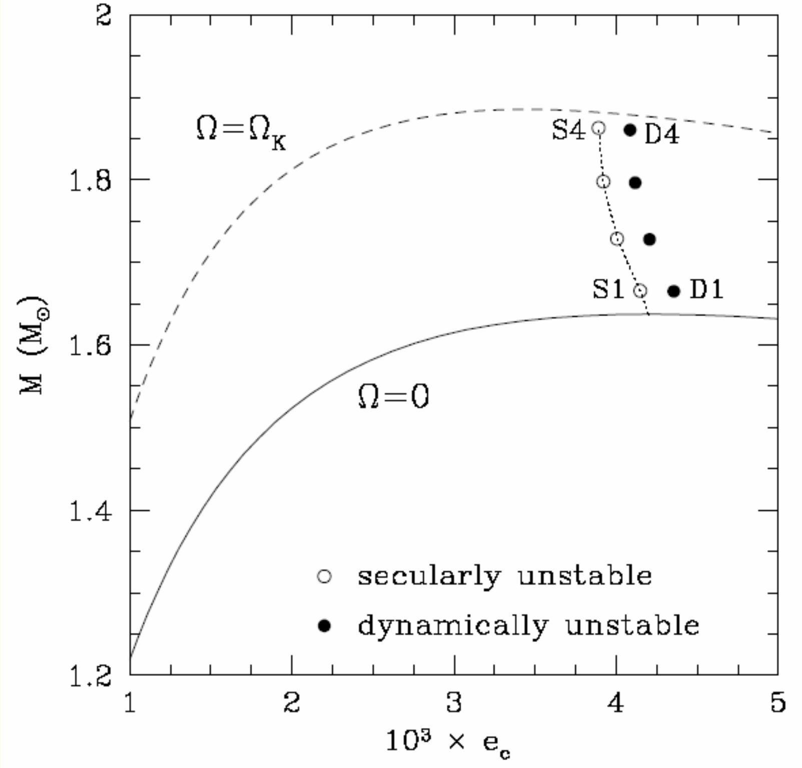
A star can collapse to a Kerr BH during:

a) *Core collapse* of massive star

b) *Accretion-induced collapse* of a compact star

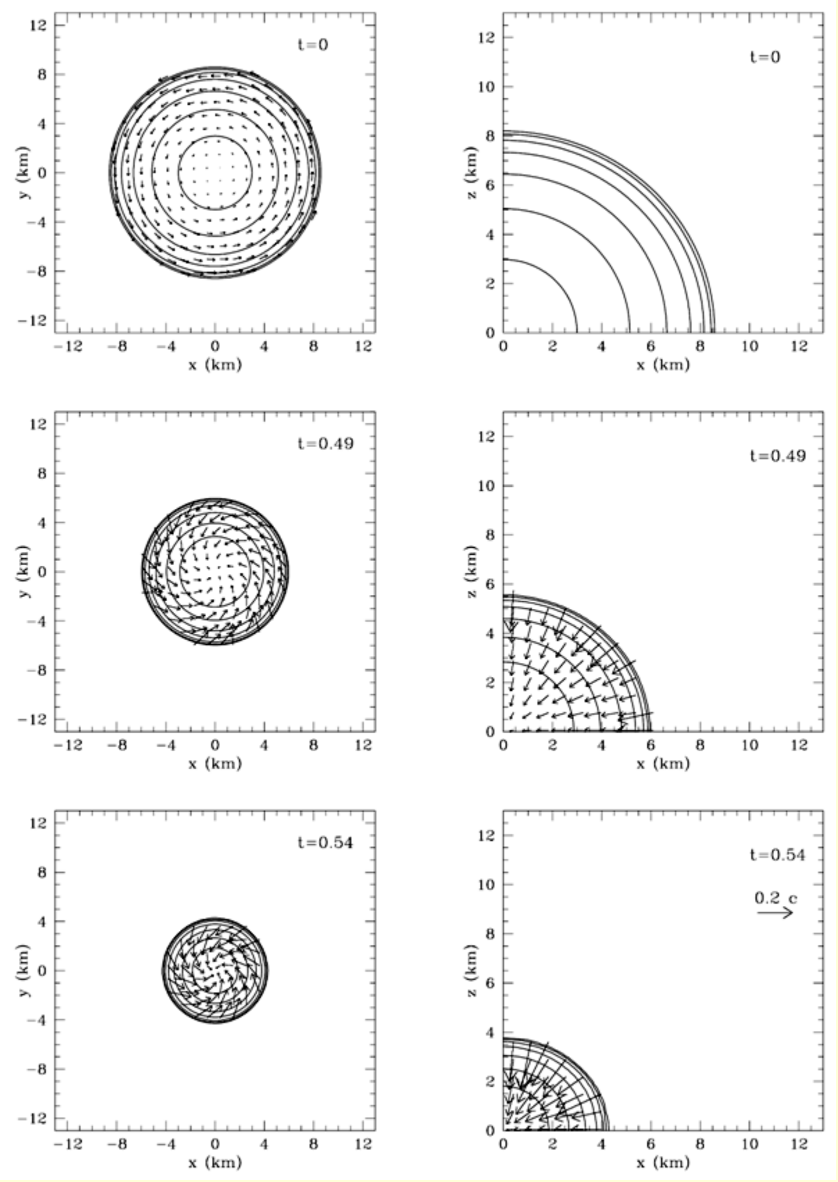
c) *Merger* of two compact stars in a binary system

d) *Spin-down* of a supramassive compact star

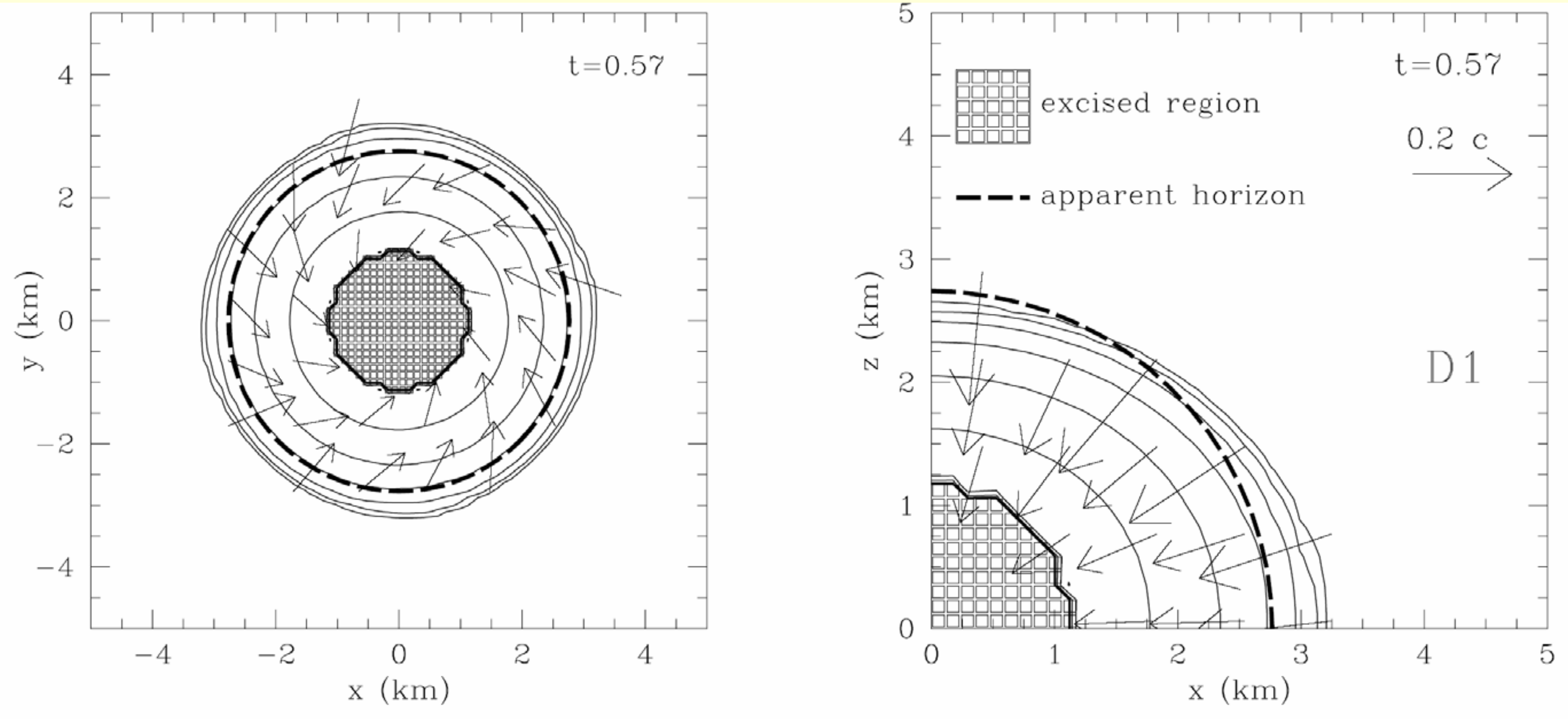


Dynamical instability soon after onset of secular instability.

Collapse of Slowly Rotating Model

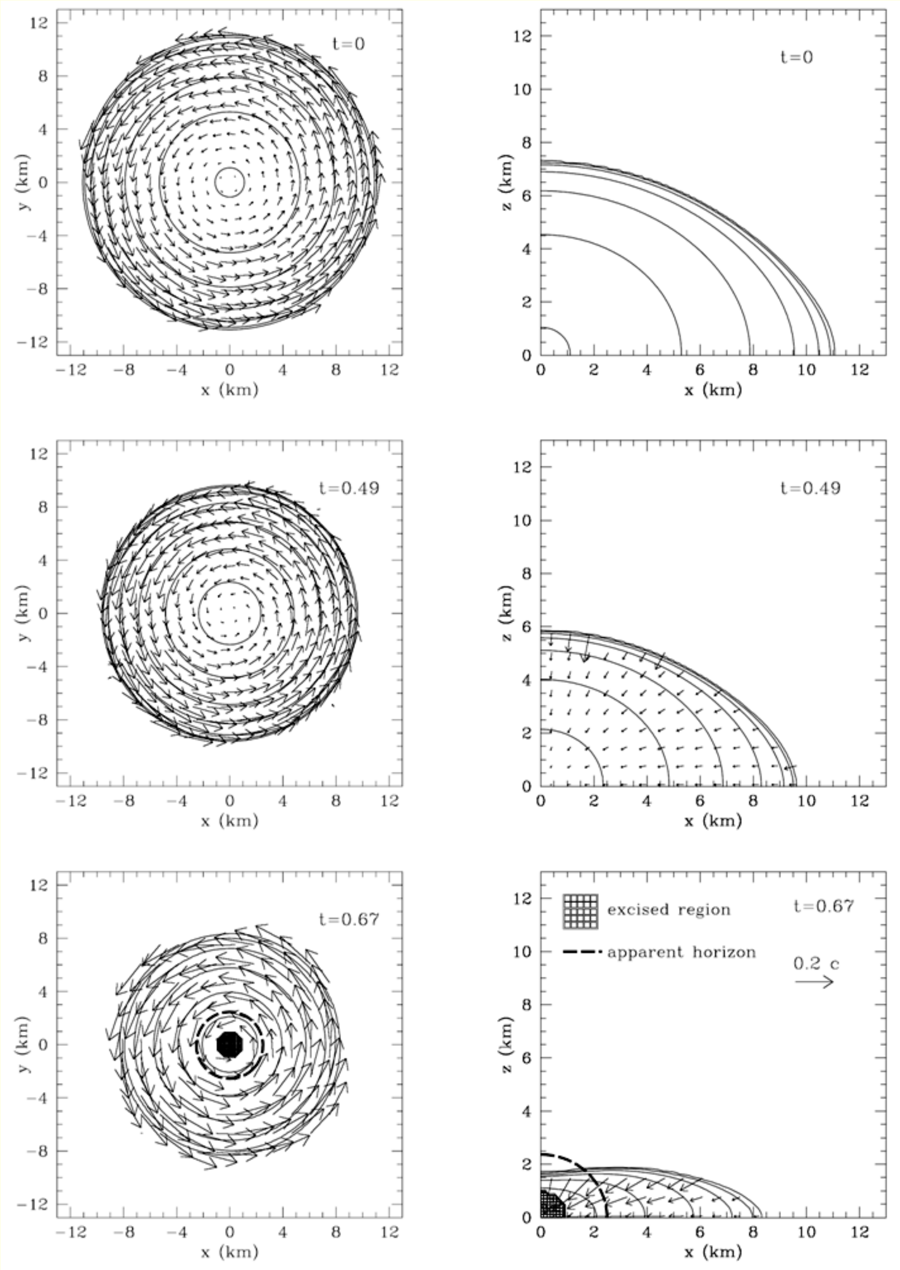


No Disk Formation

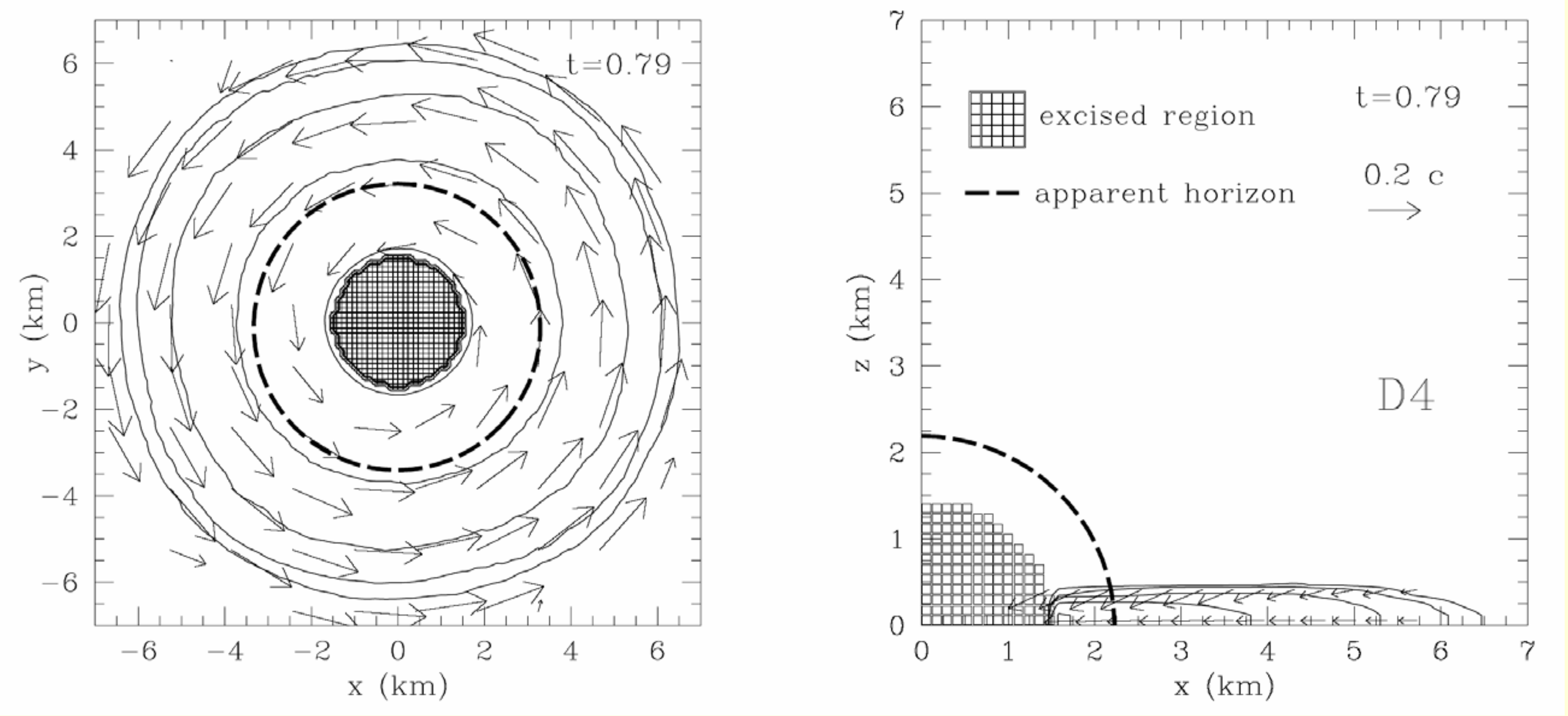


The collapse proceeds nearly spherically symmetric until all matter has enter the horizon.

Collapse of Rapidly Rotating Model



Formation of Unstable Disk



Due to *centrifugal hangup*, a thin disk forms outside the horizon.

If the initial star is uniformly rotating, the disk is *unstable* and accretes rapidly onto the BH.

Event Horizon Mass

The mass of the event horizon for a stationary Kerr BH can be computed directly from the *equatorial circumference*

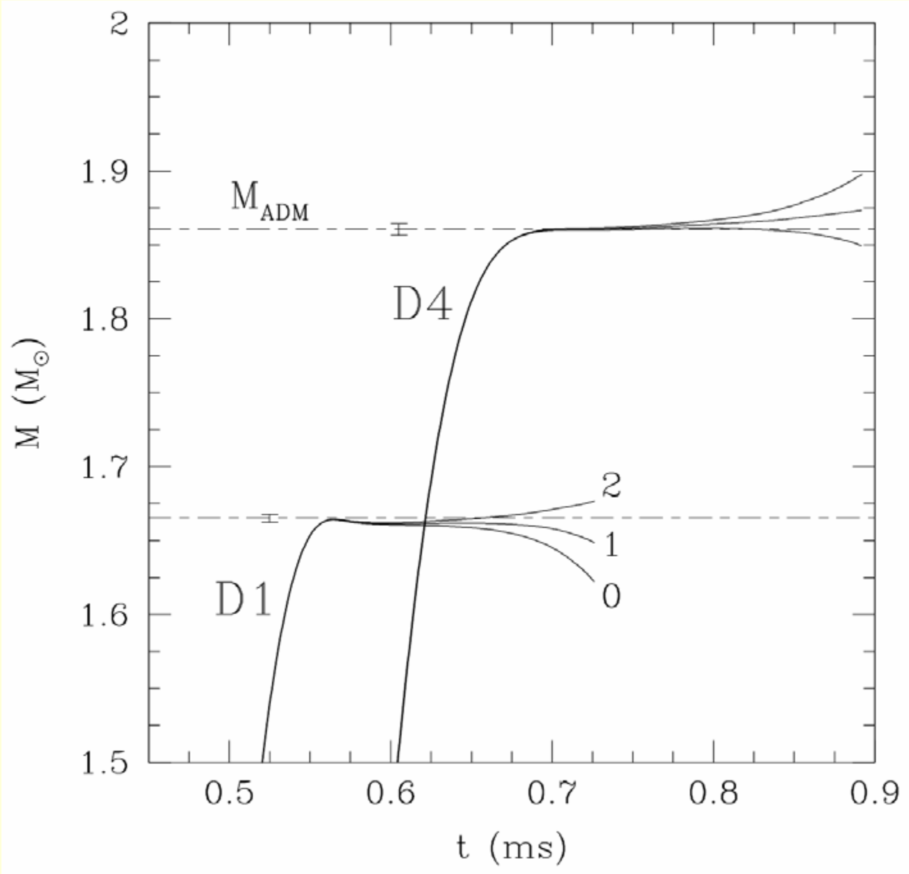
$$\mathcal{C}_{\mathrm{eq}} = \int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi$$

as

$$M = \frac{\mathcal{C}_{\mathrm{eq}}}{4\pi}$$

The event horizon is a *global* concept. It can be found only *after* the null history is known.

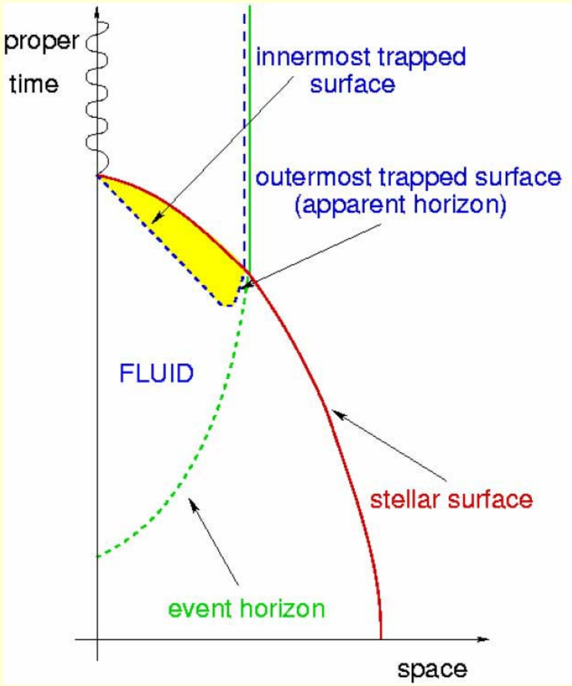
An outgoing null geodesic that starts infinitesimaly close outside the horizon, diverges exponentially away from it. Conversely, one can integrate null geodesics backwards in time, so that they *converge exponentially* onto the horizon *(Anninos et al. 1995).*



Problem: Integrating backwards several individual photons is sensitive to numerical errors.

Solution: Integrate *a whole trial null surface* with level-set method *(Diener, 2003).*

Dynamical Formation of Horizons (I)



Marginally Trapped Surfaces

Closed spacelike 2-surfaces in a t=const. slice, whose future-pointing outgoing null geodesics have zero expansion

$$\Theta \equiv n^i{}_{;i} + K_{ij} n^i n^j - K = 0$$

Apparent Horizon

Is defined as the *outermost* marginally trapped surface.

- The AH forms within the event horizon
- The AH is a local concept, an elliptic PDE is solved.

To locate the apparent horizon, we are using the fast 3-D solver of *Thornburg (2003).*

Dynamical Horizons

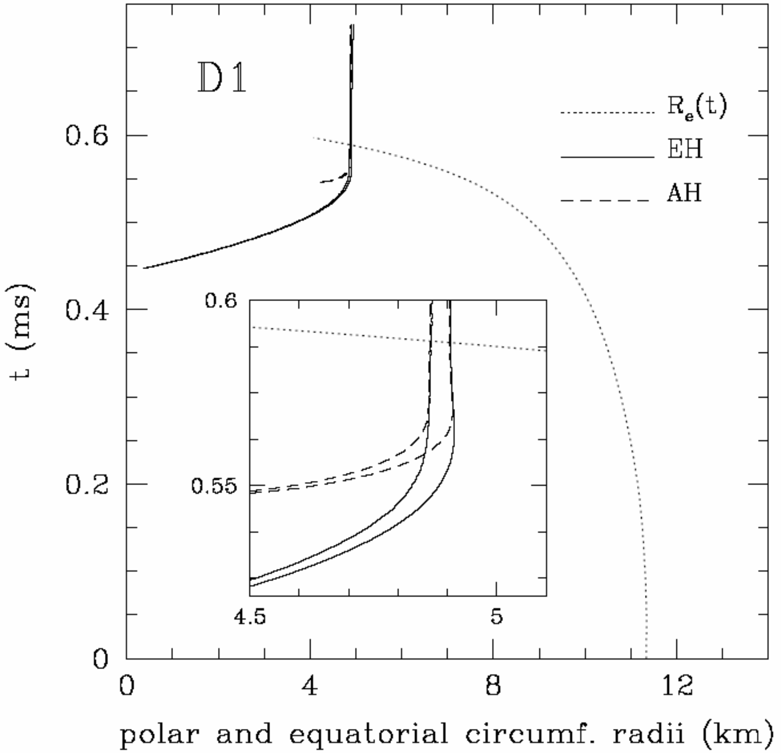
Ashtekar & Krishnan (2003) have generalized the 1st law of BH mechanics to nonlinear, time-dependent dynamical horizons, defining generalized mass and angular-momentum formulae. The *Christodoulou relation* still holds:

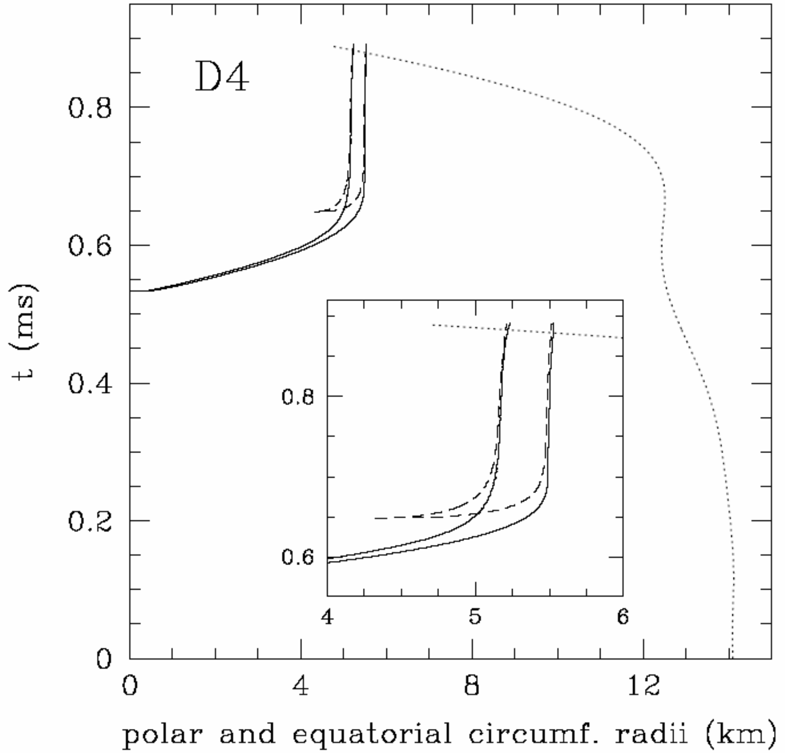
$$M = \frac{A}{16\pi} + \frac{4\pi J^2}{A}$$

Dynamical Formation of Horizons (II)

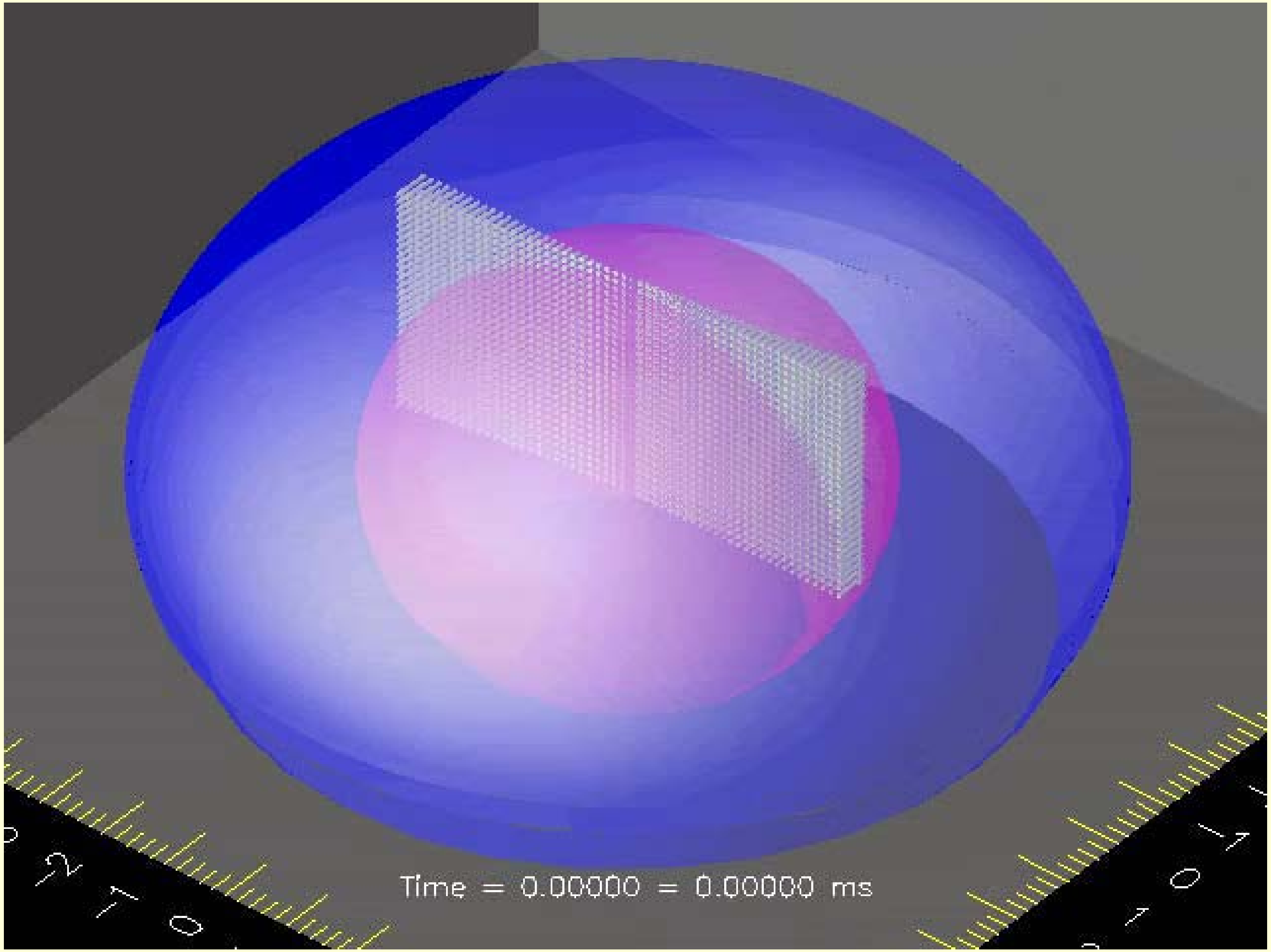
Slowly rotating model

Rapidly rotating model

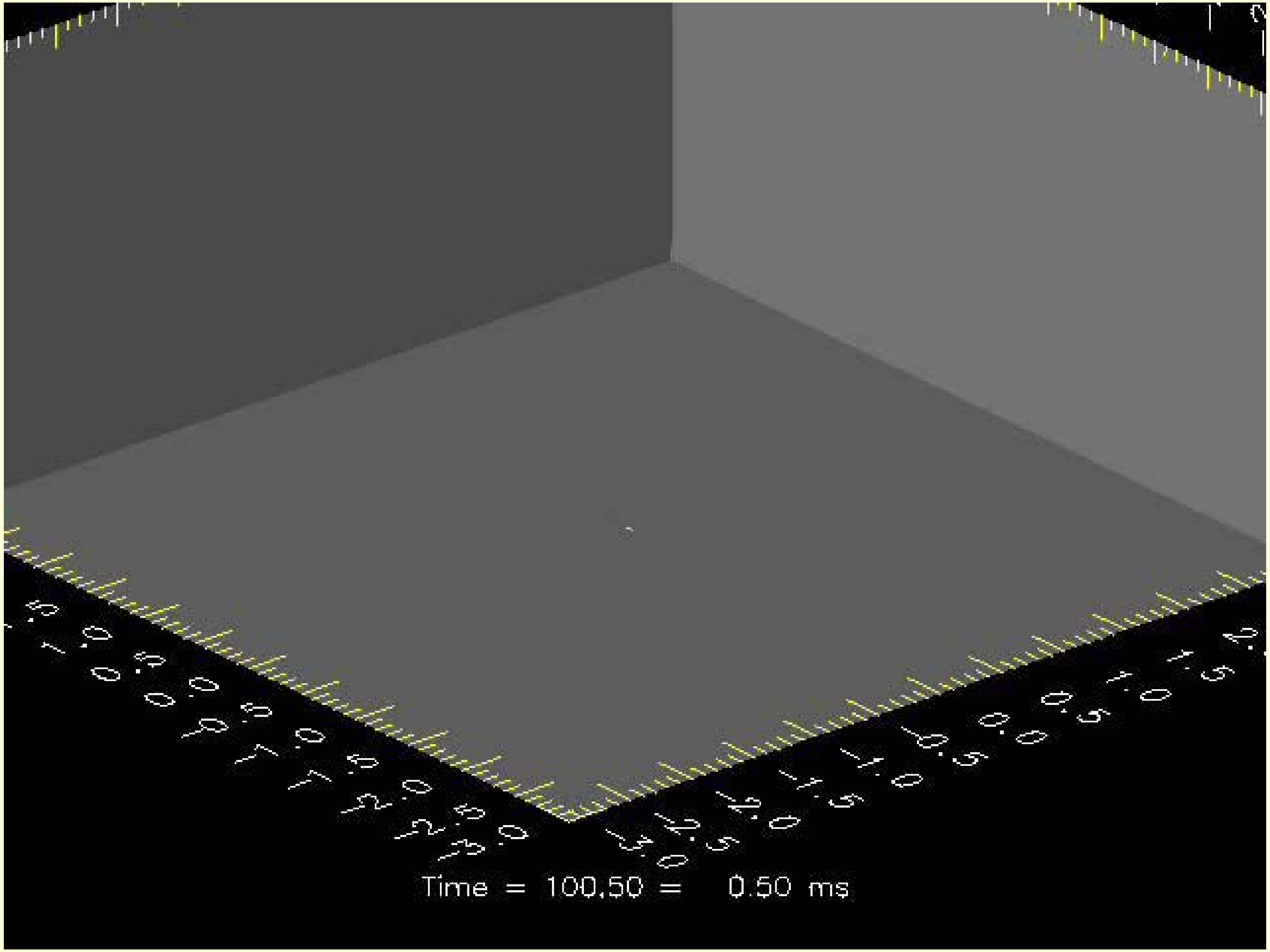




Dynamical Formation of Horizons (III)



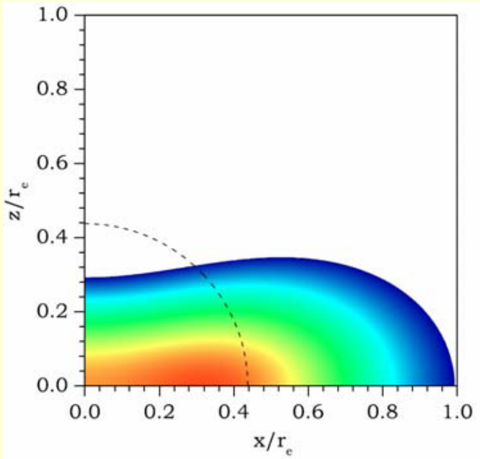
Dynamical Formation of Horizons (IV)

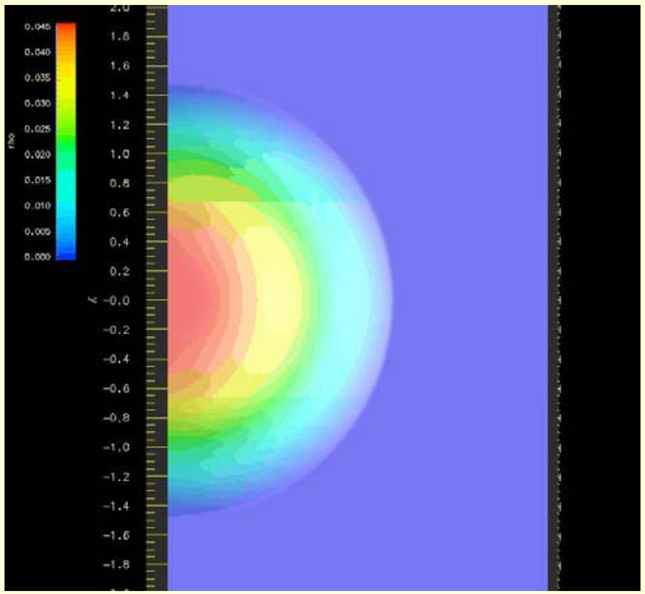


Current Developments

Differentially Rotating Initial Models

N.S., Apostolatos & Font (2004)

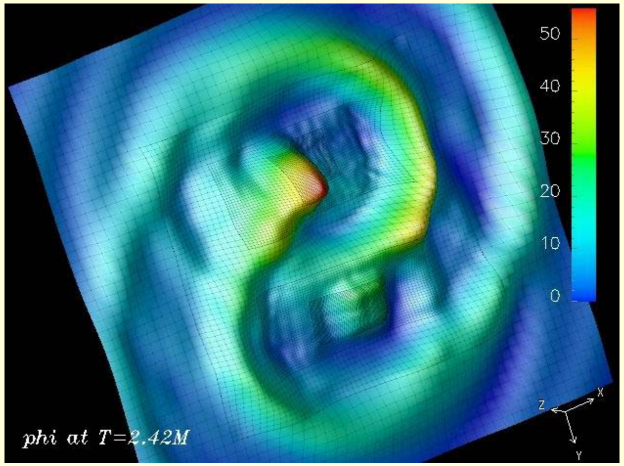


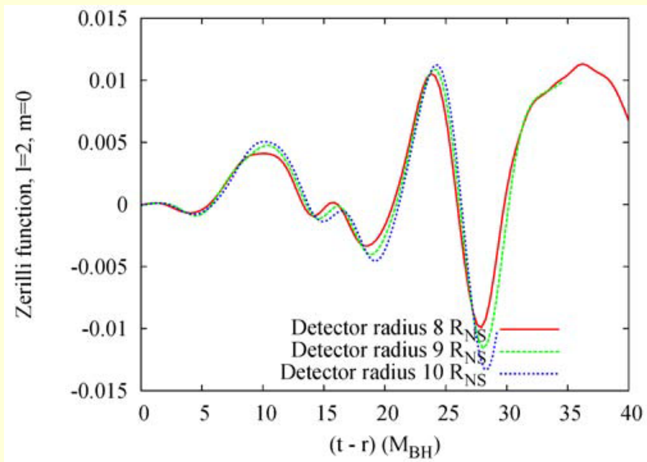


Fixed Mesh Refinement

Schnetter, Hawley, Hawke (2004)

Accurate waveforms!





Magnetic Field Evolution (I)

Electric field 4-vector $E^\alpha = F^{\alpha\beta}u_\beta = 0$

Magnetic field 4-vector $B^\alpha = {}^*F^{\alpha\beta}u_\beta$ $b^\alpha = B^\alpha/(4\pi)$

Conserved variables *(see Miralles, 2004)*

$$
\begin{aligned}
D &= \sqrt{\gamma}W\rho \\
S_i &= \sqrt{\gamma}\left(\rho h W^2 v_i + b^2 W^2 v_j - \alpha b^t b_j\right) \\
\tau &= \sqrt{\gamma}\left[\rho h W^2 - P - W\rho + b^2(W^2 - 1/2) - \alpha^2 (b^t)^2\right]
\end{aligned}
$$

1st-order flux-conservative hyperbolic system

$$
\vec{F}^i = \begin{bmatrix} \alpha\left(v^i - \beta^i/\alpha\right) D \\ \alpha\left[\left(v^i - \beta^i/\alpha\right)(S_j + b^2 W^2 v_j) + \sqrt{\gamma}[(P + b^2/2)\delta^i_j - b_j b^i]\right] \\ \alpha\left[\left(v^i - \beta^i/\alpha\right)[\tau + b^2(W^2 - 1/2)] + \sqrt{\gamma}[v^i(P + b^2/2) - \alpha b^t b^i]\right] \end{bmatrix}
$$

$$
\vec{U} = \begin{bmatrix} D \\ S_j \\ \tau \end{bmatrix} \qquad \vec{S} = \begin{bmatrix} 0 \\ \alpha\sqrt{\gamma}T^{\mu\nu}g_{\nu\sigma}\Gamma^\sigma{}_{\mu j} \\ \alpha\sqrt{\gamma}(T^{\mu t}\partial_\mu\alpha - \alpha T^{\mu\nu}\Gamma^t{}_{\mu\nu}) \end{bmatrix}
$$

Magnetic Field Evolution (II)

Maxwell's equations

$$^*F^{\alpha\beta}{}_{;\beta} = (u^\alpha B^\beta - u^\beta B^\alpha)_{;\beta} = 0$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$$

Magnetic field 3-vector $\tilde{B}^i$

$$b^t = \frac{W}{\alpha} v_i \tilde{B}^i$$

$$b^i = \frac{\tilde{B}^i + W^2 (v_i \tilde{B}^i)(v^i - \tilde{B}^i/\alpha)}{W}$$

Then, the first set of Maxwell's equations gives:

$$\partial_t \left(\sqrt{\gamma} \tilde{B}^i \right) + \partial_j \left\{ \sqrt{\gamma} \left[(\alpha v^j - \beta^j) \tilde{B}^i - (\alpha v^i - \beta^i) \tilde{B}^j \right] \right\} = 0$$

(induction equation in 1st-order flux-conservative hyperbolic form)

and

$$\partial_i \left(\sqrt{\gamma} \tilde{B}^i \right) = 0$$

"divergence-free" constraint equation

must be enforced exactly in a numerical scheme! (see Toth, 2000)

Construction of Initial Data

Metric: $$ds^2 = -e^{\gamma+\rho}dt^2 + e^{\gamma-\rho}r^2\sin^2\theta(d\phi - \omega dt)^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2)$$

Field Equations: integral form

$$\Delta[\rho e^{\gamma/2}] = S_\rho(r,\mu),$$

$$\left(\Delta + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\mu\frac{\partial}{\partial\mu}\right)\gamma e^{\gamma/2} = S_\gamma(r,\mu),$$

$$\left(\Delta + \frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\mu\frac{\partial}{\partial\mu}\right)\omega e^{(\gamma-2\rho)/2} = S_\omega(r,\mu),$$

$$\rho = -\frac{1}{4\pi}e^{-\gamma/2}\int_0^\infty dr'\int_{-1}^{1}d\mu'\int_0^{2\pi}d\phi' r'^2 S_\rho(r',\mu')\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

$$r\sin\theta\gamma = \frac{1}{2\pi}e^{-\gamma/2}\int_0^\infty dr'\int_0^{2\pi}d\theta' r'^2\sin\theta' S_\gamma(r',\theta')\log|\mathbf{r}-\mathbf{r}'|,$$

$$r\sin\theta\cos\phi\,\omega = -\frac{1}{4\pi}e^{(2\rho-\gamma)/2}\int_0^\infty dr'\int_0^\pi d\theta'\int_0^{2\pi}d\phi' r'^3\sin^2\theta'\cos\phi' S_\omega(r',\theta')\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

RNS code

N.S. & J. Friedman, 1995

Komatsu, Eriguchi & Hachisu method

Cook, Shapiro & Teukolsky compactified radial coordinate

$$\begin{aligned}
\alpha_{,\mu} = & -\nu_{,\mu} - \{(1-\mu^2)(1+rB^{-1}B_{,r})^2 + [\mu-(1-\mu^2)B^{-1}B_{,\mu}]^2\}^{-1} \\
& [\![\tfrac{1}{2}B^{-1}\{r^2B_{,rr} - [(1-\mu^2)B_{,\mu}]_{,\mu} - 2\mu B_{,\mu}\}[-\mu + (1-\mu^2)B^{-1}B_{,\mu}] \\
& + rB^{-1}B_{,r}[\tfrac{1}{2}\mu + \mu rB^{-1}B_{,r} + \tfrac{1}{2}(1-\mu^2)B^{-1}B_{,\mu}] \\
& + \tfrac{3}{2}B^{-1}B_{,\mu}[-\mu^2 + \mu(1-\mu^2)B^{-1}B_{,\mu}] - (1-\mu^2)rB^{-1}B_{,\mu r}(1+rB^{-1}B_{,r}) \\
& - \mu r^2\nu_{,r}^2 - 2(1-\mu^2)r\nu_{,\mu}\nu_{,r} + \mu(1-\mu^2)\nu_{,\mu}^2 - 2(1-\mu^2)r^2B^{-1}B_{,r} \\
& \times \nu_{,\mu}\nu_{,r} + (1-\mu^2)B^{-1}B_{,\mu}[r^2\nu_{,r}^2 - (1-\mu^2)\nu_{,\mu}^2] + (1-\mu^2)B^2e^{-4\nu} \\
& \times [\tfrac{1}{4}\mu r^4\omega_{,r}^2 + \tfrac{1}{2}(1-\mu^2)r^3\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}\mu(1-\mu^2)r^2\omega_{,\mu}^2 + \tfrac{1}{2}(1-\mu^2) \\
& \times r^4B^{-1}B_{,r}\omega_{,\mu}\omega_{,r} - \tfrac{1}{4}(1-\mu^2)r^2B^{-1}B_{,\mu}[r^2\omega_{,r}^2 - (1-\mu^2)\omega_{,\mu}^2]\}]\!],
\end{aligned}$$

Comparison of Different Codes

N.S., Living Reviews in Relativity (2003)

	AKM	Lorene/ rotstar	SF (260 × 400)	SF (70 × 200)	BGSM	KEH
$\bar{p}_c$	1.0					
r_p/r_e	0.7				1×10^{-3}	
$\bar{\Omega}$	1.41170848318	9×10^{-6}	3×10^{-4}	3×10^{-3}	1×10^{-2}	1×10^{-2}
$\bar{M}$	0.135798178809	2×10^{-4}	2×10^{-5}	2×10^{-3}	9×10^{-3}	2×10^{-2}
$\bar{M}_0$	0.186338658186	2×10^{-4}	2×10^{-4}	3×10^{-3}	1×10^{-2}	2×10^{-3}
$\bar{R}_{circ}$	0.345476187602	5×10^{-5}	3×10^{-5}	5×10^{-4}	3×10^{-3}	1×10^{-3}
$\bar{J}$	0.0140585992949	2×10^{-5}	4×10^{-4}	5×10^{-4}	2×10^{-2}	2×10^{-2}
Z_p	1.70735395213	1×10^{-5}	4×10^{-5}	1×10^{-4}	2×10^{-2}	6×10^{-2}
Z_{eq}^{f}	−0.162534082217	2×10^{-4}	2×10^{-3}	2×10^{-2}	4×10^{-2}	2×10^{-2}
Z_{eq}^{b}	11.3539142587	7×10^{-6}	7×10^{-5}	1×10^{-3}	8×10^{-2}	2×10^{-1}
$\lvert$GRV3$\rvert$	4×10^{-13}	3×10^{-6}	3×10^{-5}	1×10^{-3}	4×10^{-3}	1×10^{-1}

AKM: Ansorg et al.

Lorene/rotstar + BGSM: Meudon group

SF: RNS code

KEH: original KEH code (not compactified)

CONCLUSIONS

• GR-Hydro code accurate and stable

• Extend to GR-MHD using MHD equations in conservative form and enforcing divergence-free constraint

• Initial data of rotating magnetized stars under construction

Applications

• Nonlinear oscillations of magnetized relativistic stars

• Collapse of magnetized rotating relativistic stars

• Magnetic breaking of differential rotation by MHD turbulence

Future extensions

• Include viscosity, radiation transport, finite conductivity etc.