On the theoretical basis of kappa distributions

George Livadiotis and David J. McComas

Southwest Research Institute
San Antonio, Texas USA

glivadiotis@swri.edu  david.mccomas@swri.org
The distribution of the published papers in Space Physics and Astrophysics since 1980 that are related to kappa distributions (Title / Abstract).
We show how kappa distributions arise naturally from Tsallis Statistical Mechanics.

We expose the general relation between kappa and the spectral indices commonly used to parameterize space plasmas.

We develop the concept of physical temperature for stationary states out of equilibrium.
Space plasmas are non-equilibrium systems, tending slowly to stationary states.

A system whose distribution function has stabilized to a Boltzmann-Maxwell distribution would be in thermal equilibrium.

However, which would be the expression of probability distribution for systems relaxing into stationary states out of equilibrium?

Entropy: From the Greek word “Εντροπία”
- Ev-: in, towards + -Τροπή: a turning, change

Towards a turning ⇒ Entropy increases
When the probability distribution is stabilized ⇔ Entropy is maximized
Equilibrium ...

Boltzmann-Gibbs
Statistical Mechanics
Boltzmann-Gibbs Statistical Mechanics

- Discrete probability distribution \( \{p_k\}_{k=1}^{W} : p_1, p_2, \ldots, p_W \), associated with a conservative physical system of energy spectrum, \( \{\varepsilon_k\}_{k=1}^{W} : \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_W \).

- Entropy: 
  \[
  S_{BG}^{BG}\left(\{p_k\}_{k=1}^{W}\right) = -\sum_{k=1}^{W} p_k \ln(p_k)
  \]

- Entropy maximization:
  \[
  \frac{\partial}{\partial p_j} S_{BG}^{BG}\left(\{p_k\}_{k=1}^{W}\right) = 0 \quad \forall \ j = 1, \ldots, W
  \]

- \( \{p_k\}_{k=1}^{W} \): Not independent variables
  - (i) Normalization, \( \sum_{k=1}^{W} p_k = 1 \)
  - (ii) Known internal energy \( U \), \( \sum_{k=1}^{W} p_k \varepsilon_k = U \)
The Lagrange method involves maximizing the functional

\[ G\left(\{p_k\}_{k=1}^W\right) = S\left(\{p_k\}_{k=1}^W\right) + \lambda_1 \sum_{k=1}^W p_k + \lambda_2 \sum_{k=1}^W p_k \varepsilon_k \]

Lagrangian temperature \[-\lambda_2 \equiv \beta \equiv \frac{1}{k_B T}\] \[\Rightarrow\] \[p_k \sim e^{\lambda_2 \varepsilon_k} \equiv e^{\varepsilon_k / k_B T}\]

Continuous energy spectrum \[p(\varepsilon; T) \sim e^{-\varepsilon / k_B T}\]

Maxwellian…

\[\varepsilon = \frac{1}{2} \mu \cdot u^2\]

\[p(u; \theta) \sim e^{-(u/\theta)^2}\]

\[\theta \equiv \sqrt{\frac{2k_B T}{\mu}}\]
Out of Equilibrium …

Empirically …
Kappa distribution: An empirical approach

- The low-energy (L-E) region of ion distributions is primarily Maxwellian.

\[ p_{\text{L-E}}(\vec{u}) \sim e^{-\left(\frac{|\vec{u} - \vec{u}_b|}{\theta}\right)^2} \]

- The high-energy (H-E) (or suprathermal) region is non-Maxwellian: power-law tails.

\[ p_{\text{H-E}}(\vec{u}) \sim \left|\vec{u} - \vec{u}_b\right|^{-2(\gamma+1)} \]

\( \vec{u} \) and \( \vec{u}_b \): ion and bulk flow velocities.

- Vasyliūnas (1968): An empirical functional form for describing the distribution over the whole energy spectrum, both the L-E Maxwellian core and the H-E power-law tail.

\[ p(\vec{u}; \theta_{\kappa}; \kappa) \sim \left[ 1 + \frac{1}{\kappa} \cdot \left(\frac{\left|\vec{u} - \vec{u}_b\right|}{\theta_{\kappa}}\right)^2 \right]^{-\kappa-1} \]
Kappa distribution: An empirical approach

Why up to \(- (\kappa + 1)\)?

\[
p(\bar{u}; \theta_\kappa; \kappa) \sim \left[ 1 + \frac{1}{\kappa} \left( \frac{|\bar{u} - \bar{u}_b|}{\theta_\kappa} \right)^2 \right]^{-\kappa-1}
\]

Because of the coincidence of the spectral index \(\gamma\) with \(\kappa\).

\[
j_{H-E}(\varepsilon) \sim \varepsilon^{\frac{1}{2}} \cdot p_{H-E}(\varepsilon) \cdot g_E(\varepsilon) \sim \varepsilon^{-\kappa} \equiv \varepsilon^{-\gamma} \quad \Rightarrow \quad \kappa = \gamma
\]
in 3-dim systems

What if the power was \(\kappa^*\)?

\[
p(\bar{u}; \theta_\kappa^*; \kappa^*) \sim \left[ 1 + \frac{1}{\kappa^*} \left( \frac{|\bar{u} - \bar{u}_b|}{\theta_\kappa^*} \right)^2 \right]^{-\kappa^*}
\]

Then we have the same coincidence, \(\kappa^* = \gamma\).

\[
j_{H-E}(\varepsilon) \sim \varepsilon^{\frac{1}{2}} \cdot p_{H-E}(\varepsilon) \cdot g_E(\varepsilon) \sim \varepsilon^{-\kappa^*} \equiv \varepsilon^{-\gamma} \quad \Rightarrow \quad \kappa^* = \gamma
\]
in 1-dim systems
Relation between the 2 kinds

1\textsuperscript{st} kind

\[ p^{(1)}(\vec{u}; \theta_{\text{eff}} ; \kappa^*) \sim \left[ 1 + \frac{1}{\kappa^* - \frac{5}{2}} \left( \frac{|\vec{u} - \vec{u}_b|}{\theta_{\text{eff}}} \right)^2 \right]^{-\kappa^*} \]

2\textsuperscript{nd} kind

\[ p^{(2)}(\vec{u}; \theta_{\text{eff}} ; \kappa) \sim \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( \frac{|\vec{u} - \vec{u}_b|}{\theta_{\text{eff}}} \right)^2 \right]^{-\kappa - 1} \]

where

\[ \theta_{\text{eff}} \equiv \sqrt{\frac{2k_B T_{\text{KE}}}{\mu}} \]

Hence:

\[ \kappa^* = \kappa + 1 \]

Thermal parameters independent of \( \kappa, \kappa^* \)
Out of Equilibrium …

Tsallis Statistical Mechanics
Generalized Statistical Mechanics

- **Tsallis Entropy**

\[
S_q\left(\{p_k\}_{k=1}^W; q\right) = \frac{1 - \sum_{k=1}^W p_k^q}{q - 1}
\]

\[
S_q(q \to 1) = -\sum_{k=1}^W p_k \ln(p_k) \equiv S^{BG}
\]

- **Escort expectation value**

\[
U_q = \frac{\sum_{k=1}^W p_k^q \varepsilon_k}{\sum_{k=1}^W p_k^q}
\]

\[
U_q (q \to 1) = \sum_{k=1}^W p_k \varepsilon_k = U
\]

\[
P_k = \frac{p_k^q}{\sum_{k=1}^W p_k^q} \quad \Leftrightarrow \quad P_k = \frac{P_k^{1/q}}{\sum_{k=1}^W P_k^{1/q}}
\]

The escort probabilities characterize the system after its relaxation in stationary states out of equilibrium.
Tsallis Statistical Mechanics

- Entropy maximization:
  \[ \frac{\partial}{\partial p_j} S_q \left( \{ p_k \}_{k=1}^W ; q \right) = 0 \quad \forall \ j = 1, \ldots, W \]

- Constraints:
  (i) Normalization,
  \[ \sum_{k=1}^W p_k = 1 \]
  (ii) Known internal energy, \( U_q \),
  \[ \sum_{k=1}^W p_k \varepsilon_k = U_q \]

The Lagrange method involves maximizing the functional

\[ G_q \left( \{ p_k \}_{k=1}^W ; q \right) = S_q \left( \{ p_k \}_{k=1}^W ; q \right) + \lambda_1 \sum_{k=1}^W p_k + \lambda_2 \sum_{k=1}^W P_k \varepsilon_k \]

leading to

\[ P(\varepsilon; T_q ; q) \sim p(\varepsilon; T_q ; q)^q \sim \left[ 1 - (1 - q) \cdot \frac{\varepsilon - U_q}{k_B T_q} \right]^{\frac{q}{1-q}} \]

\[ T_q \equiv T \cdot \sum_{k=1}^W p_k^q \]

\( T_q \): Physical temperature

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Livadiotis and McComas
\[ \varepsilon = \frac{1}{2} \mu \cdot (\bar{u} - \bar{u}_b)^2 \]

\[ U_q = \left\langle \frac{1}{2} \mu \cdot u^2 \right\rangle_q = \frac{3}{2} k_B T_q \]

Tsallis-Maxwellian distribution

\[ P(\varepsilon; T_q ; q) \sim \left[ 1 - \frac{2(1-q)}{5-3q} \left( \frac{\bar{u} - \bar{u}_b}{\theta_{\text{eff}}} \right)^2 \right]^{\frac{q}{1-q}} \]

\[ \kappa \equiv \frac{1}{q-1} \]

kappa distribution

\[ p^{(2)}(\vec{u}; \theta_{\text{eff}} ; \kappa) \sim \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \left( \frac{\bar{u} - \bar{u}_b}{\theta_{\text{eff}}} \right)^2 \right]^{-\kappa-1} \]
\[ \kappa = \kappa^* - 1 = \gamma = \gamma_E - \frac{1}{2} = \frac{1}{2} \gamma_V \]

\( \gamma_V, \gamma_E, \gamma \): exponents in power laws of velocities, energy, flux

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<th>( \gamma_E )</th>
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The physical temperature $T_q$

3 definitions of temperature that coincide in equilibrium

$$T_S \equiv \left( \frac{\partial S}{\partial U} \right)^{-1}$$

$$T \equiv -\frac{1}{k_B \lambda_2}$$

$$T_{KE} \equiv \frac{2U}{3k_B}$$

In Tsallis Statistics, $T_{KE}$ differs out of equilibrium

$$T = T_S \equiv \left( \frac{\partial S_q}{\partial U_q} \right)^{-1}$$

$$T \neq T_{KE} \equiv \frac{2U_q}{3k_B}$$

… but coincides with the physical temperature $T_q$

$$T_q = T_S \equiv \left( \frac{\partial S_q}{\partial U_q} \right)^{-1} \left[ 1 + (1 - q) \cdot S_q / k_B \right]$$

$$T_q = T_{KE} \equiv \frac{2U_q}{3k_B}$$

Now the Lagrangian $T$ is expressed in terms of $T_q$, $T = T(T_q; q)$
Two hypothetical routes of transient (metastable) stationary states towards the equilibrium

The relation of physical temperature $T_q$ with the Lagrangian temperature $T$
Conclusions

- We showed how kappa distributions arise naturally from Tsallis Statistical Mechanics
- We developed the concept of physical temperature out of equilibrium, which differs significantly from the classical, equilibrium temperature
- We extracted the general relation between the basic types of kappa distributions and the spectral indices commonly used to parameterize space: $\kappa = \kappa^* - 1 = \gamma = \gamma_{E} - \frac{1}{2} = \frac{1}{2} \gamma_{V}$
– Tsallis Statistical Mechanics offer a consistent theoretical framework for describing complex systems in stationary states out of equilibrium.

– The Tsallis-like Canonical probability distribution is derived by following along the Gibbs path, by extremizing the Tsallis entropy under constraints.

– This Canonical probability distribution reads the kappa distribution that describes the solar plasmas.

– Both the two kinds of kappa distributions can describe solar plasmas. However the 2\textsuperscript{nd} kind is primary. It is connected with the escort probability and the physical temperature.