On the theoretical basis of kappa distributions

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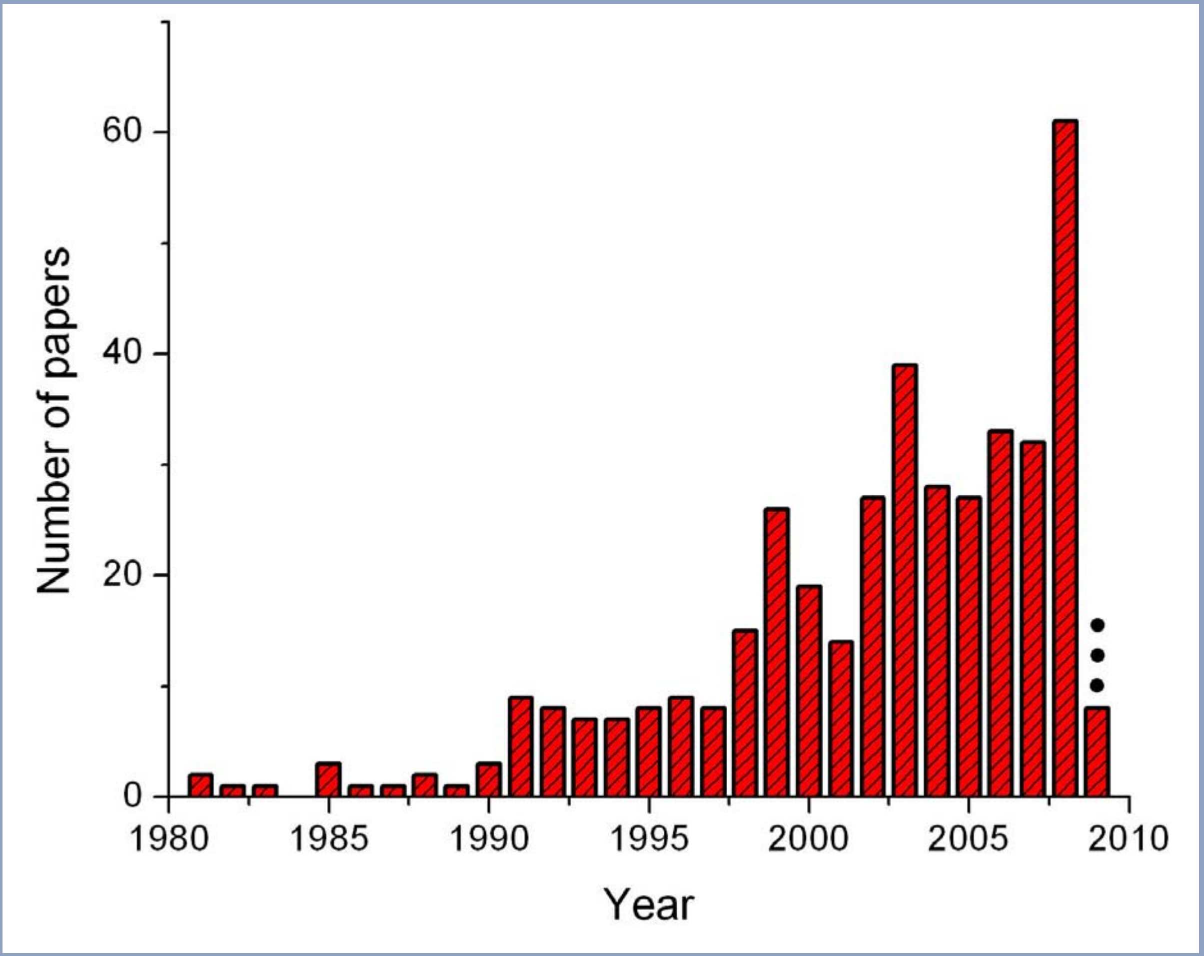


The distribution of the published papers in Space Physics and Astrophysics since 1980 that are related to kappa distributions (Title / Abstract).

- **We show how kappa distributions arise naturally from Tsallis Statistical Mechanics**

- **We expose the general relation between kappa and the spectral indices commonly used to parameterize space plasmas**

- **We develop the concept of physical temperature for stationary states out of equilibrium**

- Space plasmas are non-equilibrium systems, tending slowly to stationary states.
- A system whose distribution function has stabilized to a Boltzmann-Maxwell distribution would be in

  thermal equilibrium.
- However, which would be the expression of probability distribution for systems relaxing into

  stationary states out of equilibrium ?
- **Entropy**: From the Greek word "**Εντροπία**"
- **Εν-** : in , towards + **-Τροπή**: a turning, change
- Towards a turning $\Rightarrow$ Entropy increases
- When the probability distribution is stabilized $\Leftrightarrow$ Entropy is maximized

Equilibrium …

Boltzmann-Gibbs Statistical Mechanics

Boltzmann-Gibbs Statistical Mechanics

- Discrete probability distribution $\{p_k\}_{k=1}^{W} : p_1, p_2, \ldots, p_W$, associated with a conservative physical system of energy spectrum, $\{\varepsilon_k\}_{k=1}^{W} : \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_W$.
- Entropy: $S^{BG}\left(\{p_k\}_{k=1}^{W}\right) = -\sum_{k=1}^{W} p_k \ln(p_k)$
- Entropy maximization: $\frac{\partial}{\partial p_j} S^{BG}\left(\{p_k\}_{k=1}^{W}\right) = 0$ $\quad \forall\ j = 1, \ldots, W$
- $\{p_k\}_{k=1}^{W}$: Not independent variables
  - (i) Normalization, $\sum_{k=1}^{W} p_k = 1$
  - (ii) Known internal energy U, $\sum_{k=1}^{W} p_k\ \varepsilon_k = U$

The Lagrange method involves maximizing the functional

$$G\left(\{p_k\}_{k=1}^{W}\right) = S\left(\{p_k\}_{k=1}^{W}\right) + \lambda_1 \sum_{k=1}^{W} p_k + \lambda_2 \sum_{k=1}^{W} p_k \ \varepsilon_k$$

Lagrangian temperature

$$-\lambda_2 \equiv \beta \equiv \frac{1}{k_B T} \quad \Rightarrow \quad p_k \sim e^{\lambda_2 \varepsilon_k} \equiv e^{-\frac{\varepsilon_k}{k_B T}}$$

Continuous energy spectrum

$$p(\varepsilon;T) \sim e^{-\frac{\varepsilon}{k_B T}}$$

Maxwellian…

$$\varepsilon = \frac{1}{2}\mu \cdot u^2 \qquad p(u;\theta) \sim e^{-(u/\theta)^2} \ , \quad \theta \equiv \sqrt{\frac{2k_B T}{\mu}}$$

Out of Equilibrium …

Empirically …

Kappa distribution: An empirical approach

– The low-energy (L-E) region of ion distributions is primarily Maxwellian.

$$p_{\mathrm{L-E}}(\vec{u}) \sim e^{-\left(\left|\vec{u}-\vec{u}_b\right|/\theta\right)^2}$$

– The high-energy (H-E) (or suprathermal) region is non-Maxwellian: power-law tails.

$$p_{\mathrm{H-E}}(\vec{u}) \sim \left|\vec{u}-\vec{u}_b\right|^{-2(\gamma+1)}$$

$\vec{u}$ and $\vec{u}_b$: ion and bulk flow velocities.

– Vasyliũnas (1968): An empirical functional form for describing the distribution over the whole energy spectrum, both the L-E Maxwellian core and the H-E power-law tail.

$$p(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{\left|\vec{u}-\vec{u}_b\right|}{\theta_\kappa}\right)^2\right]^{-\kappa-1}$$

Kappa distribution: An empirical approach

– Why up to – (κ + 1) ?

$$p(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]^{-\kappa-1}$$

Because of the coincidence of the spectral index γ with κ.

$$j_{\mathrm{H-E}}(\varepsilon) \sim \varepsilon^{\frac{1}{2}} \cdot p_{\mathrm{H-E}}(\varepsilon)\cdot g_{\mathrm{E}}(\varepsilon) \sim \varepsilon^{-\kappa} \equiv \varepsilon^{-\gamma} \quad \Rightarrow \quad \kappa = \gamma$$

in 3-dim systems

– What if the power was – κ?

$$p(\vec{u};\theta_\kappa^*;\kappa^*) \sim \left[1+\frac{1}{\kappa^*}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa^*}\right)^2\right]^{-\kappa^*}$$

Then we have the same coincidence, $\kappa^* = \gamma$.

$$j_{\mathrm{H-E}}(\varepsilon) \sim \varepsilon^{\frac{1}{2}} \cdot p_{\mathrm{H-E}}(\varepsilon)\cdot g_{\mathrm{E}}(\varepsilon) \sim \varepsilon^{-\kappa^*} \equiv \varepsilon^{-\gamma} \quad \Rightarrow \quad \kappa^* = \gamma$$

in 1-dim systems

Relation between the 2 kinds

- **1st kind** $$p^{(1)}\left(\vec{u};\theta_{\text{eff}};\kappa^{*}\right)\sim\left[1+\frac{1}{\kappa^{*}-\frac{5}{2}}\cdot\left(\frac{\left|\vec{u}-\vec{u}_{b}\right|}{\theta_{\text{eff}}}\right)^{2}\right]^{-\kappa^{*}}$$

$$=$$

- **2nd kind** $$p^{(2)}\left(\vec{u};\theta_{\text{eff}};\kappa\right)\sim\left[1+\frac{1}{\kappa-\frac{3}{2}}\cdot\left(\frac{\left|\vec{u}-\vec{u}_{b}\right|}{\theta_{\text{eff}}}\right)^{2}\right]^{-\kappa-1}$$

where $\theta_{\text{eff}}\equiv\sqrt{\frac{2k_{B}T_{\text{KE}}}{\mu}}$, T_{KE} Thermal parameters independent of κ, κ^{*}

Hence: $\kappa^{*}=\kappa+1$

Out of Equilibrium …

Tsallis Statistical Mechanics

Generalized Statistical Mechanics

- **Tsallis Entropy**

$$S_q\left(\{p_k\}_{k=1}^{W};q\right)=\frac{1-\sum_{k=1}^{W} p_k{}^q}{q-1}$$

$$S_q(q\to 1)=-\sum_{k=1}^{W} p_k \ln(p_k)\equiv S^{BG}$$

- **Escort expectation value**

$$U_q=\frac{\sum_{k=1}^{W} p_k{}^q\varepsilon_k}{\sum_{k=1}^{W} p_k{}^q}$$

or

$$U_q=\sum_{k=1}^{W} P_k\varepsilon_k$$

$$U_q(q\to 1)=\sum_{k=1}^{W} p_k\varepsilon_k=U$$

where

$$P_k=\frac{p_k{}^q}{\sum_{k=1}^{W} p_k{}^q}\Leftrightarrow p_k=\frac{P_k{}^{1/q}}{\sum_{k=1}^{W} P_k{}^{1/q}}$$

The escort probabilities characterize the system after its relaxation in stationary states out of equilibrium.

Tsallis Statistical Mechanics

- Entropy maximization: $\frac{\partial}{\partial p_j} S_q\left(\{p_k\}_{k=1}^{W}; q\right) = 0$ $\forall\ j = 1, \ldots, W$
- Constraints:

  (i) Normalization, $\sum_{k=1}^{W} p_k = 1$

  (ii) Known internal energy, U_q, $\sum_{k=1}^{W} P_k\ \varepsilon_k = U_q$

The Lagrange method involves maximizing the functional

$$G_q\left(\{p_k\}_{k=1}^{W}; q\right) = S_q\left(\{p_k\}_{k=1}^{W}; q\right) + \lambda_1 \sum_{k=1}^{W} p_k + \lambda_2 \sum_{k=1}^{W} P_k\ \varepsilon_k$$

leading to

$$P(\varepsilon; T_q; q) \sim p(\varepsilon; T_q; q)^q \sim \left[1 - (1-q) \cdot \frac{\varepsilon - U_q}{k_B T_q}\right]^{\frac{q}{1-q}}$$

$$T_q \equiv T \cdot \sum_{k=1}^{W} p_k{}^q$$

T_q: Physical temperature

$$\varepsilon = \frac{1}{2}\mu \cdot (\vec{u} - \vec{u}_b)^2$$

$$U_q = \left\langle \frac{1}{2}\mu \cdot u^2 \right\rangle_q = \frac{3}{2} \cdot k_B T_q \quad \Longrightarrow$$

Tsallis-Maxwellian distribution

$$P(\varepsilon; T_q; q) \sim \left[1 - \frac{2(1-q)}{5-3q} \cdot \left(\frac{|\vec{u} - \vec{u}_b|}{\theta_{\mathrm{eff}}} \right)^2 \right]^{\frac{q}{1-q}}$$

$$\kappa \equiv \frac{1}{q-1} \quad \Longrightarrow$$

kappa distribution

$$p^{(2)}(\vec{u}; \theta_{\mathrm{eff}}; \kappa) \sim \left[1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left(\frac{|\vec{u} - \vec{u}_b|}{\theta_{\mathrm{eff}}} \right)^2 \right]^{-\kappa-1}$$

$$\kappa = \kappa^* - 1 = \gamma = \gamma_E - \frac{1}{2} = \frac{1}{2}\gamma_V$$

γ_V, γ_E, γ: exponents in power laws of velocities, energy, flux

	Publication	γ_E	γ_v	γ	κ	κ^*	Comments
1	*Decker et al.* [2005]	2.13	3.26	1.63	**1.63**	2.63	2nd kappa distribution
2	*Fisk & Gloecker* [2006]	2	**3**	1.5	1.5	2.5	Suprathermal power-law tail
3	*Dialynas et al.* [2009]	>3	>5	>2.5	>2.5	**>3.5**	1st kappa distribution
4	*Dayeh et al.* [2009]	<3	<5	**<2.5**	<2.5	<3.5	Suprathermal power-law tail

The physical temperature T_q

3 definitions of temperature that coincide in equilibrium

$$T_S \equiv \left(\frac{\partial S}{\partial U} \right)^{-1} \qquad T \equiv -\frac{1}{k_B \lambda_2} \qquad T_{\mathrm{KE}} \equiv \frac{2U}{3k_B}$$

In Tsallis Statistics, T_{KE} differs out of equilibrium

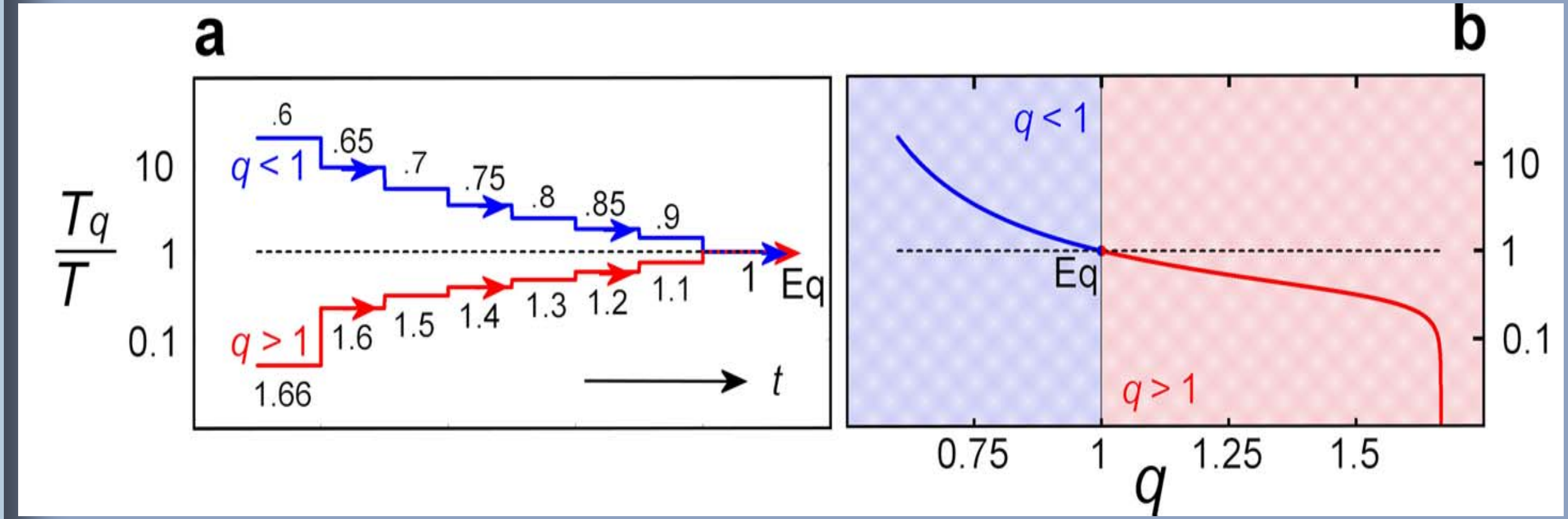
$$T = T_S \equiv \left(\frac{\partial S_q}{\partial U_q} \right)^{-1} \qquad T \neq T_{\mathrm{KE}} \equiv \frac{2U_q}{3k_B}$$

… but coincides with the physical temperature T_q

$$T_q = T_S \equiv \left(\frac{\partial S_q}{\partial U_q} \right)^{-1} \left[1 + (1-q) \cdot S_q / k_B \right] \qquad T_q = T_{\mathrm{KE}} \equiv \frac{2U_q}{3k_B}$$

Now the Lagrangian T is expressed in terms of T_q, $T=T(T_q\,;q)$

Two hypothetical routes of transient (metastable) stationary states towards the equilibrium



The relation of physical temperature T_q with the Lagrangian temperature T

Conclusions

- **We showed how kappa distributions arise naturally from Tsallis Statistical Mechanics**
- **We developed the concept of physical temperature out of equilibrium, which differs significantly from the classical, equilibrium temperature**
- **We extracted the general relation between the basic types of kappa distributions and the spectral indices commonly used to parameterize space** $\kappa = \kappa^{*} - 1 = \gamma = \gamma_{\mathrm{E}} - \frac{1}{2} = \frac{1}{2}\gamma_{\mathrm{V}}$

- Tsallis Statistical Mechanics offer a consistent theoretical framework for describing complex systems in stationary states out of equilibrium.
- The Tsallis-like Canonical probability distribution is derived by following along the Gibbs path, by extremizing the Tsallis entropy under constraints.
- This Canonical probability distribution reads the kappa distribution that describes the solar plasmas.

- Both the two kinds of kappa distributions can describe solar plasmas. However the 2nd kind is primary. It is connected with the escort probability and the physical temperature.