Self-Organization in thermally unstable plasmas

R.Chin*, E. Verwichte, G. Rowlands, V.M. Nakariakov, and C. Brady

CFSA, University of Warwick, Coventry, UK

* R.J.Chin@warwick.ac.uk

Modern Nonlinear Plasma Physics conference
Greece, June 15th-19th 2009
**Motivation**

- Regions of thermal activity
- Nonlinearity
- Dissipative mechanisms

How do waves propagate through an active medium?

**EIT/SOHO**

Covers plasmas on all scales from the astrophysical down to the laboratory.

- Solar coronal phenomena
- ELMs (Edge Localised Modes)

**UKAEA/Culham MAST**
Non-adiabaticity

Contains all the non-adiabatic terms of

- Thermal conduction
- Thermal instability. Field (1965)

\[ \frac{\partial L}{\partial T} \bigg|_{\rho_0, p_0} < 0 \]

The rate at which a plasma system radiates heat, is described by a general heat/loss function \( L \).

\[ L = L_r(\rho, p, T) - H(\rho, p) \]

Complicated \( L \) i.e. higher order derivatives

Regions of thermal instability

A typical profile of cooling functions (dashed line) Sutherland & Dopita (2003) and (solid) Mewe (1985)
The simplest mechanism
  • Activity – Thermal overstability
  • Nonlinearity
  • Dissipation – high-frequency damping

The general evolutionary eqn for waves under such mechanisms is:

\[
\frac{\partial u}{\partial t} = F(\nabla u, u) + \nabla \cdot (D \cdot \nabla u) + A(r, t, u) = 0
\]

Balance of all 3 phases leads to stationary waves in particular Autowaves!

Autowaves propagate with characteristics (such as speed and amplitude etc) independently of the initial conditions, and are instead completely determined by the plasma properties.

The latter property could be exploited to be a non-invasive probe for plasmas
Evidence of autowaves in solar flares

Numerical simulation of solar flare oscillations

Predicted thermal dampening

We use the full time-dependent MHD equations.

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]

\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}),
\]

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v},
\]

\[
\frac{dp}{dt} - \frac{\gamma p}{\rho} \frac{d\rho}{dt} = (\gamma - 1) \left[ \mathcal{L}(\rho, p) + \nabla \cdot (\kappa \nabla T) \right],
\]

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.
\]

Non-adiabatic terms

MHD equations + \nabla \mathbf{B} = 0 + p = R\rho T \text{ (equation of state)}
Assumptions of model

• We look at perturbations about the plasma equilibrium of the form \( f = f_0 + f_1(r,t) \). \( f_0 \) denotes equilibrium.
  - No background flows i.e. \( \mathbf{v_0} = 0 \).
  - Quadratic nonlinearity i.e. Ignore terms > \( O(f_1^2) \).

We extend the model as studied by Nakariakov et al. (2000) to include arbitrary heating and radiative cooling.
Combining the MHD equations & assumptions to reduce the set of equations to a single variable $V_z$.

\[
\left[ \frac{\partial^4}{\partial t^4} - \left( C_A^2 + C_s^2 \right) \frac{\partial^4}{\partial t^2 \partial z^2} + C_{Az}^2 C_s^2 \frac{\partial^4}{\partial z^4} \right] V_z = 2C_s^2 \left[ L_1 + K \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2}{\partial t^2} + C_{Az}^2 \frac{\partial^2}{\partial z^2} \right] \int \frac{\partial^2 V_z}{\partial z^2} dt' \right] \nonumber \\
- 2C_s^2 \left[ \frac{\partial^2}{\partial t^2} + C_{Az}^2 \frac{\partial^2}{\partial z^2} \right] \frac{\partial}{\partial z} \left[ \int \frac{\partial^2 V_z}{\partial z^2} dt' \right]^2 + N,
\]

\text{Magnetoacoustic operator}

\textbf{Nonlinear terms}

\begin{align*}
L_1 &= \frac{(\gamma - 1)}{2C_s^2} \left( \frac{\partial}{\partial \rho} + C_s^2 \frac{\partial}{\partial p} \right) \mathcal{L}, \\
L_2 &= \frac{(\gamma - 1) \rho_o}{4C_s^2} \left( \frac{\partial}{\partial \rho} + C_s^2 \frac{\partial}{\partial p} \right)^2 \mathcal{L}, \\
K &= \frac{(\gamma - 1)^2 \kappa_o}{2\gamma R \rho_o}.
\end{align*}

\textbf{Activity}

\textbf{Thermal conduction}

$C_s = \text{local acoustic speed, } C_A = \text{Alfven speed, } C_{Az} = C_A \cos(\theta)$

All derivatives of the cooling function is evaluated at the plasma equilibrium e.g. $L_1$
Transforming to the frame where the evolution is taking place

\[ \zeta = z - Ct \text{ and } t = \tau \text{ “slow-time”}. \] Where \( C \) is the magnetoacoustic speed. We obtain the full nonlinear evolutionary equation:

\[ \frac{\partial V_z}{\partial \tau} - \chi \frac{\partial^2 V_z}{\partial \zeta^2} + \varepsilon V_z \frac{\partial V_z}{\partial \zeta} + \alpha |\mu_1| V_z + \beta |\mu_2| V_z^2 = 0 \]

\( \alpha, \beta = \text{sgn}(\mu_1, \mu_2) \)

\[ \chi = \frac{C_s^2 (C^2 - C_{A_2}^2)}{C^2 (2C^2 - C_s^2 - C_A^2)}, \]

\[ \varepsilon = \frac{1}{2}(\gamma + 1) \frac{C_s^2 (C^2 - C_{A_2}^2)}{C^2 (2C^2 - C_s^2 - C_A^2)} + \frac{3C_s^2 C_{A_2}^2}{2(C^2 - C_{A_2}^2)(2C^2 - C_s^2 - C_A^2)}, \]

\[ \mu_1 = -L_1 \frac{C_s^2 (C^2 - C_{A_2}^2)}{C^2 (2C^2 - C_s^2 - C_A^2)}, \quad \mu_2 = -L_2 \frac{C_s^2 (C^2 - C_{A_2}^2)}{C^2 (2C^2 - C_s^2 - C_A^2)}. \]

The full evolutionary equation describes the evolution of magnetoacoustic waves propagating through an active medium i.e. \( \alpha = -1 \).

- Is the generalised Burgers-Fischer Eqn
- Also contains information about the fast mode
Normalizing the nonlinear evolutionary equation.

\[ \tau^* = \frac{\mu_1}{\mu} \tau, \quad \zeta^* = \sqrt{\frac{\mu_1}{\chi}} \zeta, \quad V_z^* = \frac{\varepsilon}{\sqrt{\chi \mu_1}} V_z. \]

\[ \frac{\partial V^*_z}{\partial \tau^*} - \frac{\partial^2 V^*_z}{\partial \zeta^{*2}} + V^*_z \frac{\partial V^*_z}{\partial \zeta^*} + \alpha V^*_z + \beta k V^*_z^2 = 0 \]

\[ k = \frac{|\mu_2| \sqrt{\chi}}{\sqrt{|\mu_1| \varepsilon}}. \text{ dimensionless } k \text{ parameter} \]

Still need to solve for 3 parameters.. \(\alpha, \beta \text{ and } k\).

Near extrema of the cooling function \(k\) becomes large.

We are looking for stationary solutions so we need to prove the existence of the stationary wave \(\rightarrow\) stationary reference frame \(s = \zeta^* - C_E \tau^*\). With

\[ \psi(s) = Vz^*(s). \]

\[ \frac{d^2 \psi}{ds^2} - (\psi - C_E) \frac{d\psi}{ds} - \alpha \psi - \beta k \psi^2 = 0 \]
Linear stability analysis

Linear growth rates for $k=1.0$

\[ \psi \sim \exp(\delta s) \]

Require $C_E > 0$

\[ \Im\{\delta\} \quad \Re\{\delta\} \]

\[ (\alpha, \beta) = (-1, 1) \]

\[ (\alpha, \beta) = (1, 1) \]

\[ (\alpha, \beta) = (-1, 1) \]

\[ (\alpha, \beta) = (1, -1) \]
In the linear regime of cooling i.e. far from extrema (see fig)

- Neglect higher order derivatives ($\mu_2$)

$$\frac{\partial V_z}{\partial \tau} - \chi \frac{\partial^2 V_z}{\partial \zeta^2} + \varepsilon V_z \frac{\partial V_z}{\partial \zeta} + \alpha |\mu| V_z = 0$$

The stationary equation is given by

$$\frac{d^2 \psi}{ds^2} - \left(\frac{\varepsilon \psi - C_E}{\chi}\right) \frac{d \psi}{ds} + \alpha \frac{|\mu|}{\chi} \psi = 0.$$

$V_z$ must remain finite as $s \to \infty$ this is only possible if the activity and nonlinearity are balanced $\therefore C_E = 0$

Bounded solutions represent stationary waves.
Numerical simulation: Linear Autowave

A sine wave was used as our initial wave profile to undergo evolution for a variety of activity ($\mu_1$). The McCormack finite difference scheme was implemented as our PDE solver. Increasing activity → increasing amplitudes.

Shock-like – but finite thermal conduction prevents multi-value solution.

Initial vs final amplitude
In the Nonlinear regime of heating/cooling i.e. far from extremums.

- Keep higher order derivatives in cooling function ($\mu_2 \neq 0$)

$$\frac{\partial V^*_z}{\partial \tau^*_\zeta} - \frac{\partial^2 V^*_z}{\partial \zeta^*_2} + V^*_z \frac{\partial V^*_z}{\partial \zeta^*_*} + \alpha V^*_z + \beta k V^*_z^2 = 0$$

we obtain a limit cycle solution i.e. the amplitude becomes constant

Autowaves can exist in the finite nonlinear heating cooling regime

$k << 1$, $\alpha = -1 \ \forall \beta$ and $C_E << 2$
If we represent the stationary nonlinear differential equations in the form of a generic equation in terms of functions $g(\psi)$ and $f(\psi)$ we can solve using a perturbation method for

\[
\frac{dP}{ds} - \varepsilon F(\psi)P - G(\psi) = 0, \quad P = \frac{d\psi}{ds}; \quad P^2(\psi) + \varepsilon a(\psi) P(\psi) + b(\psi) = 0.
\]

\[
F(\psi) = (\psi - C_E), \quad G(\psi) = \alpha \psi + \beta k \psi^2.
\]

Differentiating the phase polynomial with respect to $s$ and substituting for the equation of motion we obtain a set of ODES for determine the coefficients of the above polynomial.

\[
(2P + \varepsilon a) \frac{dP}{ds} + \varepsilon a' P^2 + b' P = 0
\]

\[
(-2\varepsilon F(\psi) + \varepsilon a') P^2 + (-\varepsilon a G(\psi) + b' - 2\varepsilon F(\psi))P - a G(\psi) = 0.
\]
Results of perturbation theory

Resulting set of ODES to solve:

\[
\frac{da}{d\psi} = -2(\psi - C_E) + \alpha \frac{a}{b} \psi(1 + \frac{\beta}{\alpha} k \psi), \\
\frac{db}{d\psi} = -2\alpha(1 + \frac{\beta}{\alpha} k \psi) + \varepsilon^2 \left[ \alpha \frac{a}{b} \psi(1 + \frac{\beta}{\alpha} k \psi) - (\psi - C_E) \right].
\]

Solving term by term we obtain:

\[
C_E = \frac{6}{7k} \left[ 1 + \varepsilon^2 \frac{3}{7^2 k^2} + \ldots \right]
\]

Can determine relative velocity with the observed wavelengths, and vice versa.

\[
P^2 + \frac{6}{7k} \psi \left( 1 - \frac{2k}{3} \psi \right) P - \psi \left( 1 - \frac{2k}{3} \psi \right) = 0
\]
Conclusions/Future

• Bounded solutions for the a priori stationary nonlinear ODEs were found to exist analytical and via phase-plane analysis for both linear and non-linear thermally unstable regimes with the parameter range also determined.

• Using the McCormack FD scheme the full time dependent evolutionary equation was solved numerically for both cooling regimes with stationary solutions satisfying the Autowave condition for $a = -1$.

• Developed analytical solutions for the limit cycle scenario of stationary solutions with nonlinear H/C.

• Develop a numerical code to solve the full evolutionary equation to retrieve both autosolitary and linear autowaves.

Long term goal:
• Extend to 2D to study ELMs within our theory framework.