# ANALYTICAL ESTIMATES OF NONLINEAR WAVE-PARTICLE DYNAMICS (IN THE RADIATION BELTS)

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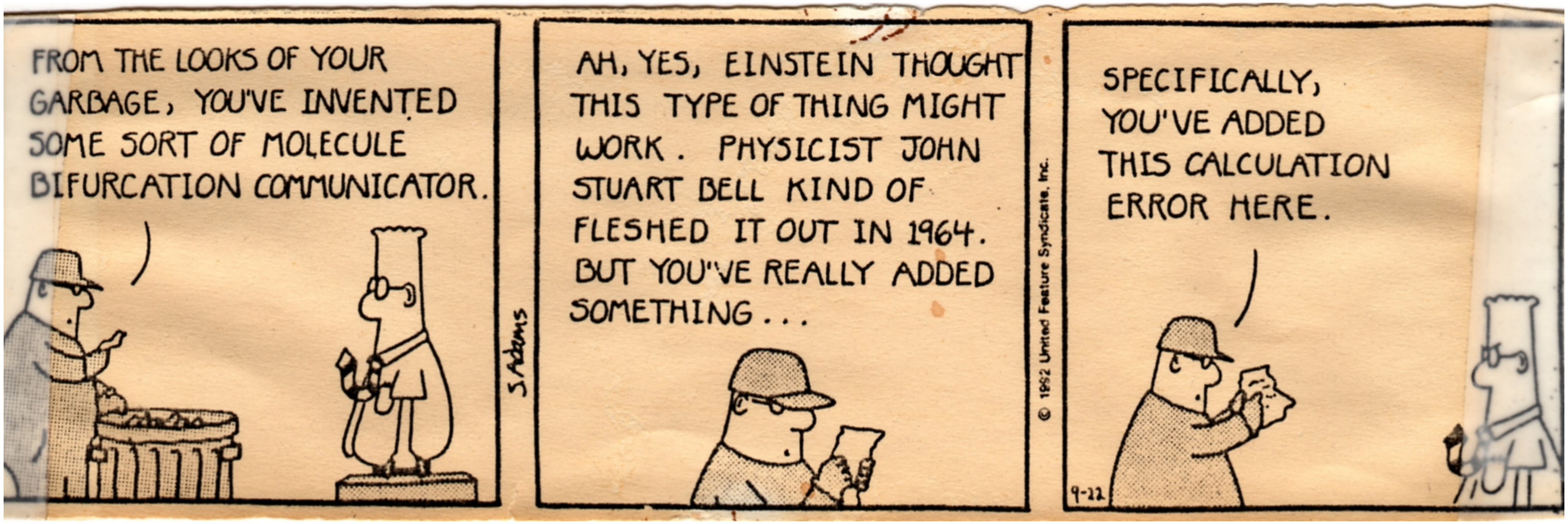
Wave-particle interactions are considered crucial for understanding the radiation belts. Often, quasilinear theory is used.

But recent reports of RBWWs$^{(TM)}$ (Really Big Whistler Waves) [Cattell et al.; Cully et al.] raise fresh doubts about this.

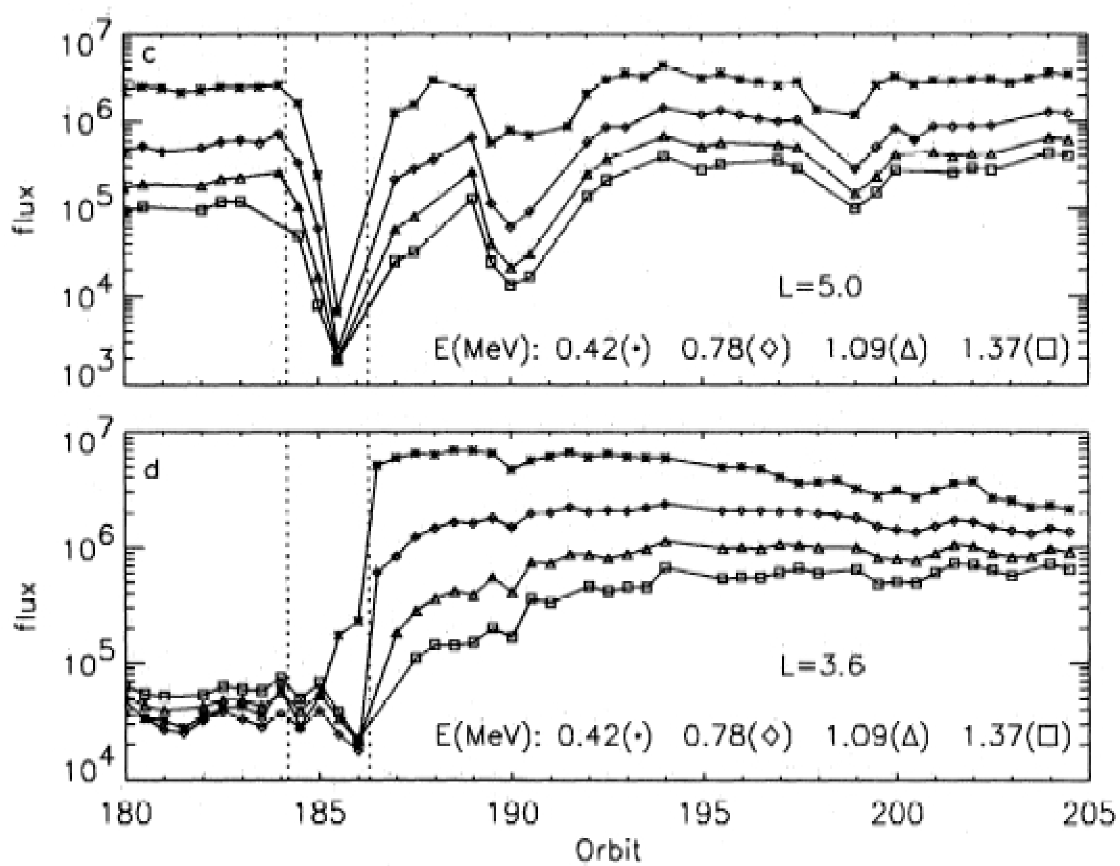
Recent advances in nonlinear simulations are very timely [Nunn, Omura et al., Gibby, ...] but are very demanding.

Existing theoretical ideas – diffusion, phase bunching, and phase trapping – can be described by transport coefficients, practical in global modeling studies.

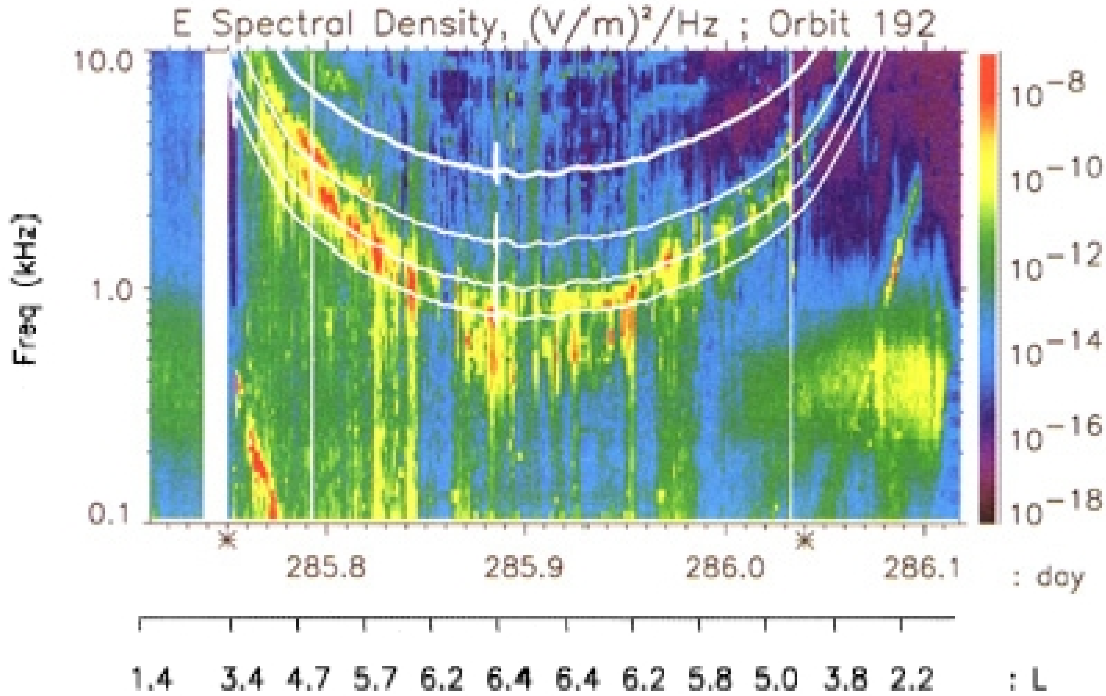



Disclaimer:

- physics content
- graphics quality




Current picture: relativistic electrons are produced in the outer radiation belts during magnetic storms ...




by local interactions with cyclotron-resonant waves combined with radial transport by time-varying fields and drift-resonant waves.

## Motion of a particle resonant with one fixed wave (not self-consistent)

Start with the Hamiltonian of a particle in a **B** field:

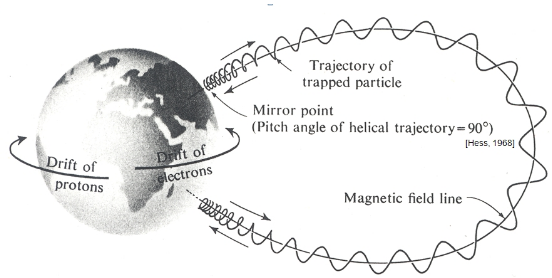
$$H(\mathbf{x}, \mathbf{P}; t) = mc^2 \sqrt{1 + \left( \frac{\mathbf{P} - q\mathbf{A}(\mathbf{x})/c}{mc} \right)^2}$$

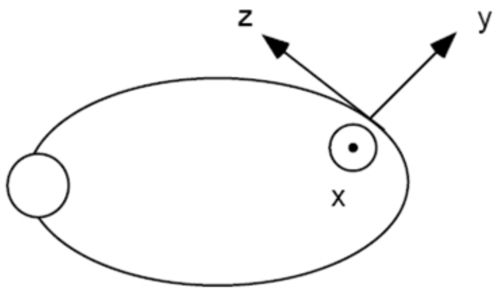
where $\mathbf{P} = \mathbf{p} + q\mathbf{A}/c$ is the canonical momentum and $\mathbf{A} = \mathbf{A}_o + \mathbf{A}_w$.

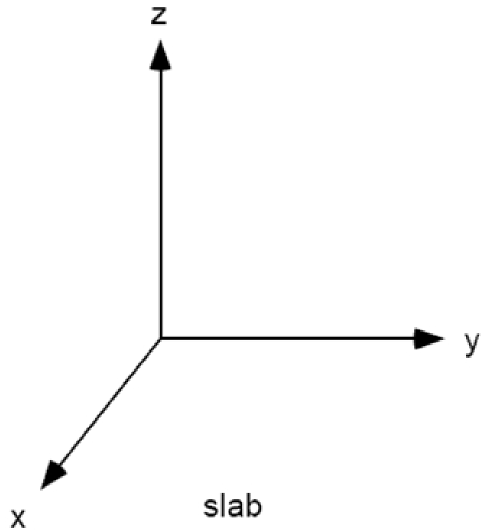
Recall:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{P}}, \qquad \frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}$$

is equivalent to $\mathbf{F} = d\mathbf{p}/dt$.




slab geometry: $z \sim$ distance along field line

$$\mathbf{A}_o = -yB_o g(z)\hat{x} \;\;\Rightarrow\;\; \mathbf{B}_o = -yB_o g'\hat{y} + B_o g\hat{z}$$

$\nabla \cdot \mathbf{B}_o = 0$ exactly for any $g(z)$

For a dipole, near the equator, $g(z) \approx 1 + g_2 z^2$.

Change variables from $(x, P_x, y, P_y, z, P_z)$
to $(X, P_X, \phi, I, \tilde{z}, \tilde{P}_z)$, using the generating function.

$I$ is essentially the first adiabatic invariant $\mu = p_\perp^2/2mB$,
$\phi$ is the gyroangle, and $\tilde{z} = z$.

Rewrite $H$ in the new variables and
- Taylor expand (to $1^{st}$ order) in $qA_w/mc^2$
- use the expansion $\sin(a \sin\theta) = \sum_{n=-\infty}^{\infty} J_n(a) \sin n\theta$
- normalize the variables

After "a little" algebra ...

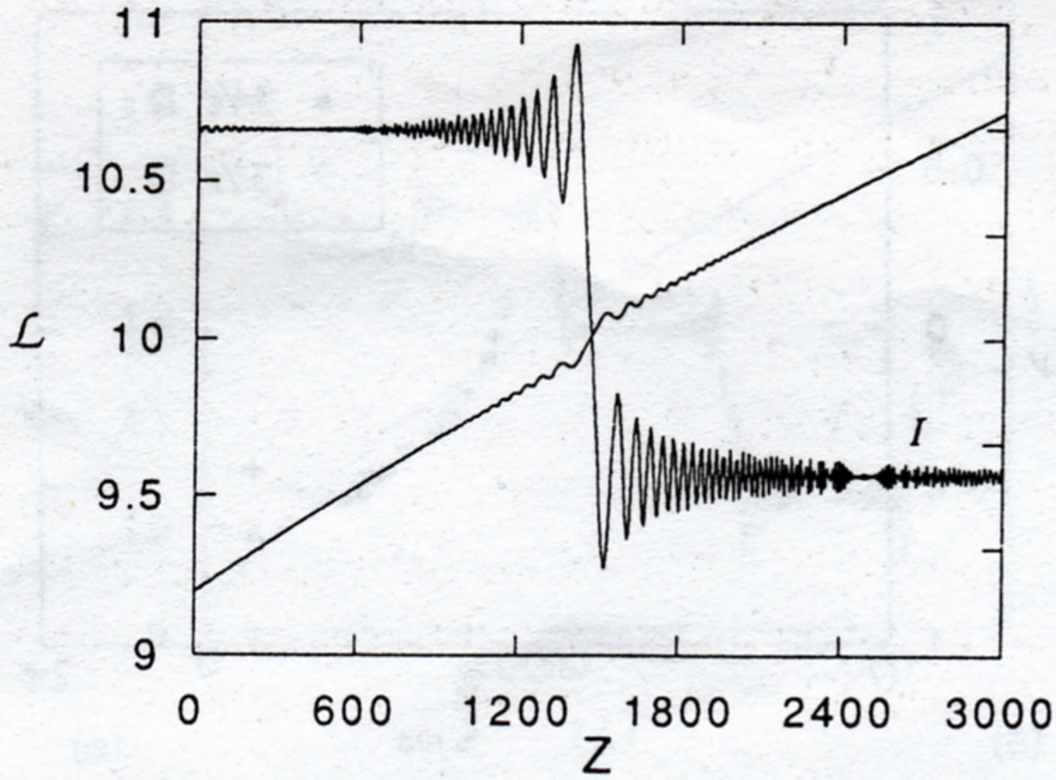
To lowest order,

$$\frac{H}{mc^2} = H_0 + \epsilon \sum_{n=-\infty}^{\infty} H_n \sin \xi_n,$$

with

$$\frac{d\xi_n}{dt} = \omega - k_\| v_\| - sn\frac{\Omega_c}{\gamma}.$$

Near the $\ell^{th}$ resonance, all terms except $n = \ell$ can be dropped by gyroaveraging over $\phi$.



11
10.5
10
9.5
9
$\mathcal{L}$
I
0
600
1200
1800
2400
3000
Z

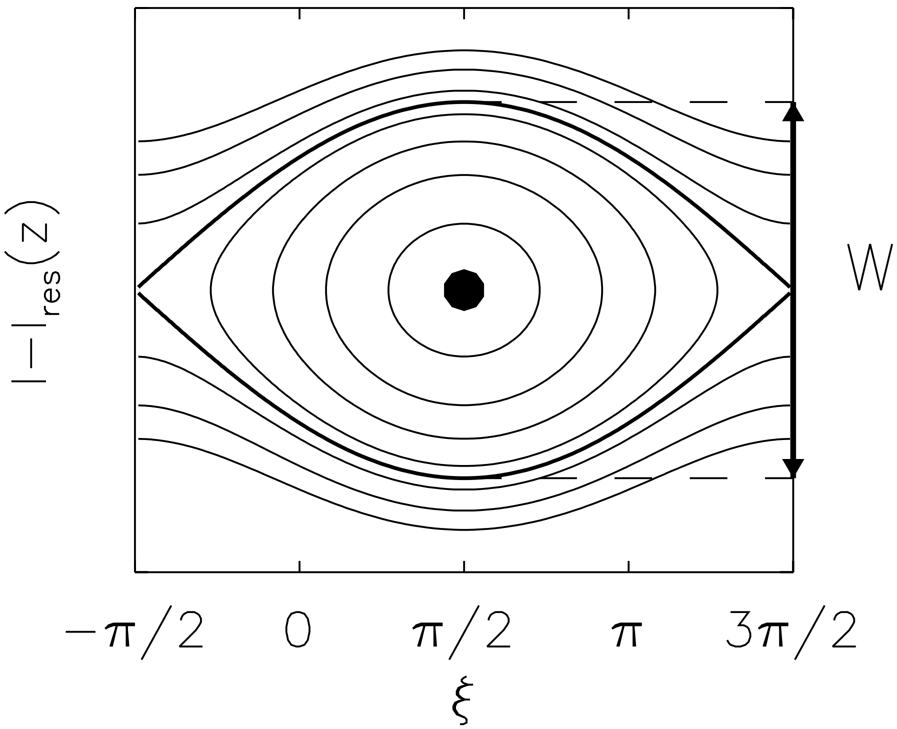
Then $dH/dt = \partial H/\partial t \;\; \Rightarrow \;\; \omega \; dI/dt = s\ell \; d\gamma/dt$.

If $\omega$ is constant, $\omega I = s\ell\gamma(I, P_z, z)$ eliminates $P_z$ and leads to

$K(I, \xi; z) = K_o(I, z) + \epsilon K_1(I, z) \sin \xi$ with "time" $z$.

The equations are now simple enough to think about.

For fixed $z$, the phase portrait is like that of a plane pendulum:




$$W \sim \sqrt{\frac{K_1}{\partial^2 K_o / \partial I^2}}, \quad \omega_0 \sim \sqrt{K_1 \frac{\partial^2 K_o}{\partial I^2}}$$

Because $K$ depends on $z$, the picture shifts as $z$ changes. Differentiating the $0^{th}$ order resonance condition

$$\frac{d}{dz}\left\{\frac{\partial H_o}{dI}(I_{res}, z) = 0\right\}$$

gives

$$\frac{dI_{res}}{dz} = -\frac{\partial^2 K_o/\partial z \partial I}{\partial^2 K_o/\partial I^2}.$$

The "time" for the island to move by its own width is

$$\tau \equiv \frac{W}{dI_{res}/dz},$$

and the inhomogeneity parameter is

$$\mathcal{R} \equiv \omega_0 \tau = \left|\frac{\partial^2 K_o/\partial z \partial I}{K_1\ (\partial^2 K_o/\partial I^2)}\right| \sim \frac{\partial B_o/\partial z}{B_w}$$

Strongly inhomogeneous case: $\mathcal{R} \gg 1$, the $z$ dependence dominates.

$$\xi \approx \xi_{res} + \frac{A}{2}(z - z_{res})^2, \;\; A \equiv \left(\frac{\partial^2 K_o}{\partial z \partial I}\right)_{res}$$

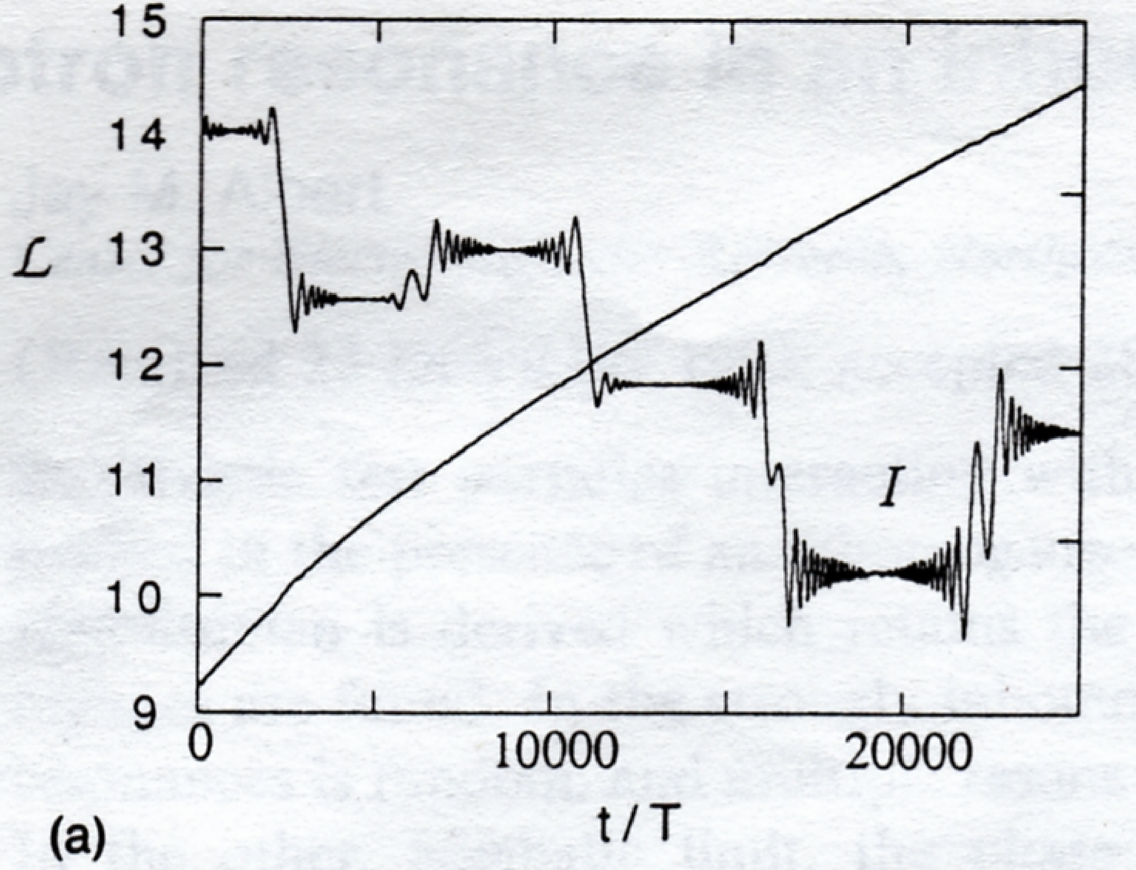
(modified at the equator).

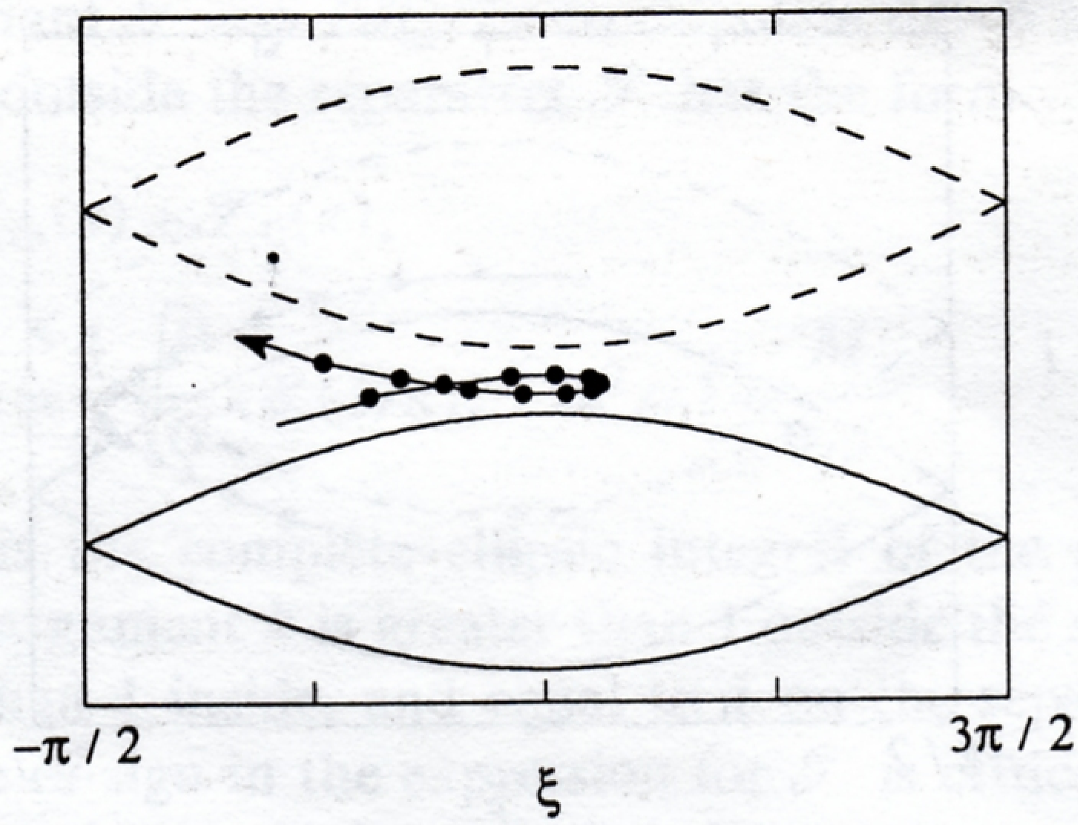
Going across the resonance,

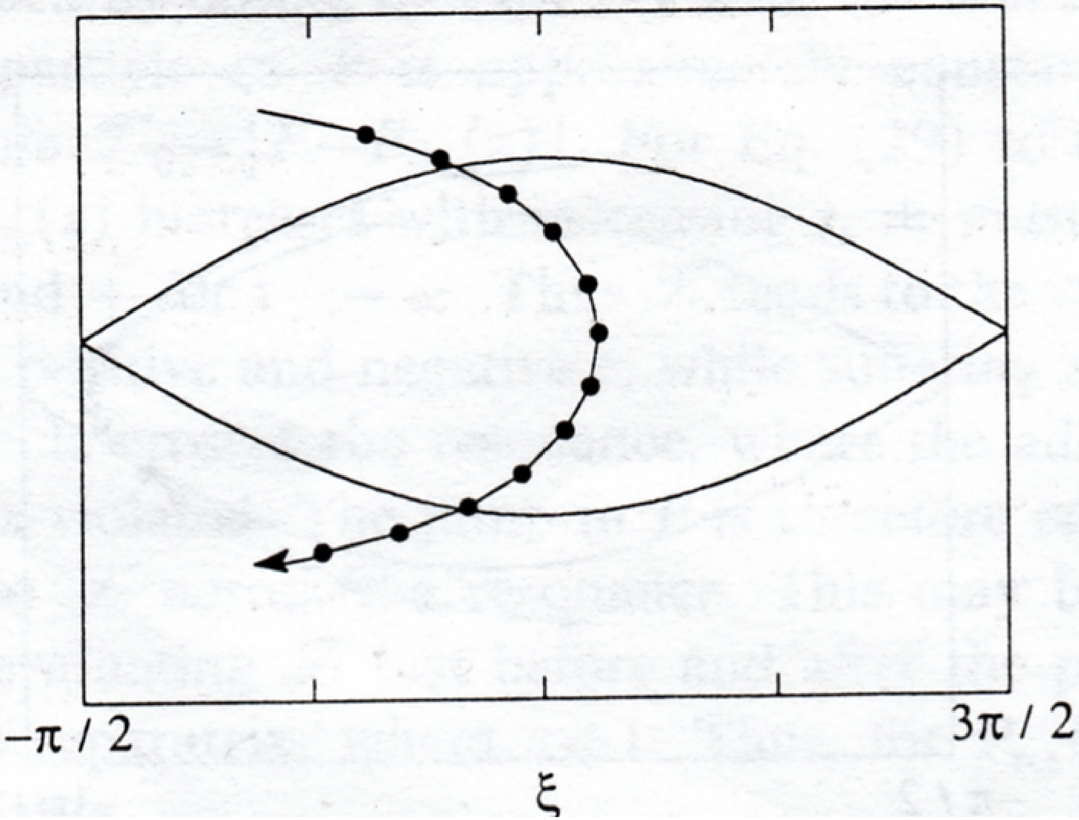
$$\delta I = \int_{-\infty}^{\infty} -\epsilon K_1 \cos \xi \; dz = -\epsilon K_1 \sqrt{\frac{2\pi}{|A|}} \cos\!\left(\xi_{res} + \frac{\pi}{4}\mathrm{sign}(A)\right).$$

$\xi_{res}$ is random over $(0, 2\pi)$, so $\delta I$ is randomly $\pm$.

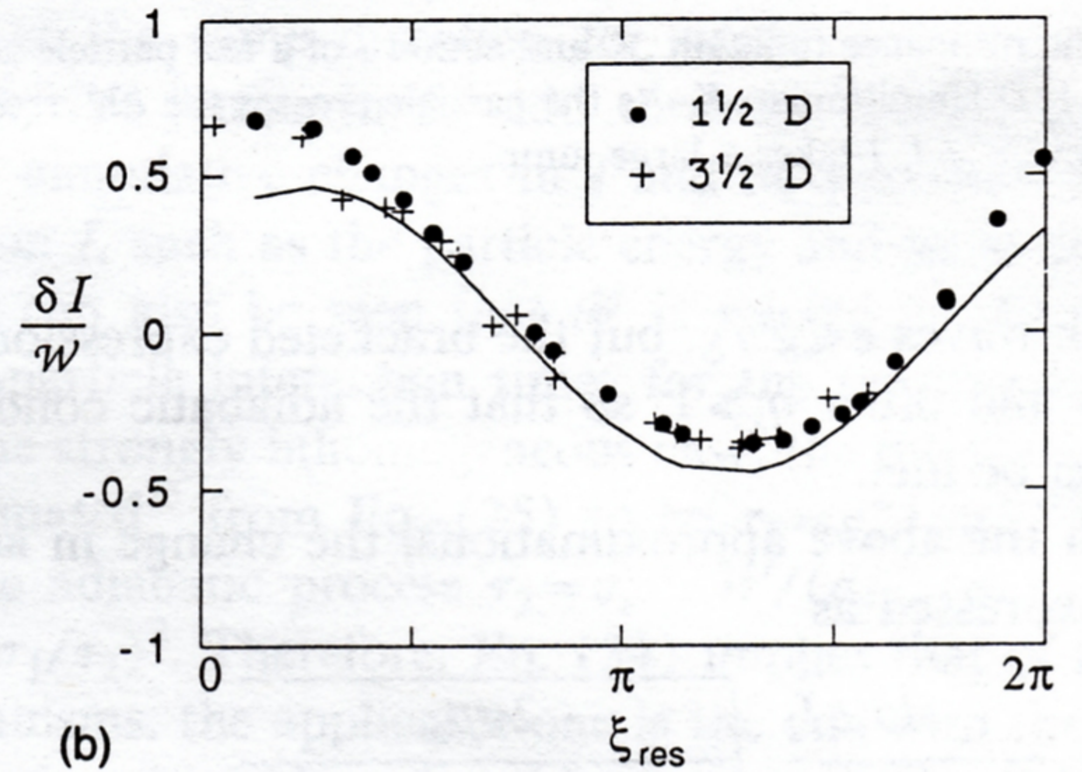
Multiple passes through the resonance: diffusion!


15
14
13
12
11
10
9
$\mathcal{L}$
$I$
0
10000
20000
t / T
(a)


−π / 2
3π / 2
ξ



−π / 2
3π / 2
ξ


1
0.5
0
-0.5
-1
$\frac{\delta I}{w}$
0
π
2π
$\xi_{res}$
● 1½ D
+ 3½ D
(b)

$$D_{II} = \frac{K_1^2}{4\tau_b}\frac{2\pi}{|A|} \Rightarrow$$

$$\begin{aligned} D_{\alpha_0\alpha_0} &= \left(\frac{\partial\alpha_0}{\partial I}\right)^2 D_{II} \\ D_{\alpha_0 p} &= \left(\frac{\partial\alpha_0}{\partial I}\right)\left(\frac{\partial p}{\partial I}\right) D_{II} \\ D_{pp} &= \left(\frac{\partial p}{\partial I}\right)^2 D_{II} \end{aligned}$$

This is consistent with

$$\begin{aligned} \frac{D_{\alpha p}}{D_{\alpha\alpha}} &= \frac{\sin\alpha\cos\alpha}{-\sin^2\alpha + s\ell\Omega_c/\omega\gamma}, \\ \frac{D_{pp}}{D_{\alpha\alpha}} &= \left(\frac{\sin\alpha\cos\alpha}{-\sin^2\alpha + s\ell\Omega_c/\omega\gamma}\right)^2 \end{aligned}$$

$$A \approx \frac{\partial}{\partial s}\left(\omega - k_{\|} v_{\|} - n\frac{\Omega_e}{\gamma}\right)$$

gives the interaction length of the resonance, $\sim \sqrt{2\pi/A}$.

For broadband waves, this is replaced by

$$\Delta k_{\|}\left|v_{\|} - \frac{\partial \omega}{\partial k_{\|}}\right|,$$

which reproduces the Kennel and Engelmann [1966] diffusion coefficients.

Surprisingly, values of the bounce-averaged broadband and single wave diffusion coefficients are often very close [JGR, 2001; 2007].

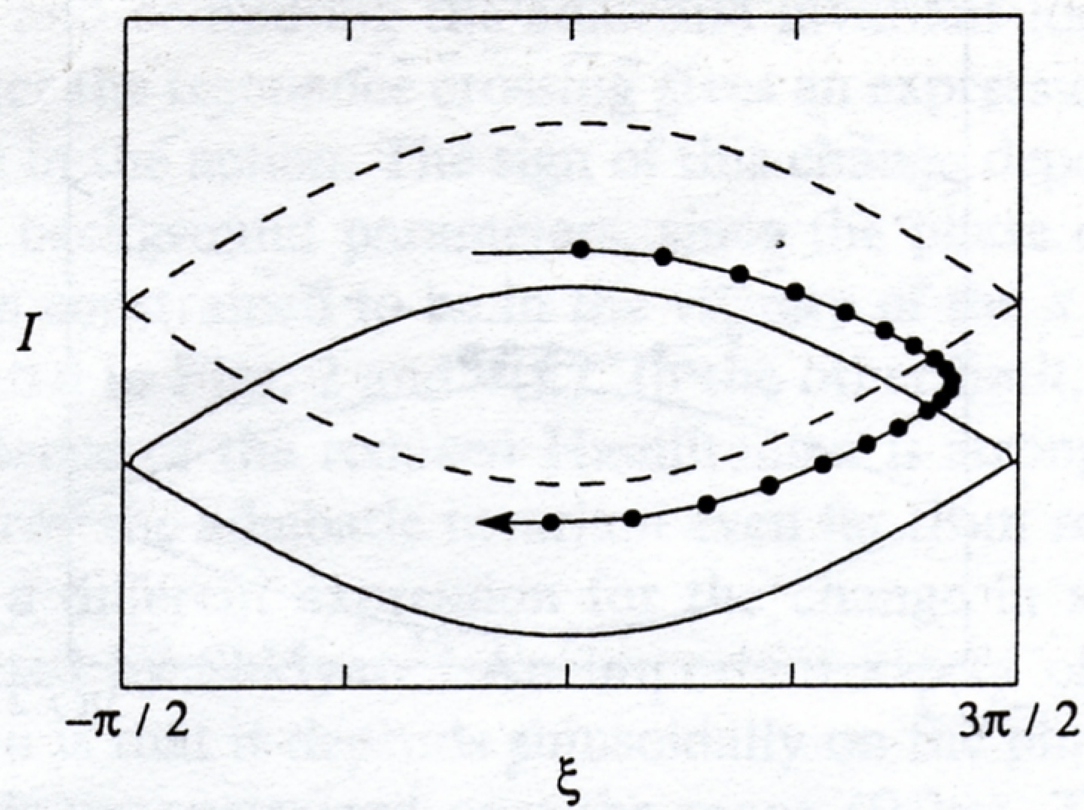
And in the single-wave limits $\delta\omega \to 0$ and $\delta\theta \to 0$, they become identical! [in preparation]

In the weakly inhomogeneous case, $\mathcal{R} \ll 1$, changes with $z$ are slow and $\mathcal{J} = \int I d\xi$ is an adiabatic invariant which is only violated near the separatrix.

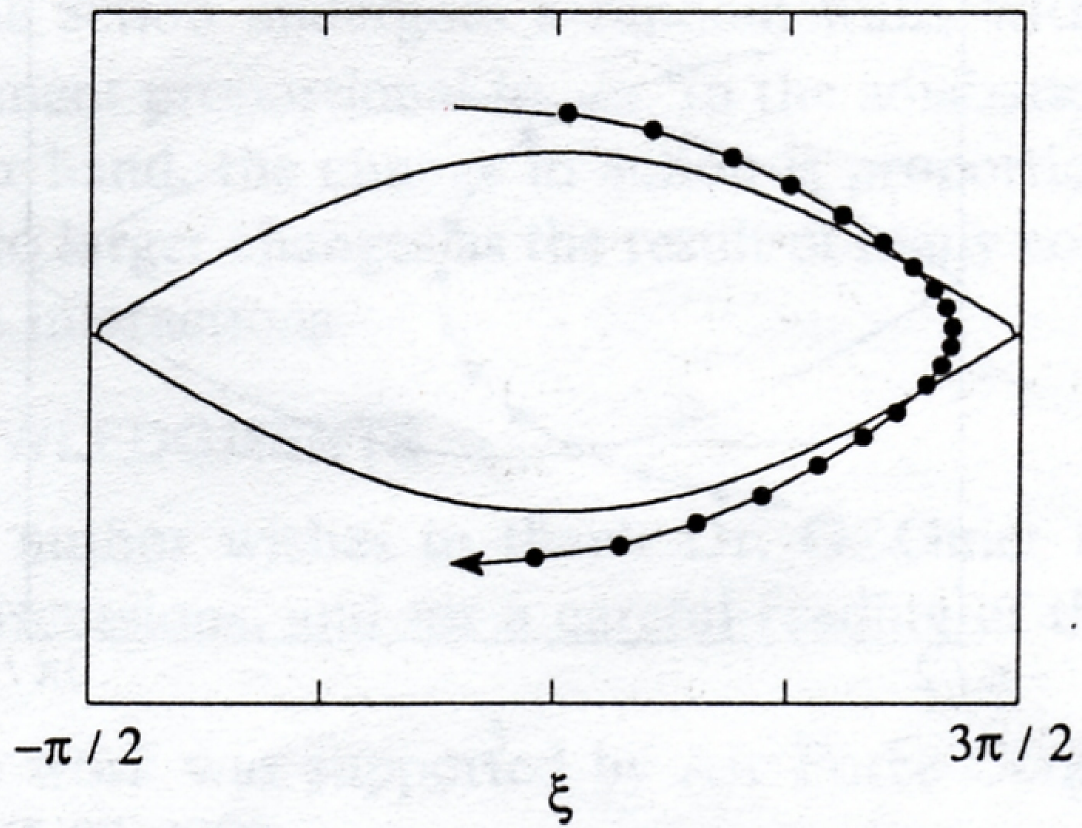
The island width gives a jump in $\mathcal{J}$ at resonance, which yields

$$\delta I = -\frac{8}{\pi}\sqrt{\left|\frac{K_1}{\partial^2 K_o/\partial I^2}\right|} \times \ \mathrm{sign}(dI_{res}/dz)$$

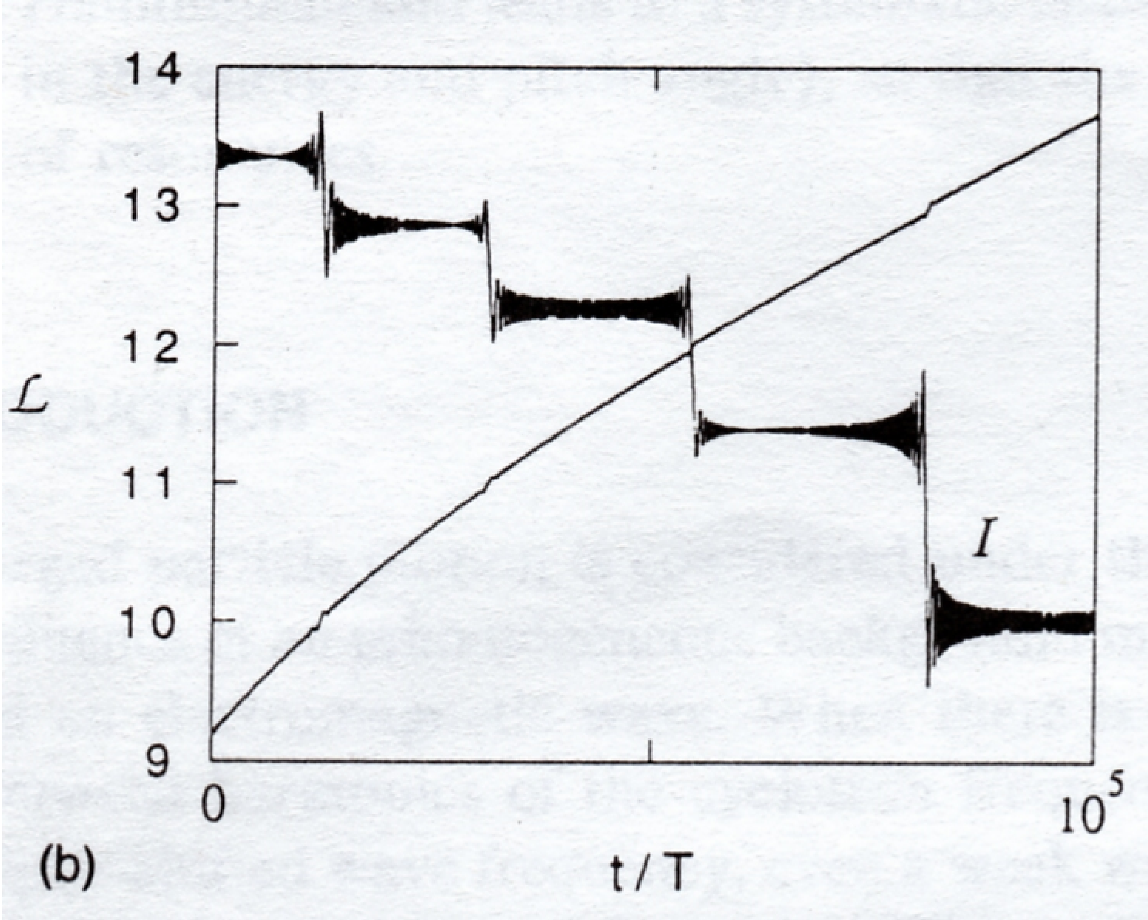
$\delta I$ is not random, because $\xi$ is determined by phase bunching.



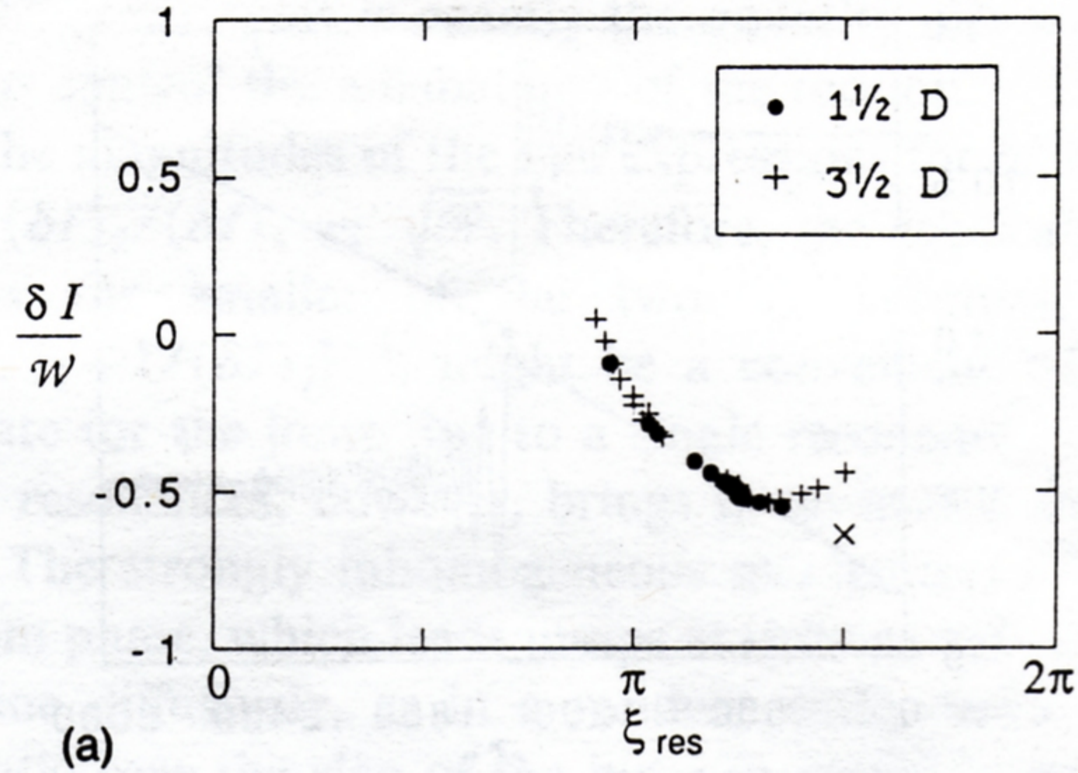
I
−π / 2
3π / 2
ξ

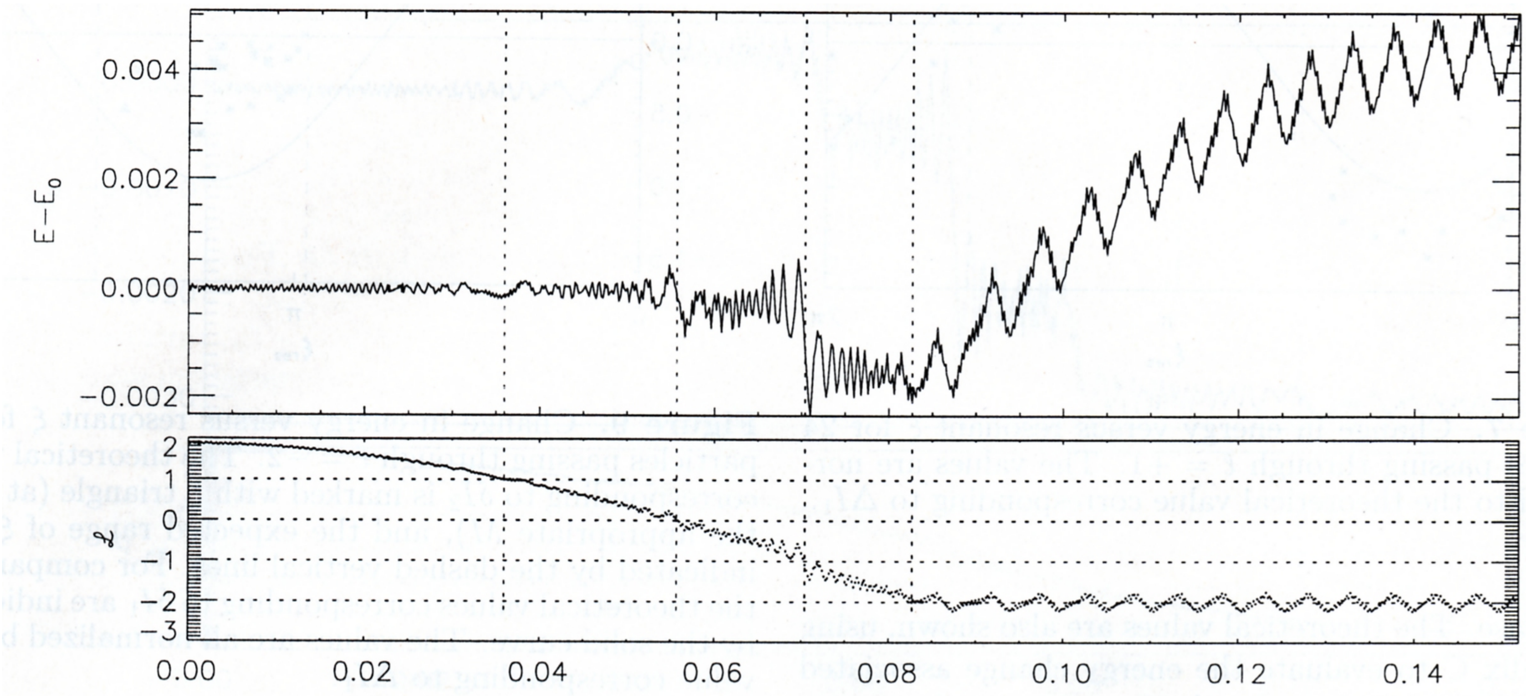


−π / 2
3π / 2
ξ



14
13
12
11
10
9
ℒ
I
0
$10^5$
t / T
(b)

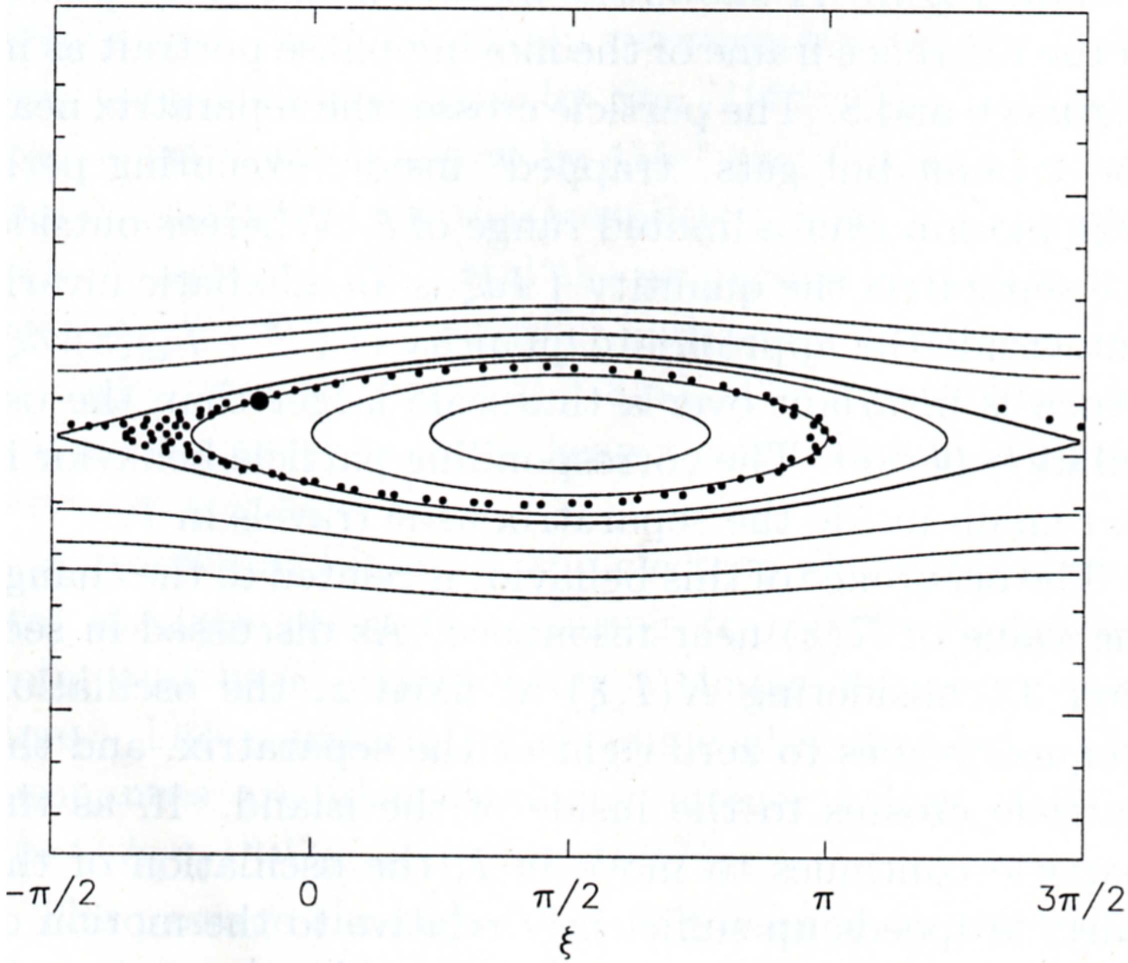

1
0.5
0
-0.5
- 1
δ I / w
0
π
2π
ξ res
1½ D
3½ D
(a)




Even more nonlinear: phase trapping. Particles can enter the separatrix and get caught there for many phase periods. $\delta I$ grows at the rate $dI_{res}/dz$.

The probability of trapping (separatrix crossing) is related to $\partial\mathcal{R}/\partial z$.

Can estimate energization if PT is assumed.



−π/2
0
π/2
π
3π/2
ξ

## Phase Trapping: Constant Frequency

There are 3 equations:

$$\gamma = \sqrt{1 + \frac{2\Omega_{eq} g I}{mc^2} + \left(\frac{P_z}{mc}\right)^2}, \qquad \text{(kinematics)}$$

$$\frac{k_z P_z}{m\gamma} - \omega + \frac{s\ell\Omega_{eq} g}{\gamma} = 0, \qquad \text{(resonance)}$$

$$\frac{\omega}{mc^2} I = s\ell\gamma, \qquad \text{(dynamics)}$$

in 4 variables: $\gamma$, $I$, $P_z$, and $z$. Solve for $\gamma(z)$:

$$\left(\frac{k_z^2 c^2}{\omega^2} - 1\right)\gamma^2 - 2\frac{s\ell\Omega_{eq} g}{\omega}\left(\frac{k_z^2 c^2}{\omega^2} - 1\right)\gamma - \left[\frac{k_z^2 c^2}{\omega^2} + \left(\frac{s\ell\Omega_{eq} g}{\omega}\right)^2\right] = 0.$$

Sustained resonance (stable PT) is assumed.

## Phase Trapping: Variable Frequency

Now there are 5 equations in 7 variables: $\gamma$, $I$, $P_z$, $\dfrac{d\gamma}{dt}$, $\dfrac{dI}{dt}$, $\dfrac{dP_z}{dt}$, and $z$, leading to a 1D ODE for $\gamma(z)$:

$$\left(\frac{k_z^2 c^2}{\omega^2} - 1\right)\frac{d\gamma}{dz} + \frac{g'}{g}\left(\frac{s\ell\Omega_{eq} g}{\omega} - \frac{k_z c}{\omega}\frac{\Omega_{eq} g I/mc^2}{P_z/mc}\right) + \frac{k_z' c}{\omega}\frac{P_z}{mc} - \frac{\omega'}{\omega}\gamma = 0,$$

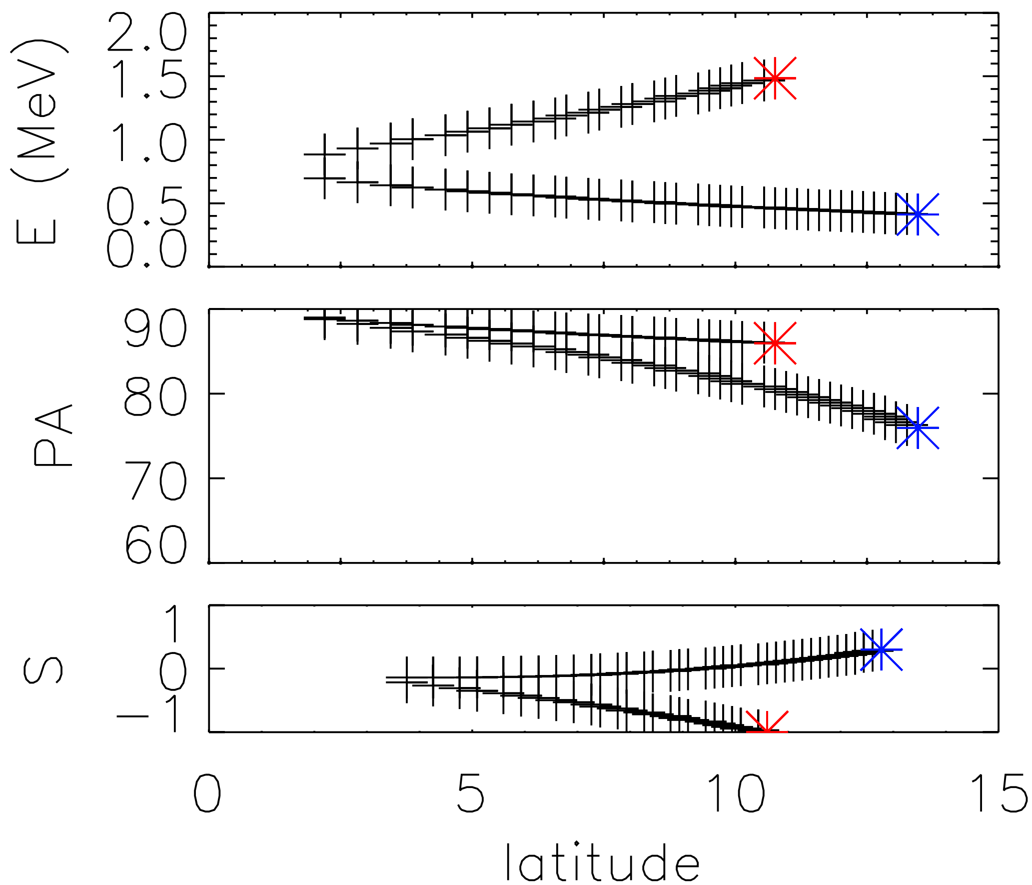
where

$$\frac{P_z}{mc} = \frac{\gamma - s\ell\Omega_{eq} g/\omega}{k_z c/\omega}, \qquad 2\frac{\Omega_{eq} g}{mc^2} I = \gamma^2 - 1 - \left(\frac{P_z}{mc}\right)^2.$$

Again, stable PT is assumed.

This is basically the procedure of *Trakhtengerts et al.* [2003] and *Demekhov et al.* [2006].

RTA can occur ($v_{\|}$ goes through 0 while maintaining PT).




This is included in the analytical treatment.

URA (resonances with $\omega > \Omega_e/\gamma$) is also included.

So: the nonlinear effects have been summarized as **advection** terms.

$$\frac{\partial f}{\partial t} = -A_{\alpha_0}\frac{\partial f}{\partial \alpha_0} - A_p \frac{\partial f}{\partial p}$$
$$+ \frac{1}{Gp}\frac{\partial}{\partial \alpha_0} G\left(D_{\alpha_0\alpha_0}\frac{1}{p}\frac{\partial f}{\partial \alpha_0} + D_{\alpha_0 p}\frac{\partial f}{\partial p}\right)$$
$$+ \frac{1}{G}\frac{\partial}{\partial p} G\left(D_{\alpha_0 p}\frac{1}{p}\frac{\partial f}{\partial \alpha_0} + D_{pp}\frac{\partial f}{\partial p}\right),$$

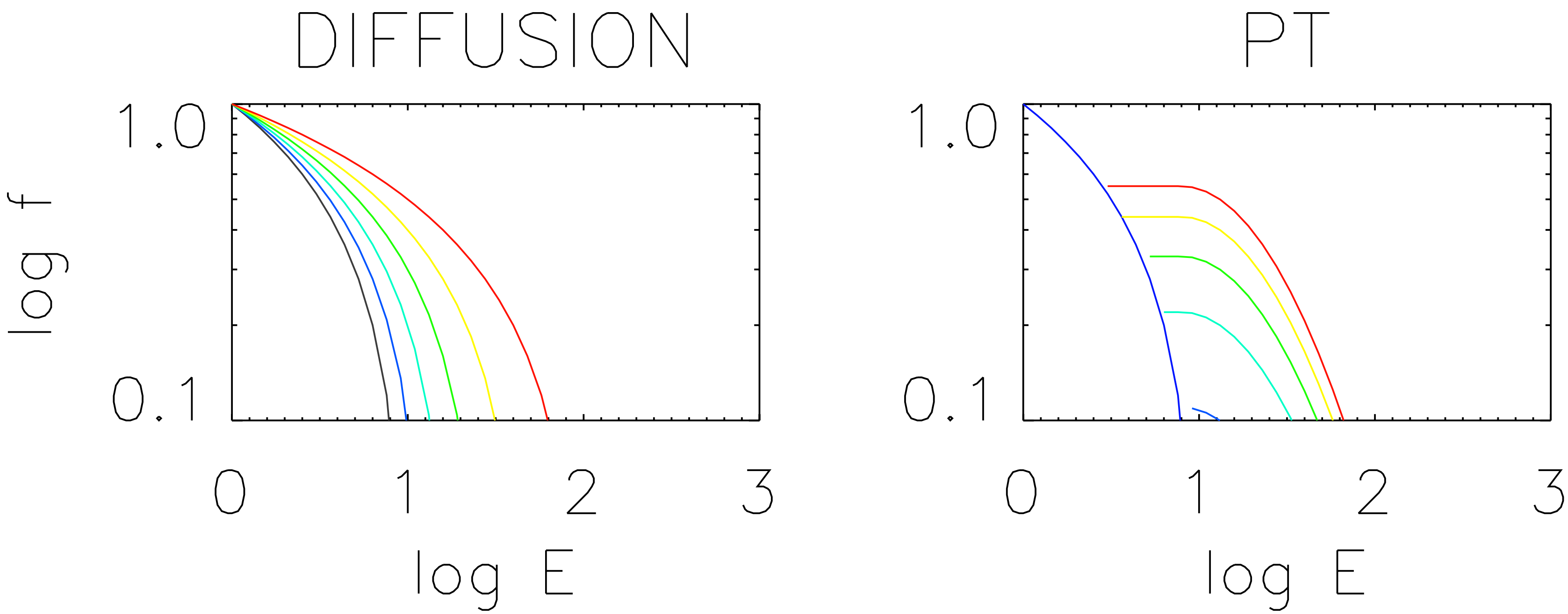
or, if you prefer,

$$\frac{\partial f}{\partial t} + \begin{bmatrix} A_{\alpha_0} \\ A_p \end{bmatrix} \begin{bmatrix} \partial f/\partial \alpha_0 \\ \partial f/\partial p \end{bmatrix} = \frac{1}{G}\begin{bmatrix} \frac{\partial}{\partial \alpha_0} & \frac{\partial}{\partial p} \end{bmatrix} G \begin{bmatrix} D_{\alpha_0\alpha_0} & D_{\alpha_0 p} \\ D_{\alpha_0 p} & D_{pp} \end{bmatrix} \begin{bmatrix} \partial f/\partial \alpha_0 \\ \partial f/\partial p \end{bmatrix},$$

where $G = p^2 T(\alpha_0)\sin\alpha_0\cos\alpha_0$. (And don't forget $D_{LL}$.)

This advection-diffusion/Fokker-Planck equation isn't so bad.

Possible evolution of $f$ (schematic):



## **Final Thoughts:**

small amplitude: linear response
"medium" amplitude: QL diffusion
large amplitude: NL behavior
very large amplitude: island overlap, QL diffusion!?

Broadband waves, homogeneous background $\Rightarrow$ diffusion
Monochromatic waves, inhomogeneous background $\Rightarrow$ $D$, PB, PT
Broadband waves, inhomogeneous background $\Rightarrow$ ???