“ I am an old man now, and when I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electro-dynamics and the other is turbulence.

About the former, I am really rather optimistic”

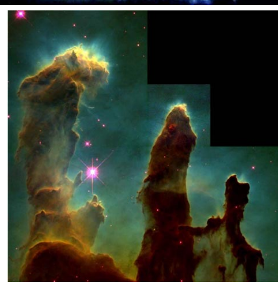
Sir Horace Lamb (1932)

From S. Goldstein, Ann. Rev. Fluid Mech, 1, 23 (1969)

“ What is turbulence ?
Turbulence is like pornography.
It is hard to define,
but if you see it, you recognize it
immediately”

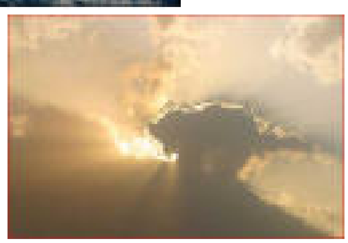
G. K. Vallis, 1999

M100 galaxy $10^{23}\ m$

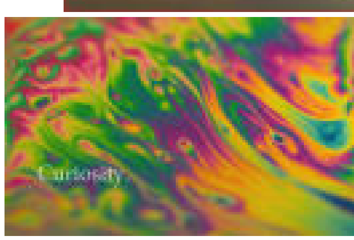


Eagle nebula $10^{18}\ m$

Earth's atmosphere $10^{7}\ m$



Clouds $10^{3}\ m$



Soap film $10^{-1}\ m$

The mathematical description of fluid motion

Leonhard Euler 1757

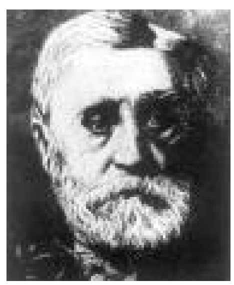
$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

Claude L. M. H. Navier 1827

George G. Stokes 1845

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

... actually, the Navier-Barré de Saint-Venant equations ...



Adhémar J. C. Barré de Saint-Venant (1843)

The dimensionless Navier-Stokes equations

$$\boldsymbol{u} \to \boldsymbol{u}/U \qquad \boldsymbol{x} \to \boldsymbol{x}/L \qquad t \to Ut/L$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + Re^{-1} \boldsymbol{\Delta} \boldsymbol{u}$$

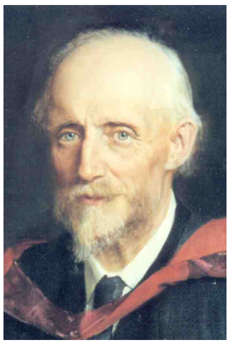
Fully developed turbulence

$$Re \to \infty$$

Typical Reynolds numbers:

$Re \sim 10^7$ atmospheric turbulence (air)
$Re \sim 10^4$ pipe/channel flow (water)

Turbulence in the lab



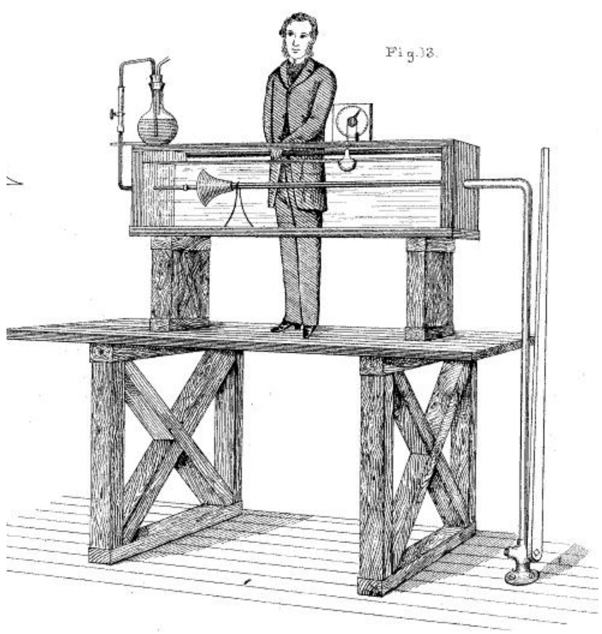
Osborne Reynolds

Phil. Trans. R. Soc., **174** 935 (1883)

XXIX. *An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels.*

By Osborne Reynolds, *F.R.S.*

Received and Read March 15, 1883.



Transition to turbulence

tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water entered the tubes with the clear water.

The general results were as follows :—

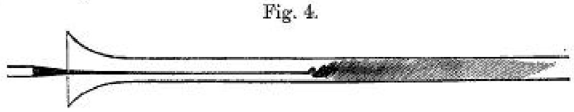
(1.) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, fig. 3.

Fig. 3.

(2.) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

(3.) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in fig. 4.

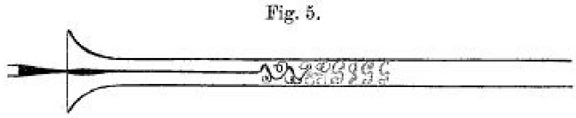
Fig. 4.



Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in fig. 5.

Fig. 5.



The experiments thus seemed to settle questions 3 and 4 in the affirmative, the existence of eddies and a critical velocity.

They also settled in the negative question 6, as to the eddies coming in gradually after the critical velocity was reached.

In order to obtain an answer to question 5, as to the law of the critical velocity, the diameters of the tubes were carefully measured, also the temperature of the water, and the rate of discharge.

(4.) It was then found that, with water at a constant temperature, and the tank as still as could by any means be brought about, the critical velocities at which the

The Friction Law and the Reynolds number

Section I.

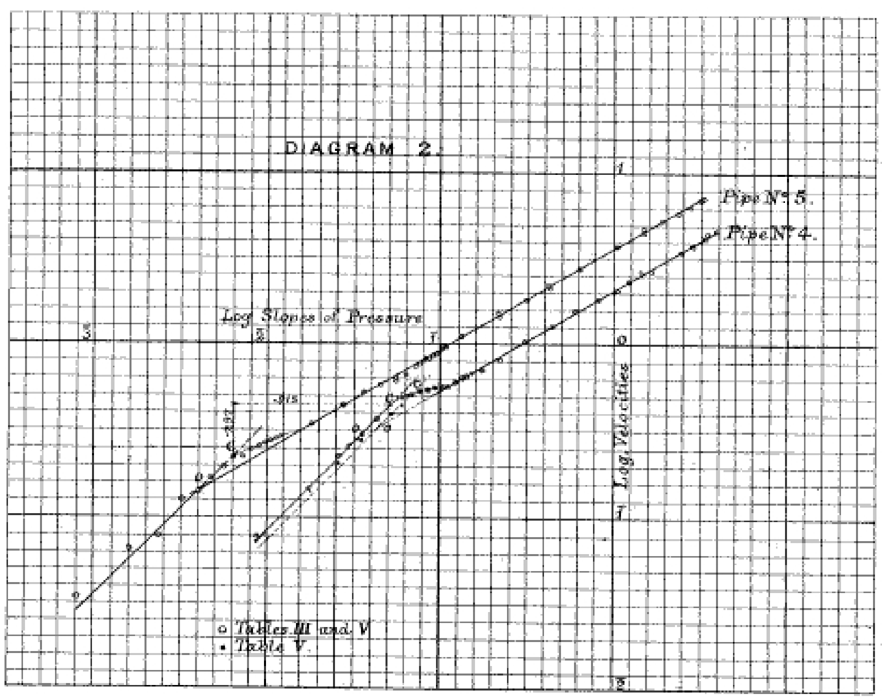
Introductory.

1. *Objects and results of the investigation.*—The results of this investigation have both a practical and a philosophical aspect.

In their practical aspect they relate to the *law of resistance to the motion of water in pipes,* which appears in a new form, the law for all velocities and all diameters being represented by an equation of two terms.

In their philosophical aspect these results relate to the fundamental principles of fluid motion; inasmuch as they afford for the case of pipes a definite verification of two principles, which are—*that the general character of the motion of fluids in contact with solid surfaces depends on the relation between a physical constant of the fluid and the product of the linear dimensions of the space occupied by the fluid and the velocity.*

The results as viewed in their philosophical aspect were the primary object of the investigation.



$U \propto \Delta p$ laminar $\quad U \propto (\Delta p)^{1/2}$ turbulent

$$Re = \frac{UL}{\nu}$$

Averages, fluctuations and the closure problem

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$$ Reynolds' averaging (1895)

$$\boldsymbol{U} = \langle \boldsymbol{u} \rangle \qquad \langle \boldsymbol{u}' \rangle = 0$$

$$\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{U} = -\boldsymbol{\nabla} P + \nu \Delta \boldsymbol{U} - \boldsymbol{\nabla} \cdot \langle \boldsymbol{u}'\boldsymbol{u}' \rangle$$

$$\partial_t \langle \boldsymbol{u}'\boldsymbol{u}' \rangle + (\boldsymbol{U} \cdot \boldsymbol{\nabla}) \langle \boldsymbol{u}'\boldsymbol{u}' \rangle + \boldsymbol{\nabla} \cdot \langle \boldsymbol{u}'\boldsymbol{u}'\boldsymbol{u}' \rangle =$$
$$\nu \Delta \langle \boldsymbol{u}'\boldsymbol{u}' \rangle - \nu \langle \boldsymbol{\nabla} \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \boldsymbol{u})^T \rangle - \langle \boldsymbol{u}'\boldsymbol{u}' \rangle \cdot (\boldsymbol{\nabla} \boldsymbol{U})^T$$

Can $\langle \boldsymbol{u}'\boldsymbol{u}' \rangle$ be expressed in terms of $\boldsymbol{U}$?

$$\langle \boldsymbol{u}'\boldsymbol{u}' \rangle = \nu_e [\boldsymbol{\nabla} \boldsymbol{U} + (\boldsymbol{\nabla} \boldsymbol{U})^T]$$

ν_e: eddy-viscosity

Joseph Boussinesq (1897)

$$\nu_e = c_\mu \frac{k^2}{\epsilon}$$ where $k = \langle |\boldsymbol{u}'|^2 \rangle$ and $\epsilon = \nu \langle |\boldsymbol{\nabla} \boldsymbol{u}'|^2 \rangle$

Energy budget for the Navier-Stokes equation

$$E = \frac{1}{2}\int |\boldsymbol{u}(\boldsymbol{x},t)|^2\, d\boldsymbol{x}$$

$$\int d\boldsymbol{x}\,\{\boldsymbol{u}\cdot[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \nu\Delta\boldsymbol{u}]\}$$

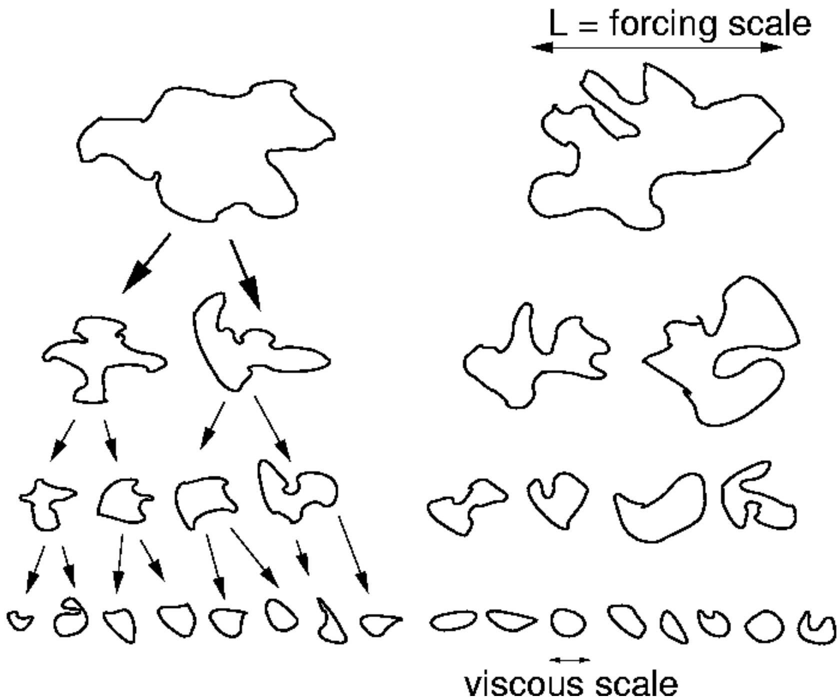
$$\boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\cdot(\tfrac{1}{2}|\boldsymbol{u}|^2\boldsymbol{u})$$
$$\boldsymbol{u}\cdot\boldsymbol{\nabla}p = \boldsymbol{\nabla}\cdot(p\boldsymbol{u})$$
$$\boldsymbol{u}\cdot\nu\Delta\boldsymbol{u} = -\nu|\boldsymbol{\nabla}\boldsymbol{u}|^2 + \boldsymbol{\nabla}\cdot(\nu\boldsymbol{u}\boldsymbol{\nabla}\boldsymbol{u})$$

$$\boxed{\frac{dE}{dt} = -\epsilon}$$

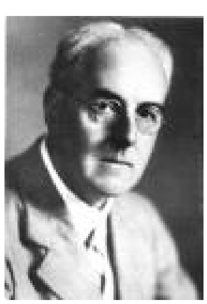
where $\epsilon = \nu\int |\boldsymbol{\nabla}\boldsymbol{u}|^2 d\boldsymbol{x}$

Statistically stationary turbulence $\rightarrow$ external forcing

The turbulent cascade

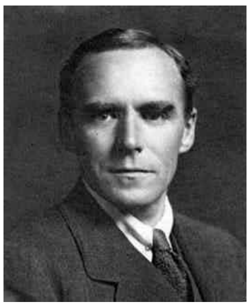


" Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity"



Lewis Fry Richardson (1920)

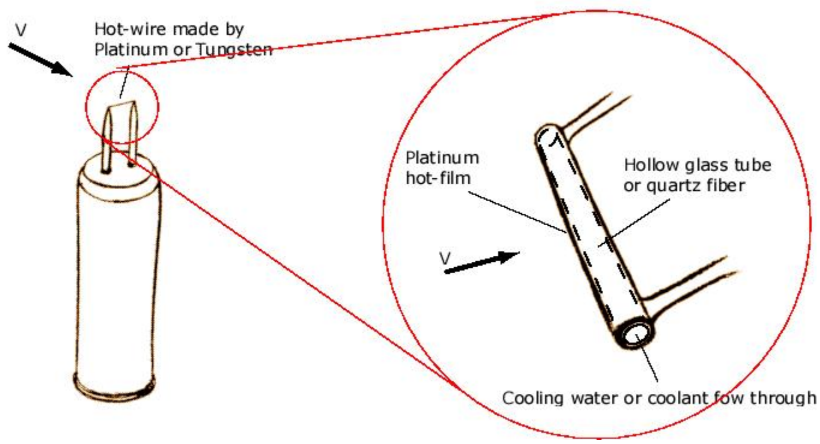
From qualitative to quantitative



Geoffrey I. Taylor

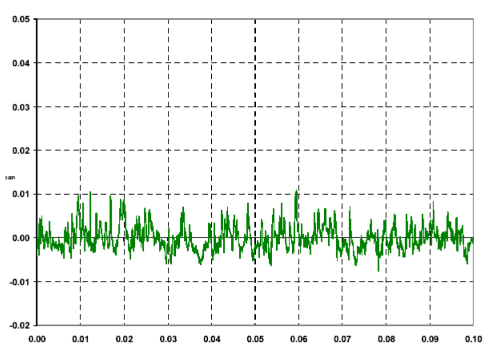
Proc. Roy. Soc. A **151**, 421 (1935)

Measuring turbulent flows



hot wire/film

anemometry



Taylor's
"frozen turbulence" hypothesis
$u_x(t) - u_x(t-\tau) = u_x(x+r) - u_x(x)$
with $r = U\tau$

The Taylor's microscale

A first definition of dissipative lengthscale

$$\lambda = \left(\frac{\epsilon}{15\, \nu\, u_{rms}^2} \right)^{1/2}$$

where $u_{rms}^2 = \langle |\boldsymbol{u}|^2 \rangle$

$$C(r) = \frac{\langle u_x(x) u_x(x+r) \rangle}{\langle u_x(x)^2 \rangle}$$

$$C(r) \simeq 1 - r^2/\lambda^2$$

for $r \ll \lambda$

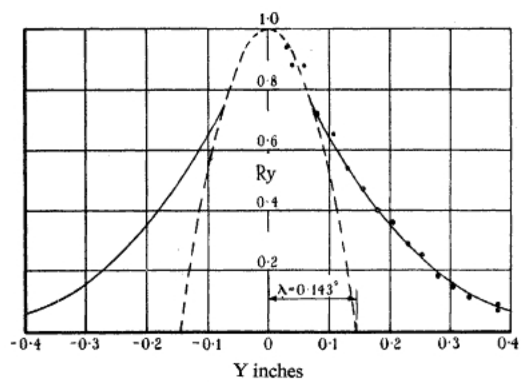


FIG. 1—Measured values of $R_y = \overline{u_0 u_y}/\overline{u_2}$ behind 0·9 inch by 0·9 inch honeycomb; $\overline{u^2} = 0·1015$ (ft sec)^2

Small-scale statistical homogeneity and isotropy

The Taylor microscale Reynolds number

$$R_\lambda = \frac{u_{rms} \lambda}{\nu}$$

Velocity fluctuations

$$\delta_r u = u_x(x+r) - u_x(x)$$

characteristic velocity of an eddy of size r

The statistical description of turbulence

$\langle \ldots \rangle$ Ensemble averaging

$\langle \mathcal{O}(\boldsymbol{x}, t) \rangle = \langle \mathcal{O}(\boldsymbol{x}, t') \rangle$ Statistical stationarity

$\langle \mathcal{O}(\boldsymbol{x}, t) \rangle = \langle \mathcal{O}(\boldsymbol{x}', t) \rangle$ Statistical homogeneity

$\langle \mathcal{O}(\boldsymbol{x}, t) \rangle = \langle \mathcal{O}(\mathsf{R}\boldsymbol{x}, t) \rangle$ Statistical isotropy, $\mathsf{R} \in \mathrm{SO}(3)$

$\langle \mathcal{O}(\boldsymbol{x}, t) \rangle = \langle \mathcal{O}(-\boldsymbol{x}, t) \rangle$ Statistical parity invariance

Examples:

- $\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}+\boldsymbol{r}) \rangle = B_T(r)(\delta_{ij} - \widehat{r}_i \widehat{r}_j) + B_L(r) \widehat{r}_i \widehat{r}_j$
  Incompressibility imposes: $B_T(r) = r B_L'(r) + 2 B_L(r)$
  $B_L(r)$: Longitudinal correlation function
  $S_L(r) = 2(B_L(0) - B_L(r)) = \langle \{ [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \widehat{\boldsymbol{r}} \}^2 \rangle$
  Longitudinal 2^{nd} order structure function

- $\langle \widehat{u}_i(\boldsymbol{k}) \widehat{u}_j(\boldsymbol{k}') \rangle = \widehat{B}(k) \delta(\boldsymbol{k}+\boldsymbol{k}')(\delta_{ij} - \widehat{k}_i \widehat{k}_j)$
  $E(k) = 4\pi k^2 \widehat{B}(k)$: Energy spectrum

Statistical theories of turbulence: Closures

$$\partial_t \langle \widehat{u}\widehat{u} \rangle + ik\langle \widehat{u}\widehat{u}\widehat{u} \rangle = -\nu k^2 \langle \widehat{u}\widehat{u} \rangle + \langle \widehat{f}\widehat{u} \rangle$$

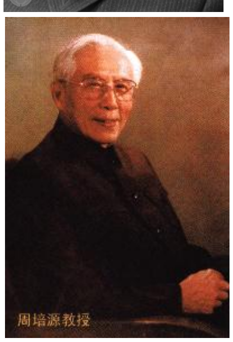
$$\partial_t \langle \widehat{u}\widehat{u}\widehat{u} \rangle + ik\langle \widehat{u}\widehat{u}\widehat{u}\widehat{u} \rangle = -\nu k^2 \langle \widehat{u}\widehat{u}\widehat{u} \rangle + \langle \widehat{f}\widehat{u}\widehat{u} \rangle$$

and so *ad infinitum*

Quasi-normal approximation

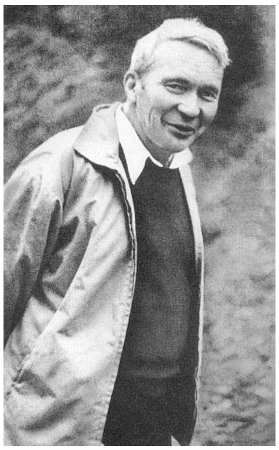
$$\langle \widehat{u}_a\widehat{u}_b\widehat{u}_c\widehat{u}_d \rangle =$$
$$\langle \widehat{u}_a\widehat{u}_b \rangle\langle \widehat{u}_c\widehat{u}_d \rangle + \langle \widehat{u}_a\widehat{u}_c \rangle\langle \widehat{u}_b\widehat{u}_d \rangle + \langle \widehat{u}_a\widehat{u}_d \rangle\langle \widehat{u}_b\widehat{u}_c \rangle$$

M. D. Millionschikov (1939)



P. Y. Chou (1940)

Kolmogorov's phenomenological theory



Andrei N. Kolmogorov (1941a)

The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers†

By A. N. Kolmogorov

The first hypothesis of similarity. *For the locally isotropic turbulence the distributions F_n are uniquely determined by the quantities ν and $\bar{\epsilon}$.*

The second hypothesis of similarity.† *If the moduli of the vectors $y^{(k)}$ and of their differences $y^{(k)} - y^{(k')}$ (where $k \neq k'$) are large in comparison with λ, then the distribution laws F_n are uniquely determined by the quantity $\bar{\epsilon}$ and do not depend on ν.*

$$B_{dd}(r) \sim C\bar{\epsilon}^{\frac{2}{3}} r^{\frac{2}{3}}.$$

$$S_L(r) = \langle \{ [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \hat{\boldsymbol{r}} \}^2 \rangle \sim C\epsilon^{\frac{2}{3}}\, r^{\frac{2}{3}}$$

$$E(k) = C'\epsilon^{\frac{2}{3}}\, k^{-\frac{5}{3}}$$

Kolmogorov's 4/5ths law (1941b)

Dissipation of energy in the locally isotropic turbulence†

By A. N. Kolmogorov

In my note (Kolmogorov 1941*a*) I defined the notion of local isotropy and introduced the quantities

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[u_d(M') - u_d(M)]^2}, \\ B_{nn}(r) &= \overline{[u_n(M') - u_n(M)]^2}, \end{aligned}\right\} \tag{1}$$

where r denotes the distance between the points M and M', $u_d(M)$ and $u_d(M')$ are the velocity components in the direction $\overline{MM'}$ at the points M and M', and $u_n(M)$ and $u_n(M')$ are the velocity components at the points M and M' in some direction, perpendicular to MM'.

In the sequel we shall need the third moments

$$B_{ddd}(r) = \overline{[u_d(M') - u_d(M)]^3}. \tag{2}$$

For the locally isotropic turbulence in incompressible fluid we have the equation

$$4\bar{E} + \left(\frac{\mathrm{d}B_{ddd}}{\mathrm{d}r} + \frac{4}{r}B_{ddd}\right) = 6\nu\left(\frac{\mathrm{d}^2 B_{dd}}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}B_{dd}}{\mathrm{d}r}\right) \tag{3}$$

Kármán

(1937)

similar to the known equation of Kármán for the isotropic turbulence in the sense of Taylor. Herein $\bar{E}$ denotes the mean dissipation of energy in the unit of time per unit of mass. The equation (3) may be rewritten in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{4}{r}\right)\left(6\nu\frac{\mathrm{d}B_{dd}}{\mathrm{d}r} - B_{ddd}\right) = 4\bar{E}, \tag{4}$$

and, in virtue of the condition $(\mathrm{d}/\mathrm{d}r)B_{dd}(0) = B_{ddd}(0) = 0$, yields

$$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r - B_{ddd} = \tfrac{4}{5}\bar{E}r. \tag{5}$$

For small r we have, as is known,

$$B_{dd} \sim \tfrac{1}{15}\bar{E}r^2/\nu, \tag{6}$$

i.e.

$$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r \sim \tfrac{4}{5}\bar{E}r.$$

Thus, the second term on the left-hand side of (5) is for small r infinitesimal in comparison with the first. For large r, on the contrary, the first term may be neglected in comparison with the second, i.e. we may assume that

$$B_{ddd} \sim -\tfrac{4}{5}\bar{E}r. \tag{7}$$

A. N. K.

(1941)

It is natural to assume that for large r the ratio

$$S = B_{ddd} : B_{dd}^{\frac{3}{2}}, \tag{8}$$

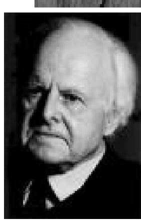
† First published in Russian in *Dokl. Akad. Nauk SSSR* (1941), **32**(1). Paper received 30 April 1941. This translation by V. Levin, reprinted here with emendations by the editors of this volume.

Proc. R. Soc. Lond. A (1991) **434**, 15–17
Printed in Great Britain

Other derivations of the Kolmogorov spectrum

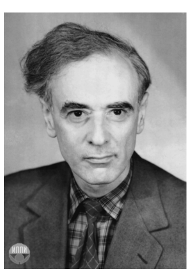
L. Onsager (1945)

W. Heisenberg (1945)



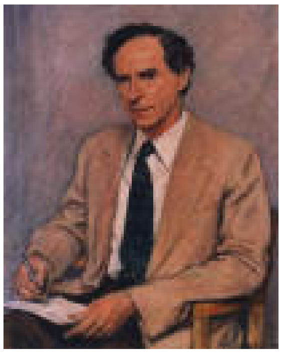
C. von Weizsacker (1945)

Early remarks to Kolmogorov's theory



L. D. Landau (1941)

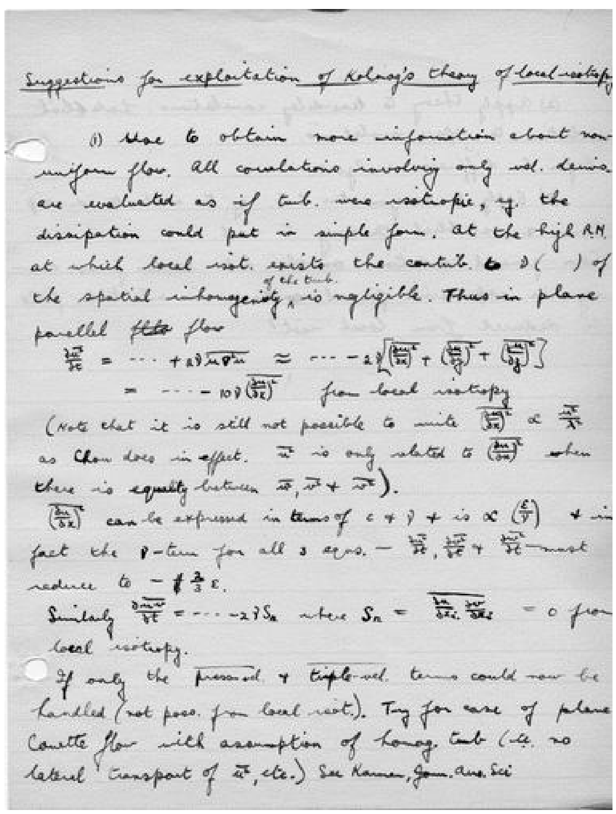
The dissemination of Kolmogorov theory



G. K. Batchelor (1947)

Kolmogorov's theory of locally isotropic turbulence

Proc. Camb. Phil. Soc. **43** 533 (1947)



Suggestions for exploitation of Kolmog's theory of local isotropy

(i) Use to obtain more information about non-uniform flow. All correlations involving only vel. derivs are evaluated as if turb. were isotropic, e.g. the dissipation could put in simple form. At the high R.N. at which local isot. exists the contrib. to $\partial(\;)$ of the spatial inhomogeneity of the turb. is negligible. Thus in plane parallel ~~flat~~ flow

$$\frac{\partial \overline{u^2}}{\partial t} = \cdots + 2\nu\,\overline{u\nabla^2 u} \approx \cdots - 2\nu\left[\overline{\left(\frac{\partial u}{\partial x}\right)^2} + \overline{\left(\frac{\partial u}{\partial y}\right)^2} + \overline{\left(\frac{\partial u}{\partial z}\right)^2}\right]$$

$$= \cdots - 10\nu\,\overline{\left(\frac{\partial u}{\partial x}\right)^2} \quad \text{from local isotropy}$$

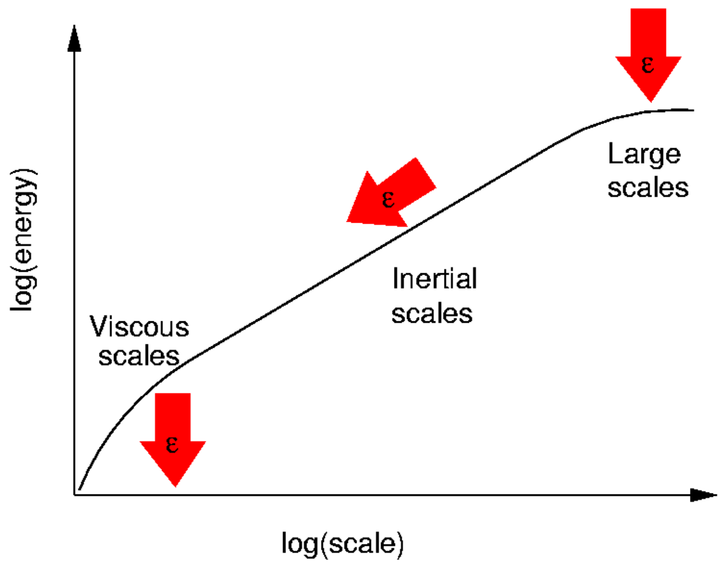
(Note that it is still not possible to write $\overline{\left(\frac{\partial u}{\partial x}\right)^2} \propto \frac{\overline{u^2}}{\lambda^2}$ as Chou does in effect. $\overline{u^2}$ is only related to $\overline{\left(\frac{\partial u}{\partial x}\right)^2}$ when there is equality between $\overline{u^2}, \overline{v^2}$ + $\overline{w^2}$).

$\overline{\left(\frac{\partial u}{\partial x}\right)^2}$ can be expressed in terms of ε + ν + is $\propto \left(\frac{\varepsilon}{\nu}\right)$ + in fact the ν-term for all 3 eqns. — $\frac{\partial \overline{u^2}}{\partial t}, \frac{\partial \overline{v^2}}{\partial t}$ + $\frac{\partial \overline{w^2}}{\partial t}$ — must reduce to $-\frac{2}{3}\varepsilon$.

Similarly $\frac{\partial \overline{uv}}{\partial t} = \cdots - 2\nu S_n$ where $S_n = \overline{\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}} = 0$ from local isotropy.

If only the $\overline{\text{pressure}}$ + $\overline{\text{triple-vel.}}$ terms could now be handled (not poss. from local isot.). Try for case of plane Couette flow with assumption of homog. turb (i.e. no lateral transport of $\overline{u^2}$, etc.) See Karman, Journ. Aero. Sci

A closer look to Kolmogorov's phenomenology



Energy flux $\varepsilon = \frac{\text{energy}}{\text{time}} \sim \frac{U^2}{L/U} \sim (U^3/L) \sim (\delta_r u)^3/r$

Viscous dissipation $\nu \frac{(\delta_r u)^2}{r^2}$

Kolmogorov scale $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

Reynolds number $Re = \frac{UL}{\nu} \sim \frac{\epsilon^{1/3} L^{4/3}}{\nu} \sim (\frac{L}{\eta})^{4/3}$

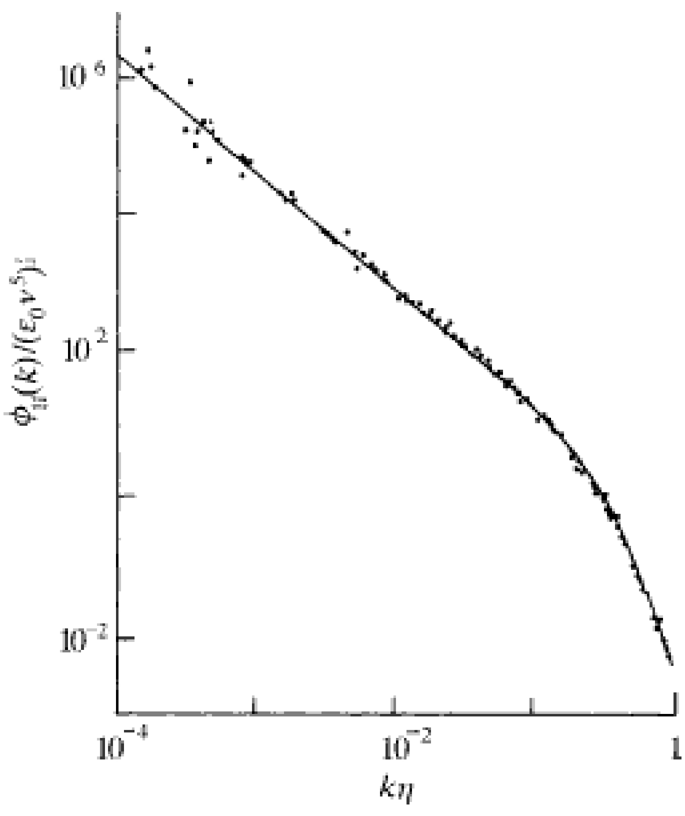
Number of active degrees of freedom $N \approx (\frac{L}{\eta})^3 \sim Re^{9/4}$

Taylor *vs* Kolmogorov scales $\frac{\lambda}{\eta} \sim \frac{L}{\eta}^{1/3} \sim Re^{1/4}$

Taylor microscale Reynolds number $Re_\lambda \sim Re^{1/2}$

Experimental verification of the Kolmogorov spectrum

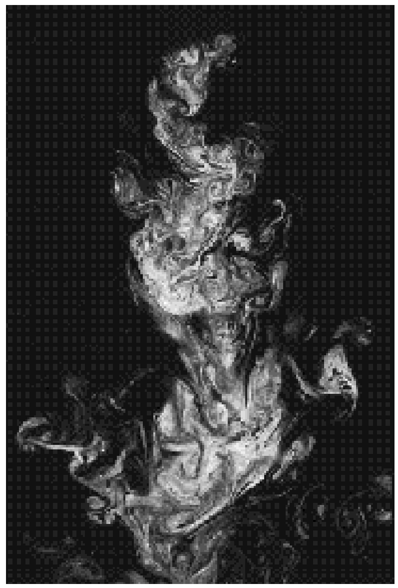
Turbulence spectra from a tidal channel



Grant H.L., Stewart R.W., and Moilliet A. (1962)

Extensions of Kolmogorov's theory

Passive scalar turbulence



$$\langle (\delta_r \theta)^2 \rangle \sim \varepsilon^{-1/3} \chi \, r^{2/3}$$

A. M. Obukhov (1949)

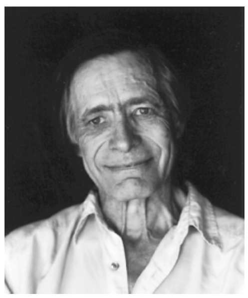
A. Yaglom (1949)

S. Corrsin (1951)

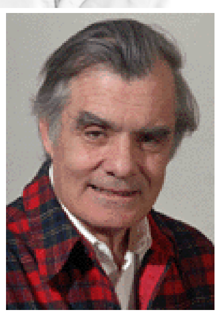
In between Navier-Stokes and Kolmogorov

(1950-1970)

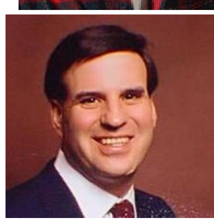
DIA, LHDIA, TFM, LET, EDQNM, and the like



R. H. Kraichnan



J. Herring



S. Orszag

... and Wyld, Edwards, Mc Comb etc. etc.

A first glimpse at intermittency

$$\frac{\langle(\frac{\partial^n u}{\partial x^n})^4\rangle}{\langle(\frac{\partial^n u}{\partial x^n})^2\rangle^2} \ vs\ Re$$

very unusual shapes to give values of the flattening factor near 4·9 and 5·9. On the simple assumption that $\partial^n u/\partial x^n$ is zero for a fraction $1-\gamma$ of the total time and varies with a Gaussian probability distribution during the remainder of the time, flattening factors 3·0, 3·9, 4·9 and 5·9 (for $n = 0$, 1, 2 and 3 respectively) correspond to $\gamma = 1{\cdot}0$, 0·77, 0·61 and 0·51 respectively. Since high orders of the velocity derivative are associated with very large wave-numbers, the results of figure 5 suggest that there is present an effect which becomes increasingly important as the wave-number is increased.

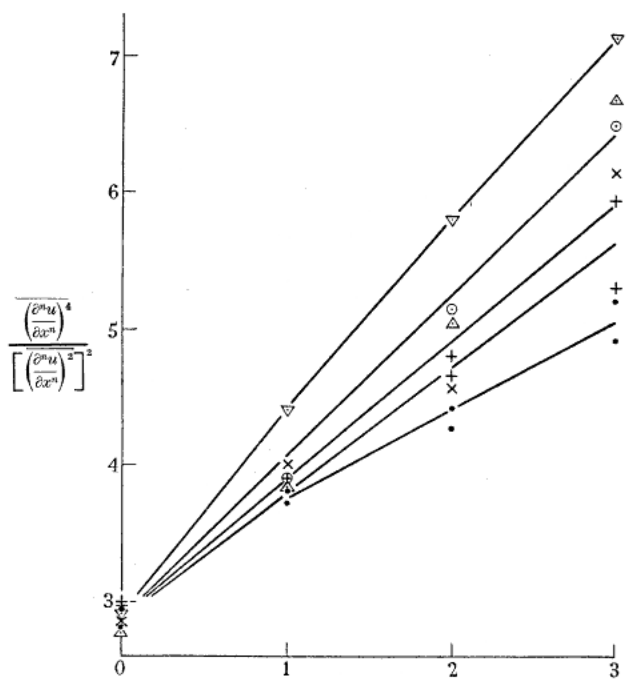
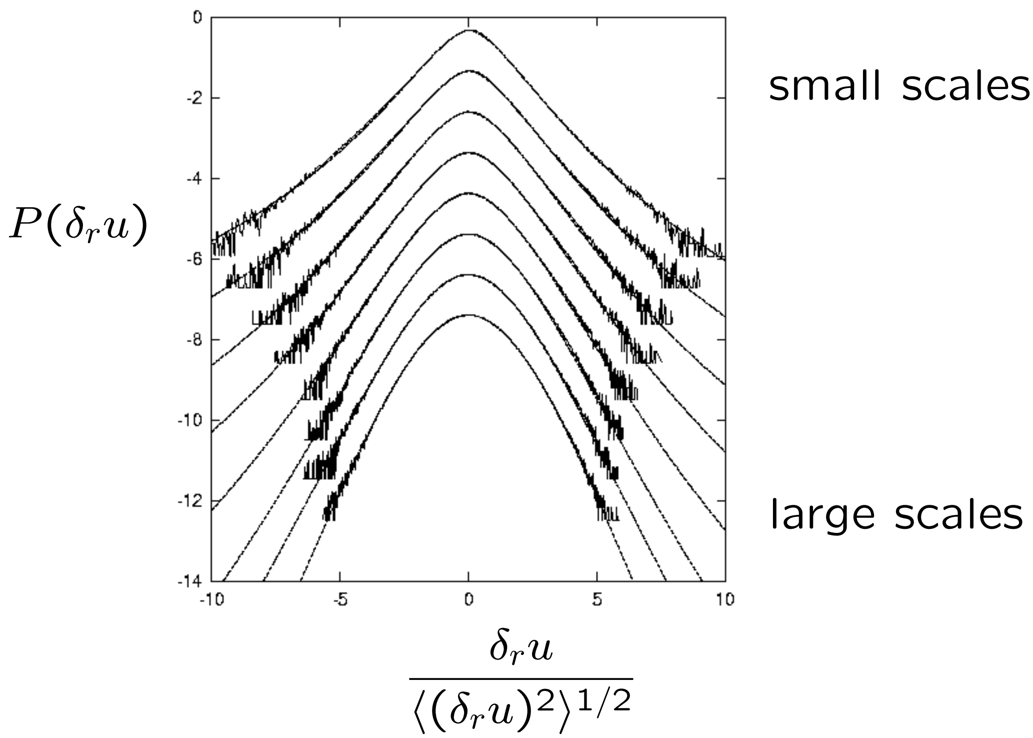


FIGURE 5. Flattening factors of velocity derivatives.

Isotropic turbulence, R_M: •, 2810; +, 5620; ×, 11,200; ⊙, 22,500.
Cylinder wake, R_d: △, 680; ▽, 4,100.

G. K. Batchelor A. A. Townsend
Proc. Roy. Soc. Lond. A bf 199, 238 (1949)

Intermittency in full glory



Kolmogorov-Obukhov theory of intermittency (1962)

Fluctuations of the energy dissipation rate

The Refined Similarity Hypothesis

(Marseille 1961)

$$\delta_r u \sim (\varepsilon_r r)^{1/3}$$

Volume-averaged dissipation $\quad \varepsilon_r = \frac{1}{\frac{4}{3}\pi r^3} \int_{S_r} \nu |\boldsymbol{\nabla} \boldsymbol{u}|^2 dV$

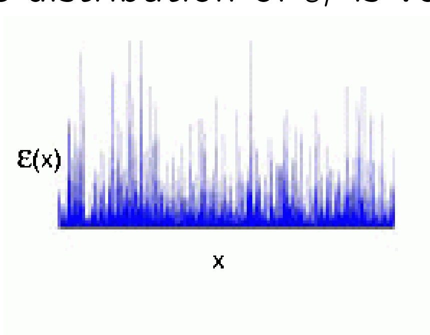
Moments of velocity differences

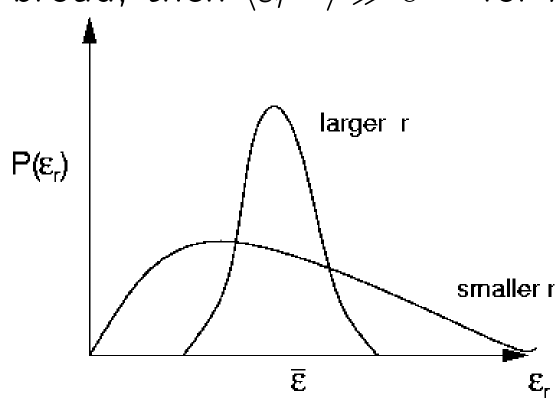
$$\langle (\delta_r u)^n \rangle \sim \langle \varepsilon_r^{n/3} \rangle r^{n/3}$$

For $n = 3$: $\quad \langle \varepsilon_r \rangle = \bar{\epsilon}$

For $n \neq 3$: $\quad \langle \varepsilon_r^{n/3} \rangle \neq \bar{\epsilon}^{n/3}$

If the distribution of ε_r is very broad, then $\langle \varepsilon_r^{n/3} \rangle \gg \bar{\epsilon}^{n/3}$ for $n > 3$





The width of the distribution of ε_r increases with decreasing r

The log-normal assumption

$$P(\ln \varepsilon_r) = \frac{1}{\sqrt{2\pi}\sigma_r}\, e^{-\frac{(\ln \varepsilon_r - m_r)^2}{2\sigma_r^2}}$$

where $m_r = \langle \ln \varepsilon_r \rangle = \ln\langle \epsilon \rangle - \frac{\sigma_r^2}{2}$

and $\sigma_r = \langle (\ln \varepsilon_r)^2 \rangle - \langle \ln \varepsilon_r \rangle^2 = A + \mu \ln(L/r)$

$$\langle \epsilon_r^{n/3} \rangle = \int e^{\frac{n}{3}\ln \varepsilon_r} P(\ln \varepsilon_r) d \ln \varepsilon_r$$
$$= A_n \langle \epsilon \rangle^{n/3} \left(\frac{L}{r}\right)^{\mu \frac{n(n-3)}{18}}$$

Scaling exponents

$$\langle (\delta_r u)^n \rangle \sim \langle \varepsilon_r^{n/3} \rangle r^{n/3} \sim r^{\frac{n}{3} - \mu \frac{n(n-3)}{18}}$$

$$\langle \varepsilon(\boldsymbol{x}) \varepsilon(\boldsymbol{x} + \boldsymbol{r}) \rangle \sim \left(\frac{L}{r}\right)^{\mu}$$

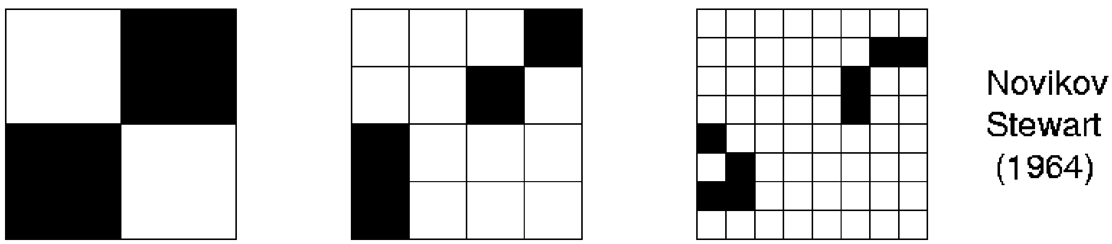
$$\frac{\langle \left(\frac{\partial u}{\partial x}\right)^4 \rangle}{\langle \left(\frac{\partial u}{\partial x}\right)^2 \rangle^2} \sim \frac{\langle \varepsilon^2 \rangle}{\langle \varepsilon \rangle^2} \sim \left(\frac{L}{\eta}\right)^{\mu} \sim Re^{3\mu/4}$$

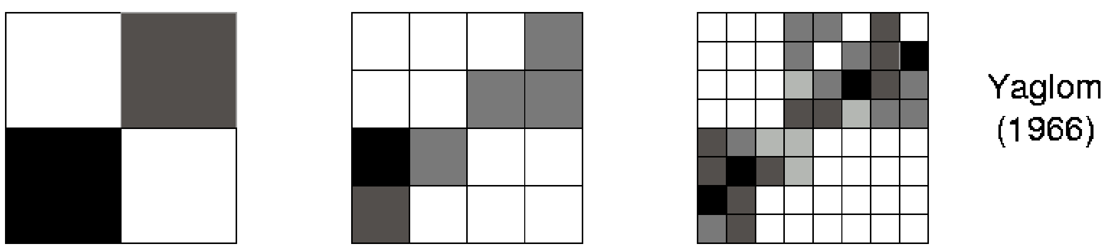
μ: the intermittency exponent

$\mu = 0.25 \pm 0.05$

Sreenivasan and Kailasnath, Phys. Fluids A **5** 512 (1993).

Multiplicative models of the turbulent cascade





<u>NS64</u>

At each step of the cascade, the dissipation is *randomly partitioned* over M cubes out of N

At scale $r/L = N^{-k/D}$ we have

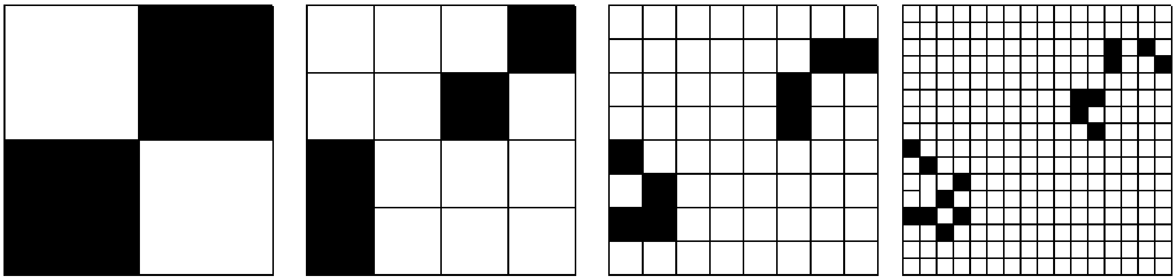
$$\langle \varepsilon_r^j \rangle = \varepsilon_0^j \left(\frac{N}{M}\right)^{kj} \left(\frac{M}{N}\right)^k = \langle \varepsilon \rangle^j \left(\frac{L}{r}\right)^{3(j-1)(1-\log_N M)}$$

<u>Y66</u>

At each step of the cascade, the dissipation is a *random fraction* of dissipation at the upper scale $\varepsilon_k = w_k \varepsilon_{k-1}$, with w_k i.i.d. r. v. (of unit mean)

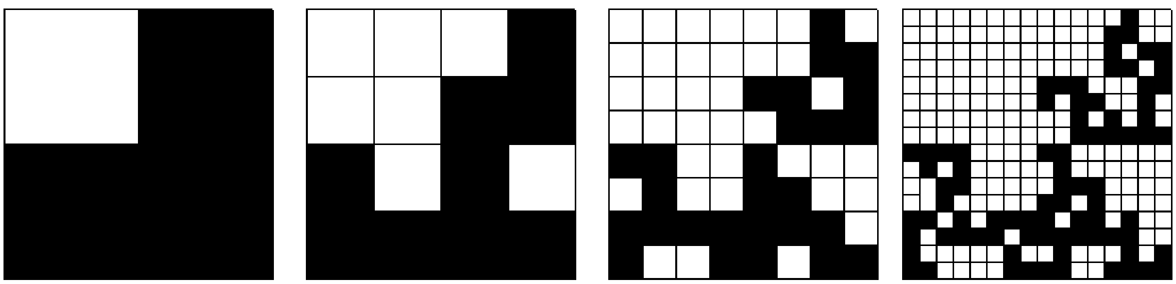
$$\langle \varepsilon_r^j \rangle = \varepsilon_0^j \langle w^j \rangle^k = \langle \varepsilon \rangle^j \left(\frac{L}{r}\right)^{3\log_N \langle w^j \rangle}$$

Fractals ...



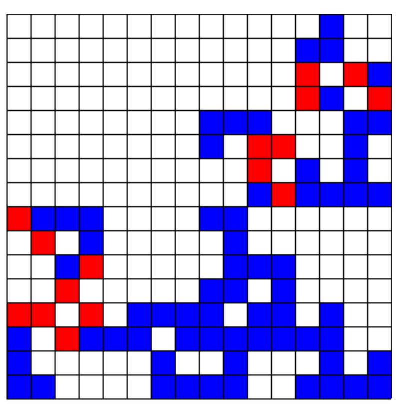
Probability of finding a black point in a box of size $r = 2^{-k}$:

$$P = \left(\tfrac{2}{4}\right)^k = r = r^{D-D_F} \qquad D_F = 1$$



$$P = \left(\tfrac{3}{4}\right)^k = r^{\log_2 \frac{4}{3}} = r^{D-D_F} \qquad D_F = 2 - \log_2 \tfrac{4}{3} \approx 1.585$$

... and multifractals



The multifractal model of intermittency

For inertial-range separations the Navier-Stokes equations admit the rescaling

$$\boldsymbol{x} \to \lambda \boldsymbol{x} \qquad \boldsymbol{u} \to \lambda^h \boldsymbol{u} \qquad t \to \lambda^{1-h} t$$

$\delta_r u = \boldsymbol{u}(\boldsymbol{x} + \frac{\boldsymbol{r}}{2}) - \boldsymbol{u}(\boldsymbol{x} - \frac{\boldsymbol{r}}{2}) \sim r^h$ for all points $\boldsymbol{x} \in S_h$
where S_h is a set of dimension $D(h)$

$$\langle (\delta_r u)^n \rangle \sim \int r^{nh} r^{D-D(h)} dh \sim r^{\inf_h [nh + D - D(h)]}$$

$D(h) = 3$ for $h = 1/3$ and $D(h) = 0$ elsewhere : K41

$$D(h) = 3 - \left[\frac{9}{2\mu} \left(h - \frac{1}{3} \right)^2 - \frac{3}{2} \left(h - \frac{1}{3} \right) + \frac{\mu}{8} \right] \quad : \text{KO62}$$

The advent of Direct Numerical Simulations

VOLUME 28, NUMBER 2 PHYSICAL REVIEW LETTERS 10 JANUARY 1972

Numerical Simulation of Three-Dimensional Homogeneous Isotropic Turbulence

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(Received 6 December 1971)

This Letter reports numerical simulations of three-dimensional homogeneous isotropic turbulence at wind-tunnel Reynolds numbers. The results of the simulations are compared with the predictions of the direct-interaction turbulence theory.

<u>Fast Fourier Transforms and Pseudo-spectral methods</u>

$$\dot{\hat{u}}(\boldsymbol{k},t) = -ik_\beta(\delta_{\alpha\gamma} - k_\alpha k_\gamma)\sum_{\boldsymbol{p}} \widehat{u}_\beta(\boldsymbol{p},t)\widehat{u}_\gamma(\boldsymbol{k}-\boldsymbol{p},t) - \nu k^2 \widehat{u}(\boldsymbol{k},t)$$

Convolutions cost $(N^3)^2$ operations !

Since $\sum_{\boldsymbol{p}} \widehat{u}_\beta(\boldsymbol{p},t)\widehat{u}_\gamma(\boldsymbol{k}-\boldsymbol{p},t) = \widehat{u_\beta u_\gamma}(\boldsymbol{k},t)$
use FFT forth and back: cost of FFT $N^3 \log N$

<u>In 1971</u> 64^3 points ($Re_\lambda \approx 35$) on NCAR CDC6600 computer took 30 sec/step

<u>In 2003</u> 1024^3 points ($Re_\lambda \approx 300$) on CINECA IBM SP-Power4 take 90 sec/step

Turbulence in flatland

vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$

$$\partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \nu \Delta \boldsymbol{\omega}$$

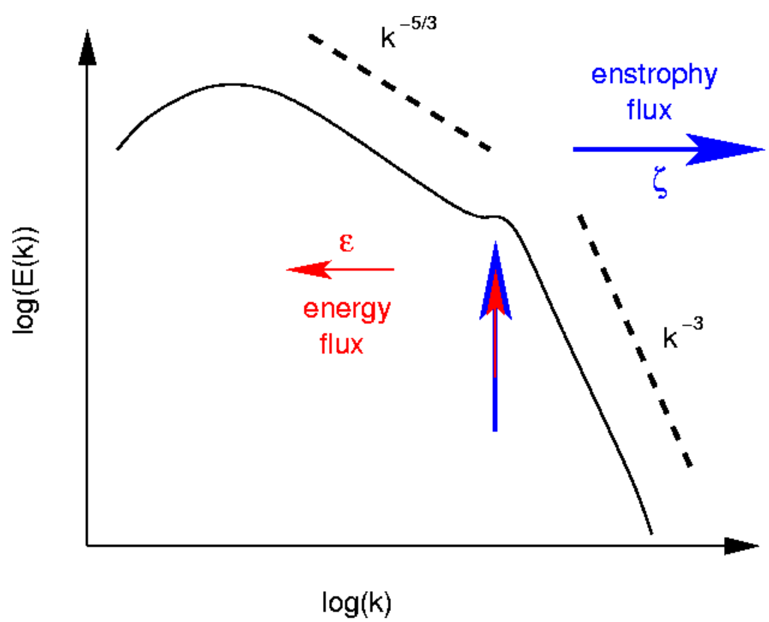
For planar flows

$$\boldsymbol{u} = (u_x(x,y,t), u_y(x,y,t), 0) \qquad \boldsymbol{\omega} = (0, 0, \omega(x,y,t))$$

$$\partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla}\omega = \nu \Delta \omega$$

Two inviscid invariants:
energy $E = \int |\boldsymbol{u}|^2 d\boldsymbol{x} = \int E(k)dk$
enstrophy $Z = \int \omega^2 d\boldsymbol{x} = \int k^2 E(k)dk$

The double cascade



R. Kraichnan, Phys. Fluids **10**, 1417 (1967)

Phenomenological arguments

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \qquad \text{for} \quad k \ll k_f$$

$$E(k) \sim \zeta^{2/3} k^{-3} \qquad \text{for} \quad k \gg k_f$$

Exact relations

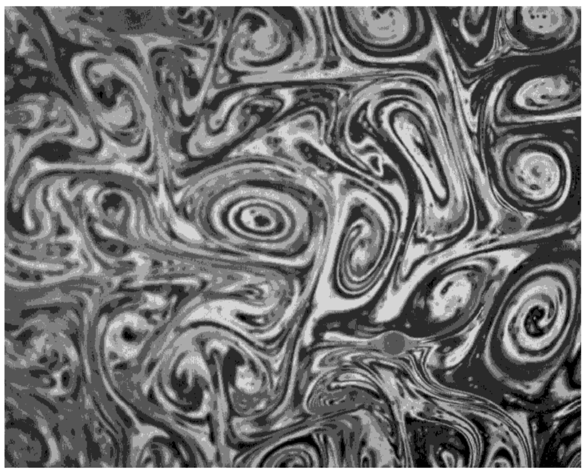
$$\langle (\delta_r u)^3 \rangle = \frac{3}{2} \varepsilon r$$

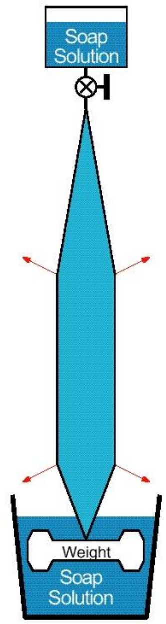
$$\langle \delta_r u (\delta_r \omega)^2 \rangle = -2 \zeta r$$

Is there intermittency in the inverse cascade ?

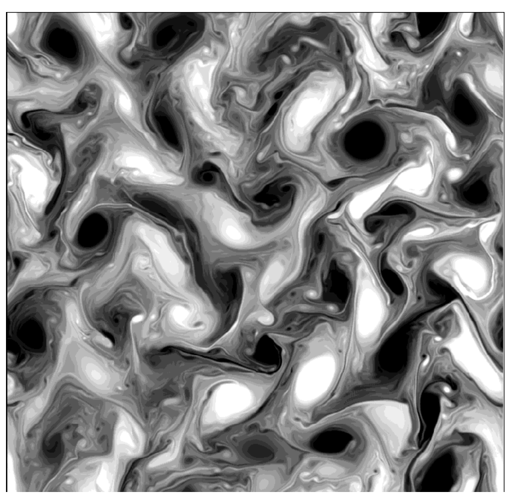
And in the direct cascade ?

Experiments





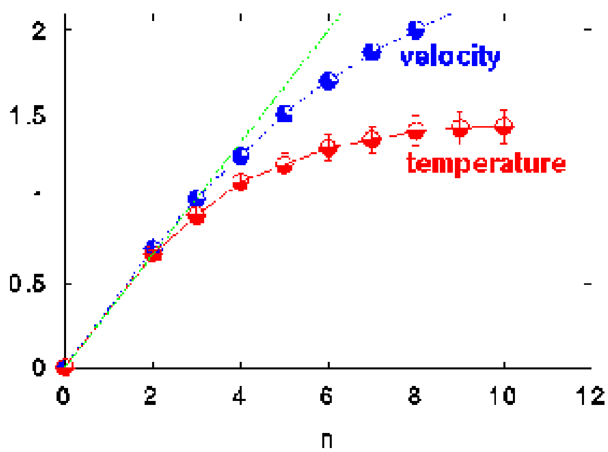
Numerics



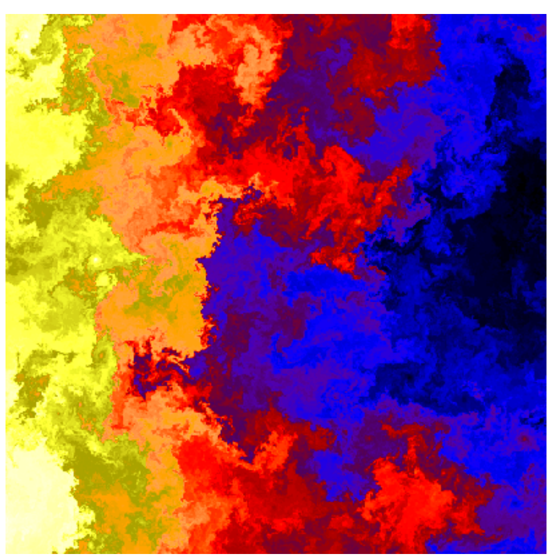
Passive scalar turbulence

$$\partial_t \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \kappa \Delta \theta + f$$

$$\langle (\delta_r \theta)^n \rangle \sim r^{\sigma_n}$$



Strong intermittency
of passive scalar



Ramps and cliffs

The Kraichnan model

$$\partial_t \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} \theta = \kappa \Delta \theta + f$$

- $\boldsymbol{u}$ Gaussian, statistically stationary, homogeneous, isotropic
- $\langle \boldsymbol{u} \rangle = \mathbf{0}$
- $\langle u_i(\boldsymbol{x},t) u_j(\boldsymbol{x}+\boldsymbol{r},t') \rangle = \left\{ D_0 \delta_{ij} - \frac{D}{2} r^\xi \left[(d-1+\xi)\delta_{ij} - \xi \widehat{r}_i \widehat{r}_j \right] \right\} \underline{\delta(t-t')}$
- f Gaussian, statistically stationary, homogeneous, isotropic
- $\langle f \rangle = 0$
- $\langle f(\boldsymbol{x},t) f(\boldsymbol{x}+\boldsymbol{r},t') \rangle = F(r/L_\theta)$

<u>Closed p.d.e.'s for scalar correlations</u>

$$C_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \langle \theta(\boldsymbol{x}_1,t) \cdots \theta(\boldsymbol{x}_n,t) \rangle$$

$$\partial_t C_n + \mathcal{M}_n C_n = F \otimes C_{n-2}$$

$$\mathcal{M}_n = \sum_{a<b}^{n} S(\boldsymbol{x}_a - \boldsymbol{x}_b) \frac{\partial}{\partial x_a^i} \frac{\partial}{\partial x_b^j}$$

The theory of passive scalar intermittency

From correlations to structure functions

$$\langle(\delta_r\theta)^4\rangle = C(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) - 4C(\boldsymbol{r},\mathbf{0},\mathbf{0},\mathbf{0}) + 6C(\boldsymbol{r},\boldsymbol{r},\mathbf{0},\mathbf{0}) - 4C(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},\mathbf{0}) + C(\boldsymbol{r},\boldsymbol{r},\boldsymbol{r},\boldsymbol{r})$$

A dead end road ?

$$\mathcal{M}_n C_n = F \otimes C_{n-2}$$

$$[\mathcal{M}_n] = \text{length}^{\xi-2} \Rightarrow [C_n] = \text{length}^{n(1-\frac{\xi}{2})}$$

Anomalous scaling and homogeneous solutions

$$C_n = C_n^{inhom} + C_n^{hom} \qquad \mathcal{M}_n C_n^{hom} = 0$$

- Scaling exponents can be computed from first principles (in some limiting cases)
- Universality of scaling exponents
- Extension to realistic velocity fields *via* the Lagrangian interpretation

Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995)
M. Chertkov *et al* Phys. Rev. E **52**, 4924 (1995)
B.I. Shraiman and E.D. Siggia, C. R. A. S. II **321**, 279 (1995)
O. Gat and R. Zeitak, Phys. Rev. E **57**, 5511 (1998)
D. Bernard *et al*, J. Stat. Phys. **90**, 519 (1998)
A. C. and M. Vergassola, Phys. Rev. Lett. **86**, 424 (2001)

References

- G. K. Batchelor, *The theory of homogeneous turbulence*, Cambridge University Press (1953)

- A. S. Monin and A. K. Yaglom, *Statistical Fluid Mechanics*, MIT Press (Vol.1 1971 – Vol.2 1975)

- H. Tennekes and J. Lumley, *A first course in turbulence*, MIT Press (1972)

- W. D. McComb, *The physics of fluid turbulence*, Oxford University Press (1992)

- U. Frisch, *Turbulence: the legacy of A. N. Kolmogorov*, Cambridge University Press (1995)

- R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 547 (1980)

- G. Falkovich, K. Gawędzki and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001)