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Random Motion in Random Fields

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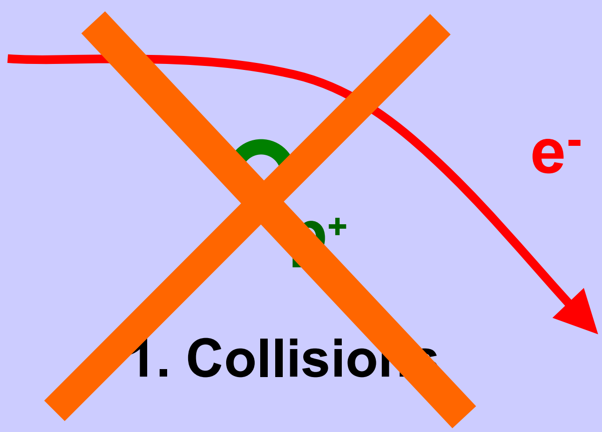
„Test“ particles

- Do not interact with each other
- Do not feed back to the plasma

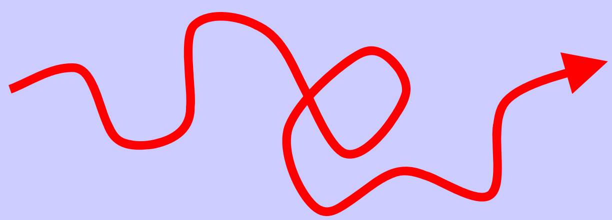
Natural applications:

- Rare species (He, O, ...)
- Collisionless high-energy tails

2 CLASSES OF FORCES:



2. Coarse-grained forces from local densities + currents



Coulomb Collisions

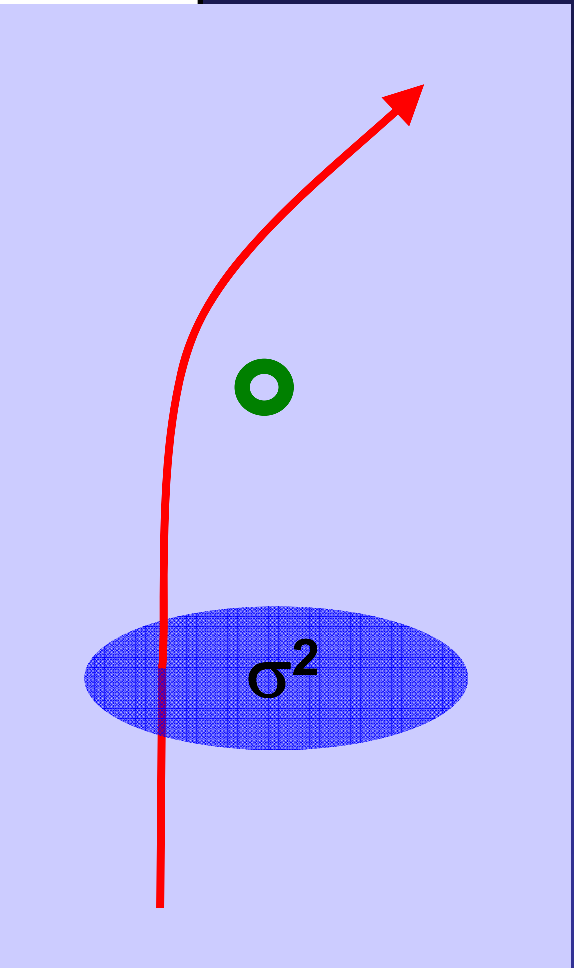
- Dynamics: $d^2\mathbf{r}/dt = k\mathbf{r}/|\mathbf{r}|^3$
- Scale $\mathbf{r}$ by α, and t by β:

  $\rightarrow$ $\alpha\beta^{-2}\, d^2\mathbf{r}/dt^2 = -\,\alpha^{-2}\, k\mathbf{r}/|\mathbf{r}|^3$

  $\rightarrow$ unchanged if $\alpha^3=\beta^2$ (Kepler).
- Since $v \sim (\alpha/\beta)$,

  $v^2\,\alpha$ = const. for <u>similar</u> orbits.
- Now, $\sigma \sim \alpha$ and thus $\boxed{\sigma^2 \sim v^{-4}}$
- Thus the collision rate

  $\boxed{\nu \sim n\,\sigma^2\, v \sim nv^{-3}}$ <u>dec</u>reases with v!

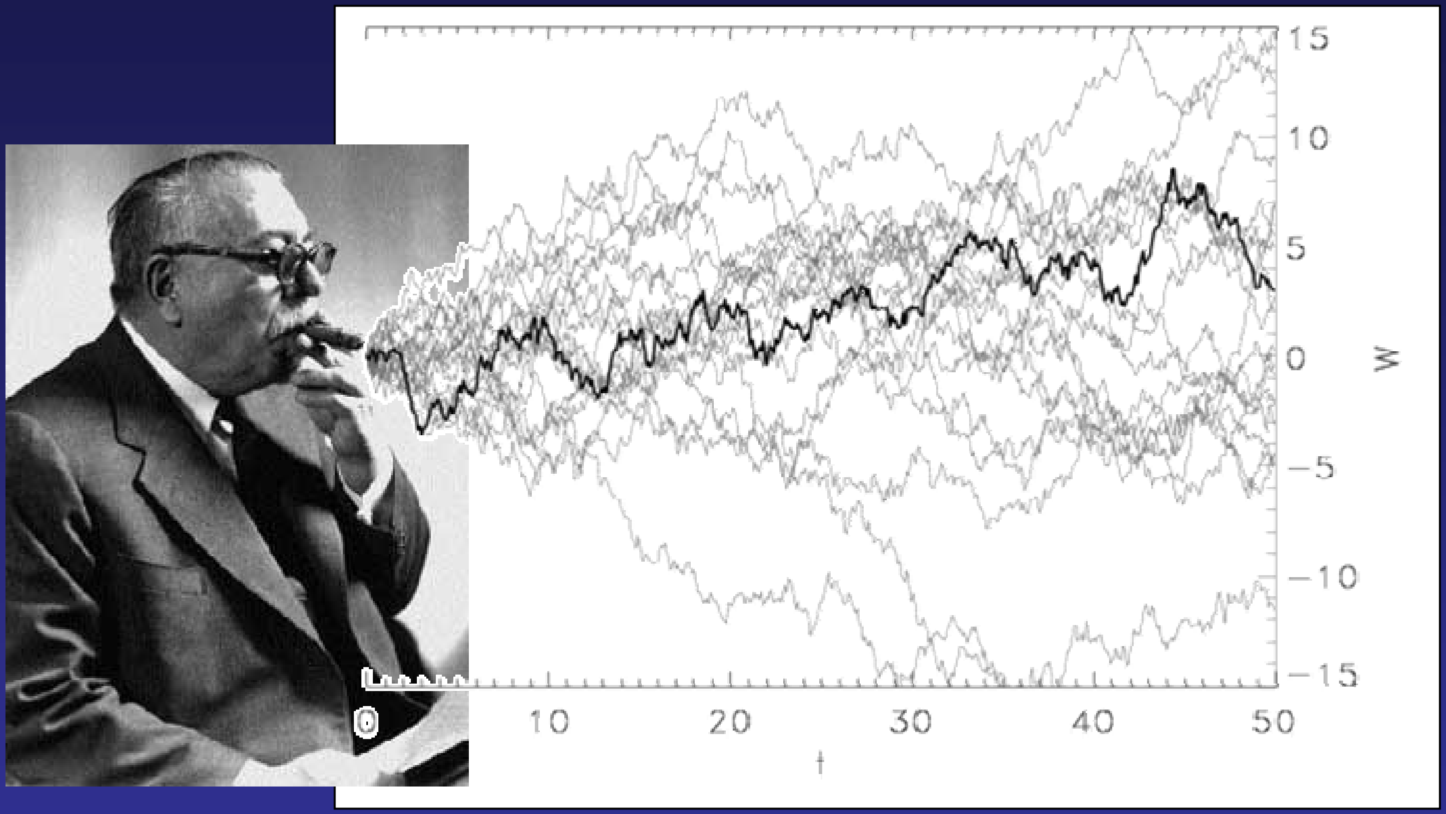


Basic Techniques for Test Particles

	Single particle: (easy)	Population: (more difficult)
exact	Ordinary differential equations (ODE): $d\mathbf{X} = F(\mathbf{X})dt$	$\{\partial_t + \partial_{\mathbf{x}} F(\mathbf{x})\}\, f(\mathbf{x},t) = 0$
approx.	Stochastic differential equations (SDE): $d\mathbf{X} = a(\mathbf{X})dt + b(\mathbf{X})d\mathbf{W}$	$\{\partial_t + \partial_{\mathbf{x}} A(\mathbf{x}) - \partial_{\mathbf{x}}^2 B(\mathbf{x})\}\, f(\mathbf{x},t) = 0$

Wiener process

Wiener Process (Brownian motion)

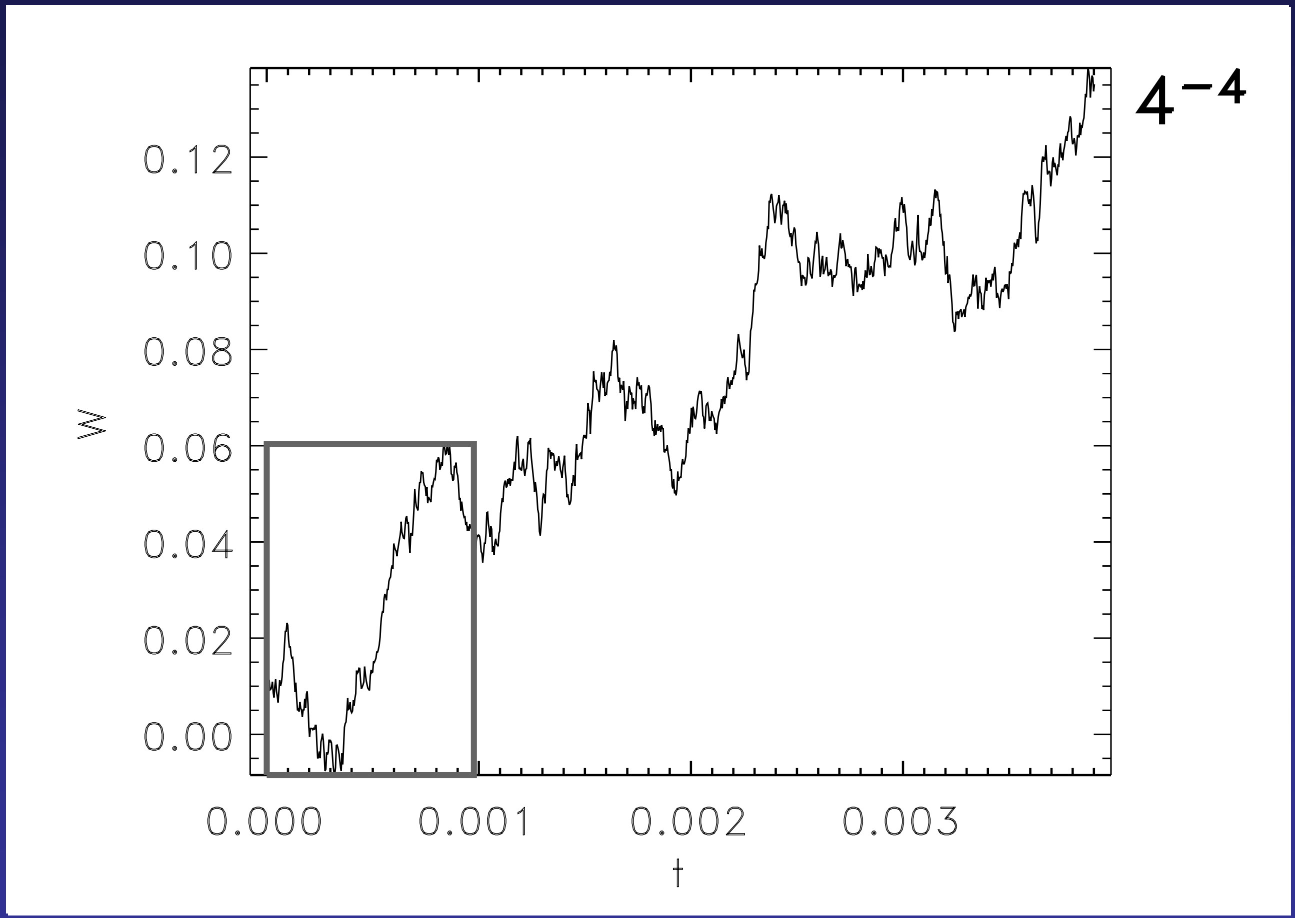


Numerical Sample paths:

$W_{n+1} = W_n + \Delta t^{1/2}\eta$
where η is standard normal

Average: $\langle W^2 \rangle = t$

In the limit $\Delta t \to 0$, $W(t)$ is continuous but nowhere differentiable:



So what does the stochastic „differential“ equation $dX = a(X)dt + b(X)dW$ really mean?

Stochastic Interpretation

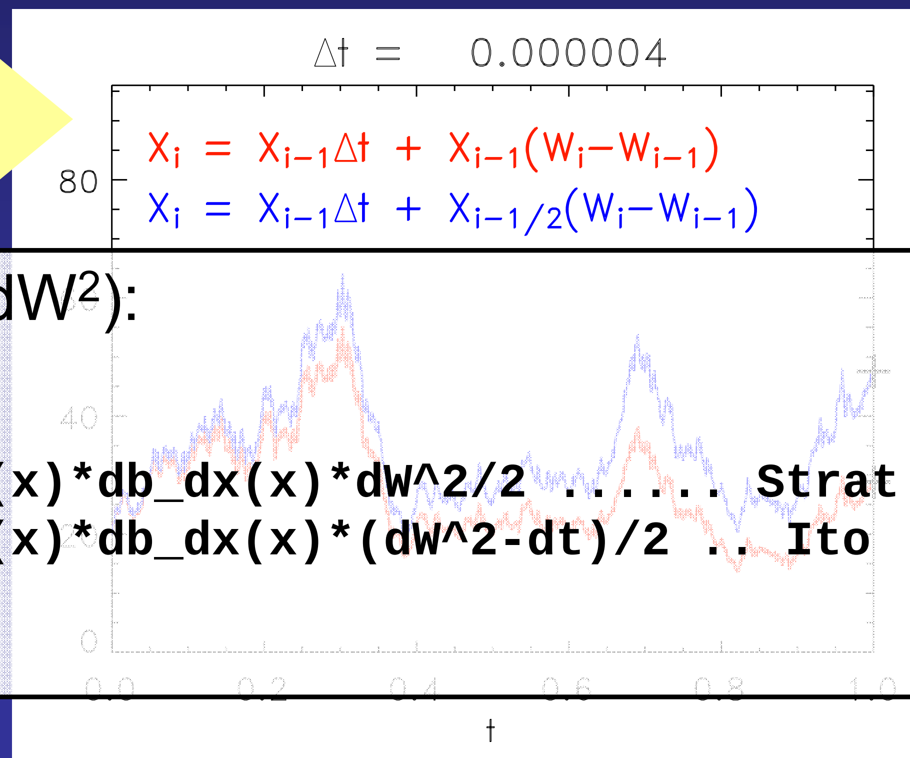
$$dX = a(X)\,dt + b(X)\,dW$$

At which X should b(X) be evaluated?

Itô: at X^-

Stratonovich: at $(X^- + X^+)/2$

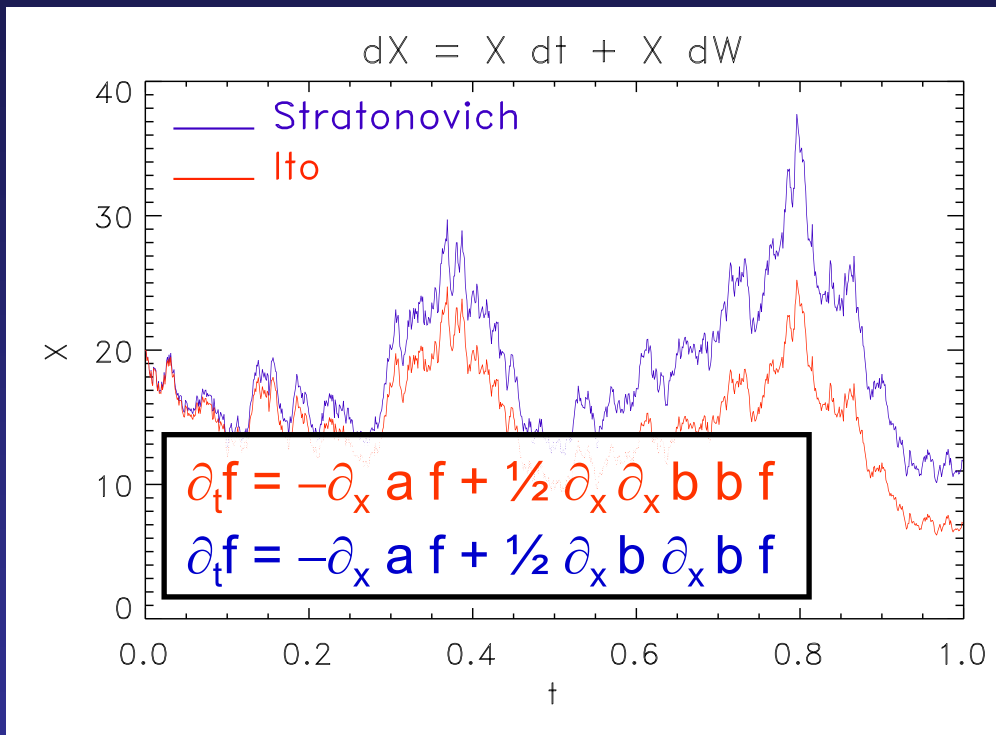
Example: $dX = X\,dt + X\,dW$
(leapfrog; same W as before)

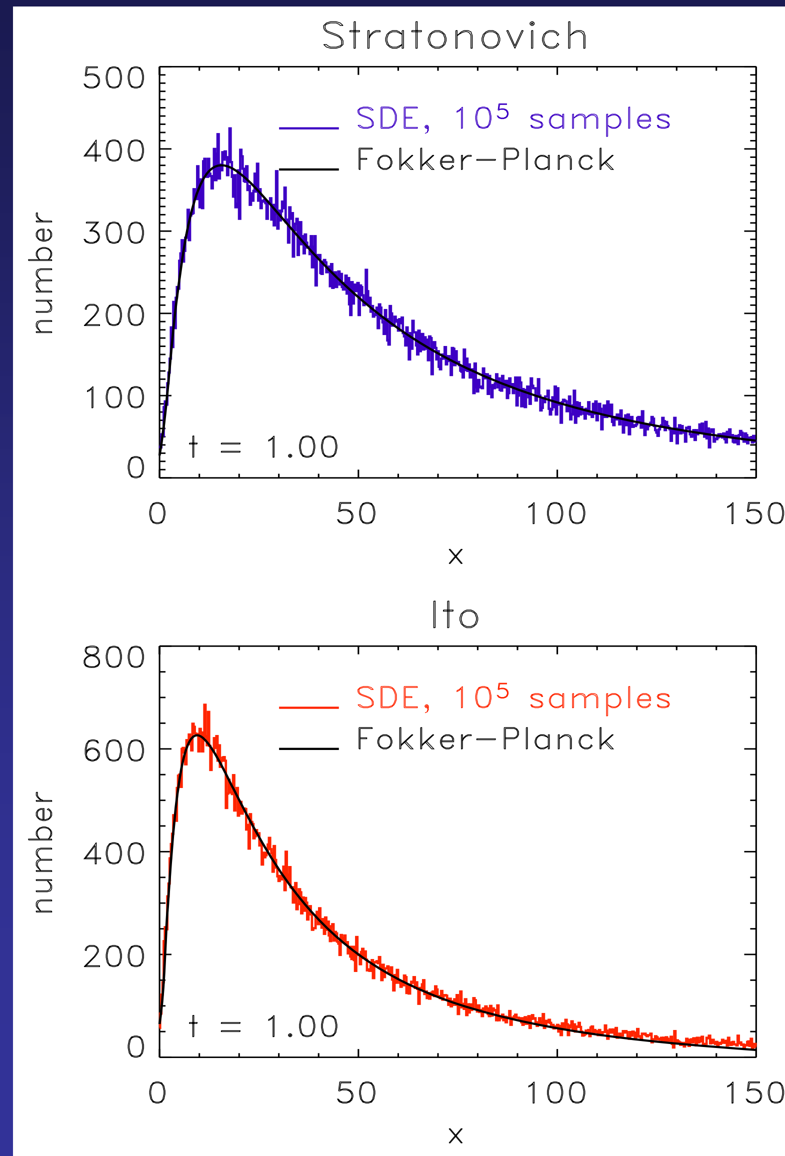


Milshtein (1974), to order dt ($=dW^2$):

```
do begin
 dW = sqrt(dt)*randomn(seed)
 x = x + a(x)*dt + b(x)*dW + b(x)*db_dx(x)*dW^2/2 ...... Strat.
 x = x + a(x)*dt + b(x)*dW + b(x)*db_dx(x)*(dW^2-dt)/2 .. Ito
 t = t + dt
end do
```

Itô and Stratonovich have the same diffusion but different drifts. The difference is not just cosmetic:

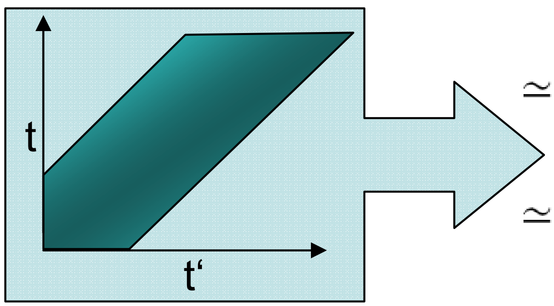




Classical physics (continuous sample paths, red noise approximated by white noise) usually leads to <u>Stratonovich</u>.

- Assume $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{f}(\mathbf{x})$, where $\mathbf{F}(\mathbf{x})$ is smooth compared to $\mathbf{f}(\mathbf{x})$, of which we know $\langle \mathbf{f}(\mathbf{x}) \rangle = 0$ and $\langle \mathbf{f}(\mathbf{x}_1)\, \mathbf{f}(\mathbf{x}_2) \rangle = \mathsf{C}(\mathbf{x}_2 - \mathbf{x}_1)$.

- Then, we may take $\mathbf{x}(0) = \mathbf{0}$ and argue that



$$
\begin{aligned}
\langle \mathbf{x}(t_1)\mathbf{x}(t_2) \rangle &= \int_0^{t_1} dt\, \mathbf{F}(\mathbf{x}(t)) \int_0^{t_2} dt'\, \mathbf{F}(\mathbf{x}(t')) + \int_0^{t_1} dt \int_0^{t_2} dt' \langle \mathbf{f}(\mathbf{x}(t)) \mathbf{f}(\mathbf{x}(t')) \rangle \\
&\simeq \int_0^{t_1} dt\, \mathbf{F}(\mathbf{X}(t)) \int_0^{t_2} dt'\, \mathbf{F}(\mathbf{X}(t')) + \int_0^{t_1} dt \int_0^{t_2} dt' \langle \mathbf{f}(\mathbf{X}(t)) \mathbf{f}(\mathbf{X}(t')) \rangle \\
&\simeq \mathbf{X}(t_1)\mathbf{X}(t_2) + \min(t_1, t_2) \underbrace{\int_{-\infty}^{\infty} d\tau\, \mathsf{C}(\mathbf{X}(\tau))}_{\doteq \mathsf{D}}
\end{aligned}
$$

where $\mathbf{X}(t)$ is the unperturbed orbit satisfying $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$. It was assumed that $\mathbf{X}(t_1) - \mathbf{X}(t_2) \simeq \mathbf{X}(t_1 - t_2)$ over the correlation length of $\mathbf{f}$, and that $\mathsf{C}(\mathbf{X}(\tau))$ decays rapidly. Thus, $\rightarrow$ requires suitable coordinates

- the field correlation C is mapped on a particle diffusion coefficient D.

- D and $\mathbf{F}$ are related to the (Stratonovich) SDE $d\mathbf{x} = \mathbf{a}\, dt + \mathbf{b}\, d\mathbf{W}$ according to $\mathbf{a} = \mathbf{F}$ and $\mathsf{D} = \mathbf{b}^T \mathbf{b}$. The latter decomposition is not unique in $d \geq 2$.

D $\rightarrow$ Fokker-Planck Equations

- **Space science applications:**

  - Cosmic rays (Parker, Jokipii, Schlickeiser, ...)

  - Particle acceleration in solar flares and in the solar wind (Miller, Petrosian, Park, Karimabadi, Le Roux, ...): diffusion in momentum space to higher and higher energies. Different wave modes (constraints on $S_{ij}(\mathbf{k})$) used for different particles.

- However, Fokker-Planck equations do not capture all features of the deterministic motion!
- Crude example: $d\mathbf{X}/dt = \mathbf{F}(\mathbf{X}) \pm \mathbf{f}(\mathbf{X})$. A change of sign does not affect $D \sim \langle \mathbf{f}\, \mathbf{f} \rangle$, and has thus no effect if $\langle \mathbf{f} \rangle = 0$.

More subtle ...

- Consider $\frac{d}{dt}(\mathbf{x}, \mathbf{v}) = (\mathbf{v}, \mathbf{F})$ with $\mathbf{F} = -\nabla\Phi(\mathbf{x})$
- If $\Phi(\mathbf{x})$ is a centered Gauss field, the mixed correlations $\langle v_i F_j \rangle$ vanish.
- If $\Phi(\mathbf{x})$ is isotropic, the velocity diffusion coefficient is

$$
\begin{aligned}
D_{ij}(\mathbf{v}) &= \frac{\partial}{\partial x'_i}\frac{\partial}{\partial x_j}\int dt\, \underbrace{\langle \Phi(\mathbf{x}')\Phi(\mathbf{x}+\mathbf{v}t)\rangle}_{[\mathcal{F}^{-1}S(\mathbf{k})](\mathbf{x}-\mathbf{x}'+\mathbf{v}t)}\Bigg|_{\mathbf{x}=\mathbf{x}'=0} \\
&= \int dt \int d\mathbf{k}\, k_i k_j e^{i\mathbf{k}\cdot\mathbf{v}t} S(|\mathbf{k}|) \\
&= \int_0^\infty dk\, k^2 S(k) \int_{4\pi} d\hat{\mathbf{k}}\, \hat{k}_i \hat{k}_j \int dt\, e^{itvk\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}} \\
&= \int dk \frac{k}{v} S(k) \int d\hat{\mathbf{k}}\, \hat{k}_j \hat{k}_j\, \delta(\hat{\mathbf{k}}\cdot\hat{\mathbf{v}}) \\
&= \int \frac{dk^2}{2v} S(k) \underbrace{\boxed{\left(\delta_{ij} - \hat{v}_i \hat{v}_j\right)}}_{\Pi_{ij}(\hat{\mathbf{v}})}
\end{aligned}
$$

projector along **v**

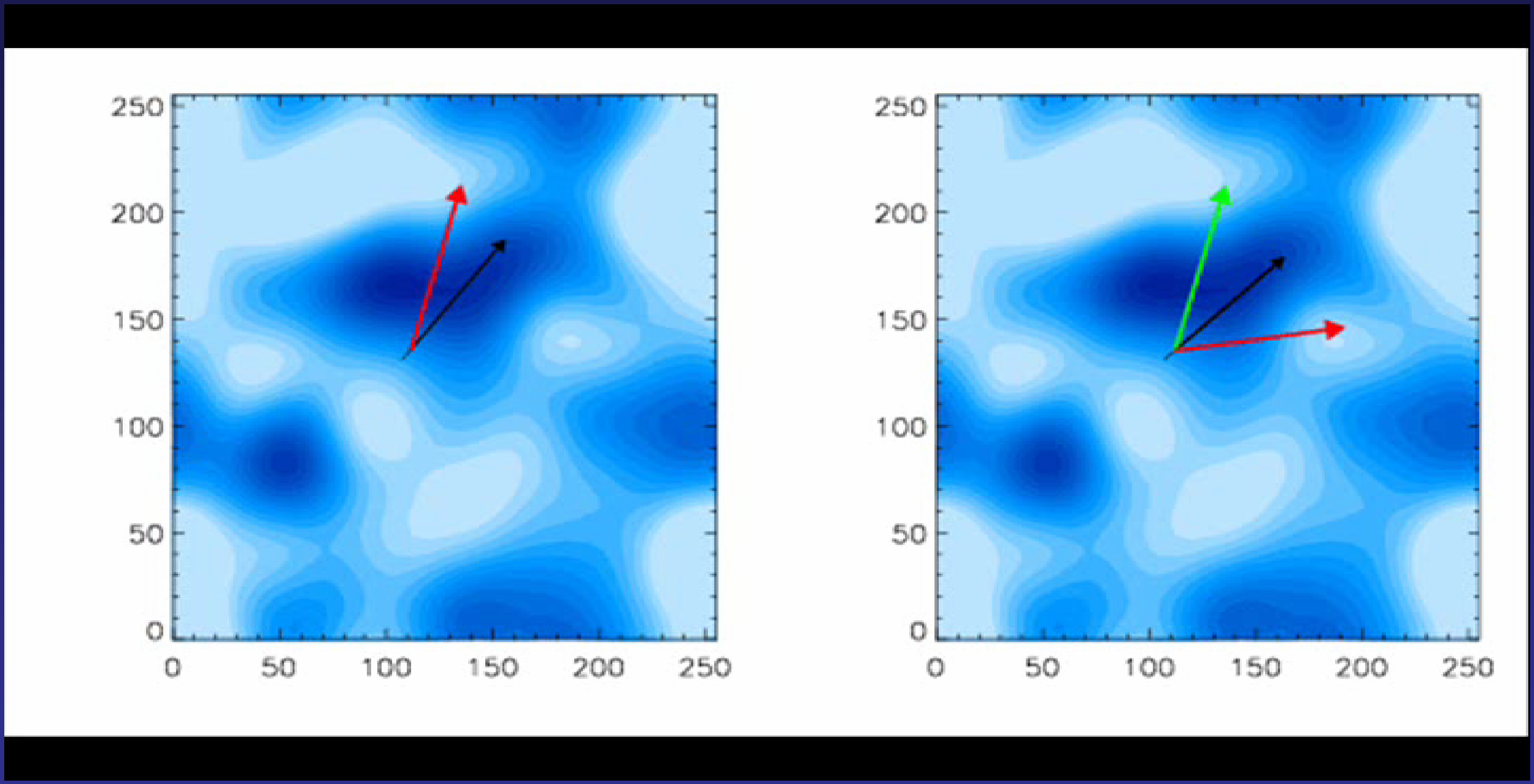
... Now ,

- Recall that $D_{ij} = \frac{c}{v}\Pi_{ij}(\hat{\mathbf{v}})$
- If $\mathbf{F} = -\nabla\Phi$ is replaced by $\mathbf{F}^* = R\,\mathbf{F}$, then the new diffision coefficient becomes

$$D^* = R\,D\,R^T\ .$$

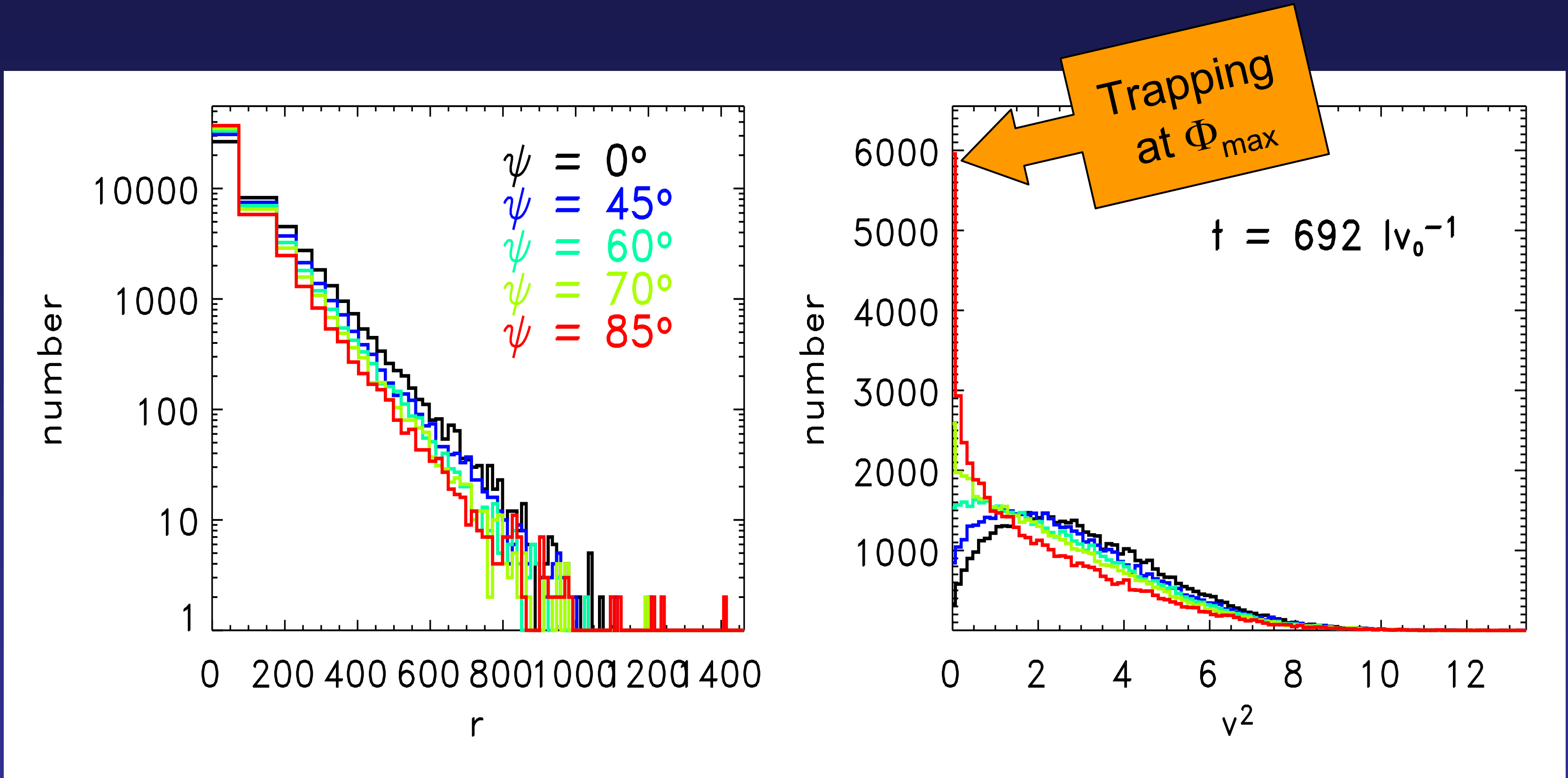
- Now, $D^* = D$ if $R^T R = 1$ and $[R, \Pi(\hat{\mathbf{v}})] = 0$, i.e. if R *is a rotation about* $\hat{\mathbf{v}}$.
- So we may modify Newton's force by rotating it around the actual velocity *without changing the velocity diffusion coefficient* if $\Phi(\mathbf{x})$ is isotropic.
- Although the energy $|\mathbf{v}|^2/2 + \Phi$ is still conserved, the orbits behave very differently!

2D case: discrete flip about **v**



$$\frac{d^2x}{dt^2} = -\nabla\Phi(x) \qquad \frac{d^2x}{dt^2} = -R(\hat{v})\nabla\Phi(x)$$

3D case: continuous rotation by ψ



Displacement **Velocity**

Remember, all ψ have the same formal velocity diffusion coefficients!

Thus,

There is need to resolve the full non-linear orbit dynamics!

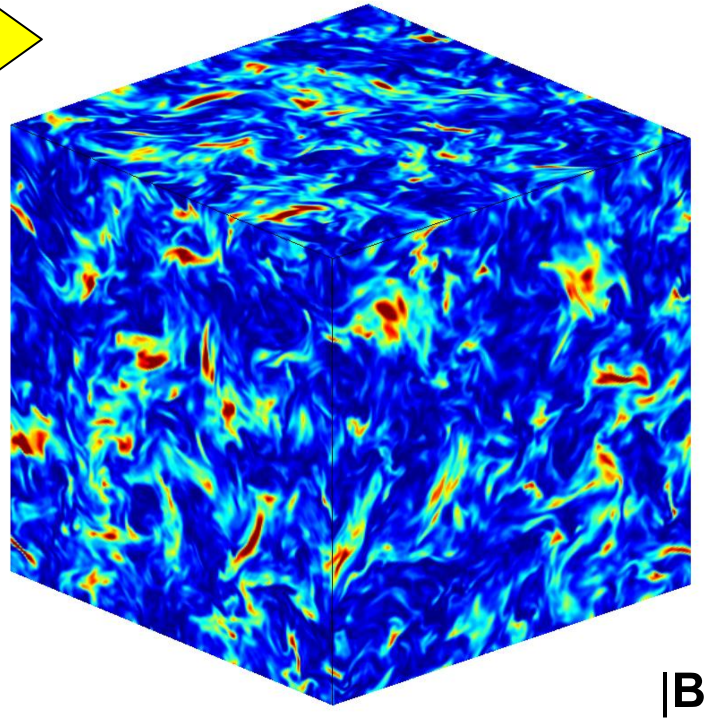
- ODE solvers: Forward-Euler, Leapfrog, Runge-Kutta Cash-Karp, Boris, symplectic, various gyrokinetic forms, etc. ...
- Benchmarking:

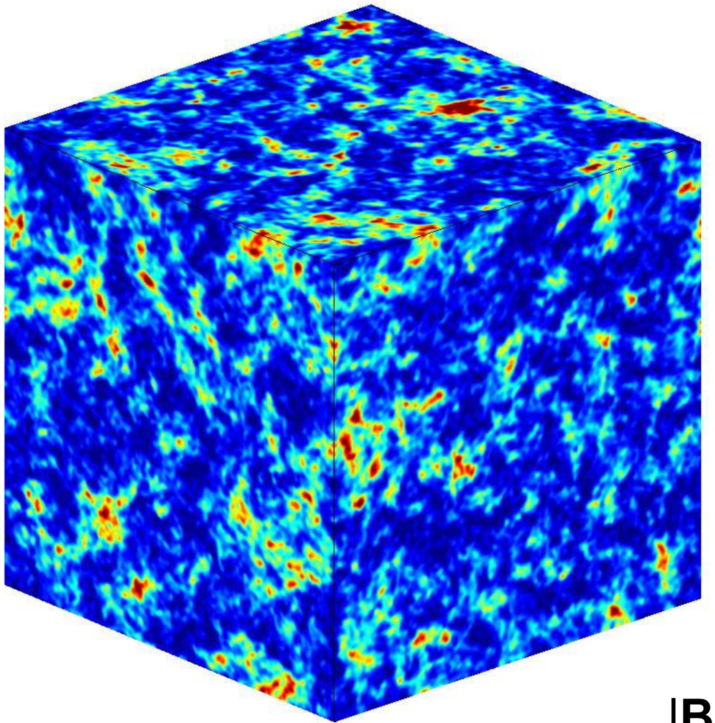
- Check convergence as dt $\rightarrow$ 0

- Systematic approach: enforce symmetry and check the corresponding invariant!

Construction of Turbulent Fields

Direct numerical simulations:





Gaussian (random-phase) proxies

Random-phase (Gauss field) proxies

- Let the fluctuations be collected in $\mathbf{z} = (\mathbf{u}, \mathbf{b}, \rho, \ldots)$
- Statistically homogeneous random-phase field:

$$\boxed{\mathbf{z}(\mathbf{x},t) = \Sigma_{\mathbf{k}\omega}\, \zeta(\mathbf{k},\omega) \cos\{\mathbf{x}.\mathbf{k} - \omega t + \psi(\mathbf{k},\omega)\}}$$

Gaussian with zero mean and covariance $S_{ij}(k)$

uniform in $[0,2\pi]$

- $S_{ij}(\mathbf{k})$ must respect the underlying physics,

$$S_{bb}(\mathbf{k},\omega).\mathbf{k} = 0 \qquad (\nabla.\mathbf{b} = 0),$$

and should account for the linearized dynamics ($\partial_t \mathbf{z} = \Lambda\mathbf{z}$):

$$S_{ij}(\mathbf{k},\omega) = 0 \quad \text{unless} \quad \det|i\omega - \Lambda(\mathbf{k})| = 0$$

$$S_{ij}(\mathbf{k},\omega).\xi = 0 \quad \text{unless} \quad (i\omega - \Lambda(\mathbf{k}))\xi = 0.$$

- <u>Example 1:</u> linearly polarized Alfvén waves: $\mathbf{z} = (\mathbf{u},\mathbf{b})$, $\omega = \mathbf{B}_0.\mathbf{k}$, and $\xi \propto (\mathbf{k} \times \mathbf{B}_0, \mathbf{k} \times \mathbf{B}_0)$.
- <u>Example 2</u>: linear force-free perturbations: $k^2 = \alpha^2$.
- $S_{ij}(\mathbf{k})$ should agree with observations.

Observational constraints

Matthaeus et al. (2005): magnetic single-time two-point functions of the solar wind, using multi-spacecraft observations (WIND, ACE, Cluster).

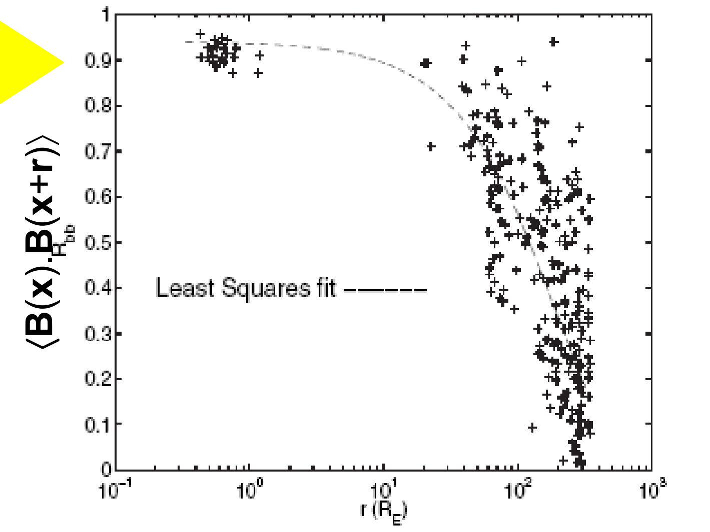
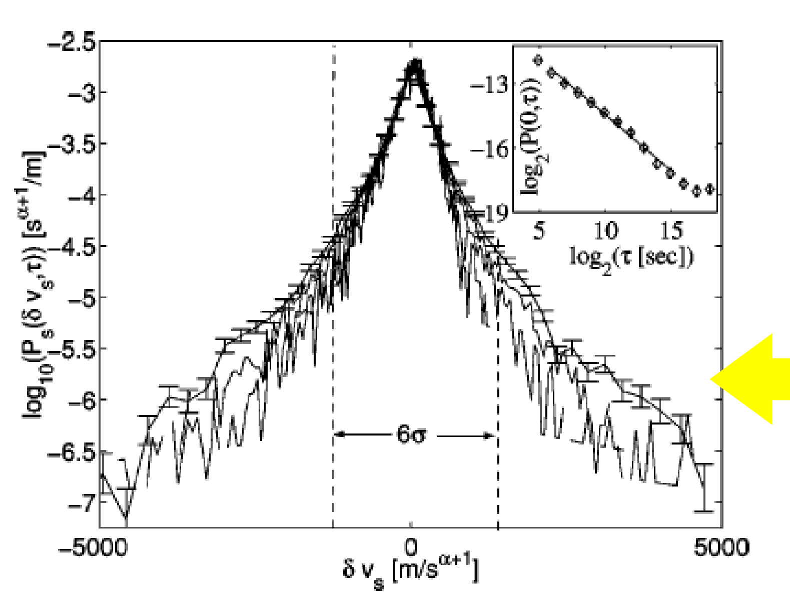


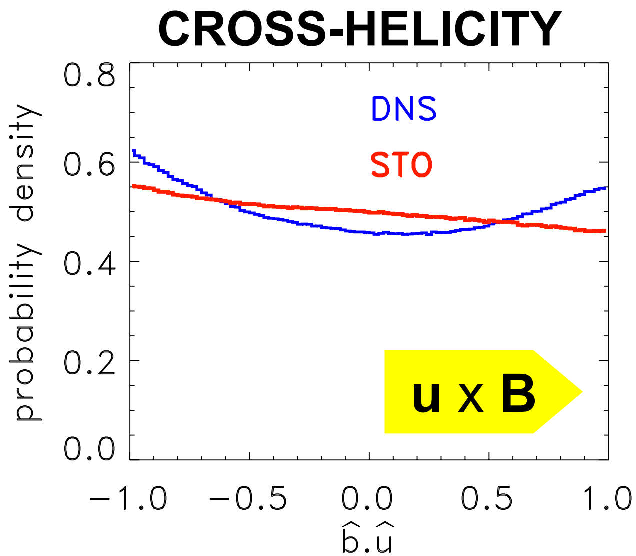
FIG. 3. Constrained exponential fit to ACE-Wind and Cluster set (2) data. This provides an estimate of $\lambda_c = 193 R_E$.

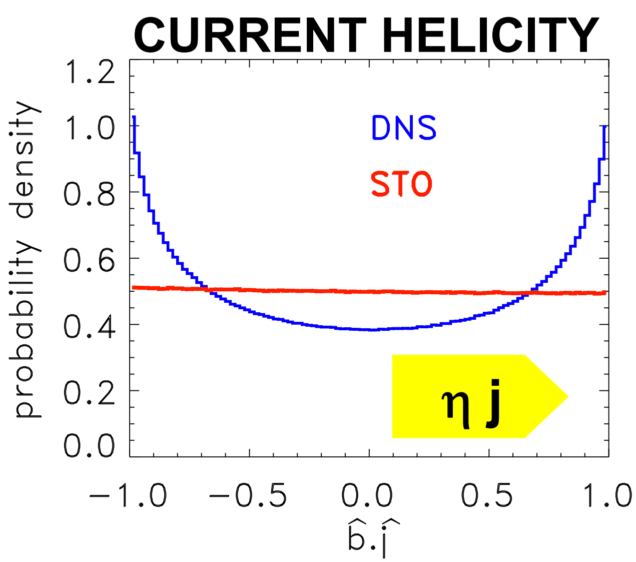


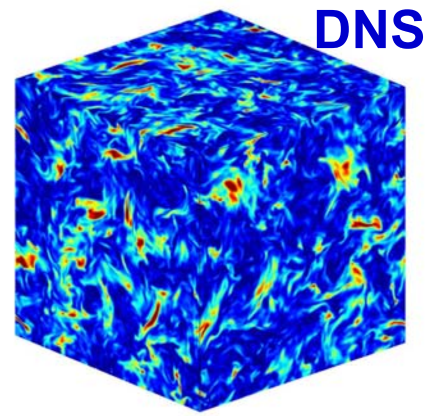
Hnat et al. (2003): distributions of $X(t+\tau)-X(t)$ with $X = |\mathbf{B}|$, $|\mathbf{v}|$, B^2, v^2, ρv^2 (WIND data); scales with $\tau^{-\alpha}$.

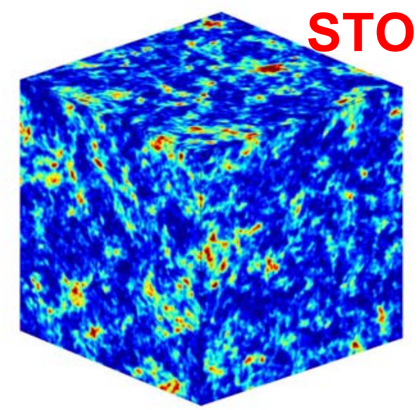
Numerical experiments: non-gaussian PDF's (Sorriso-Valvo et al. 1999, 2000) and structure functions (Politano et al. 1998).

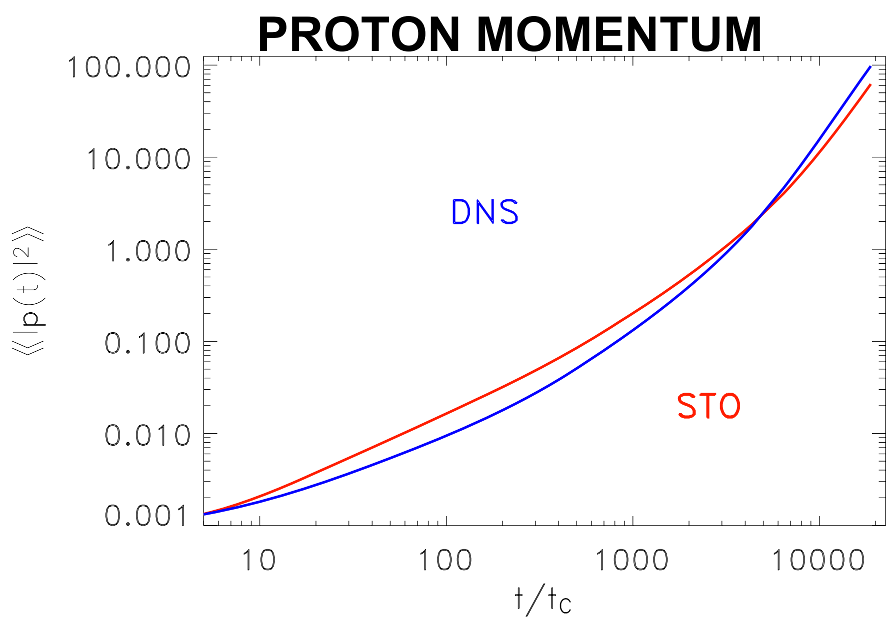
Effect of coherent MHD structures











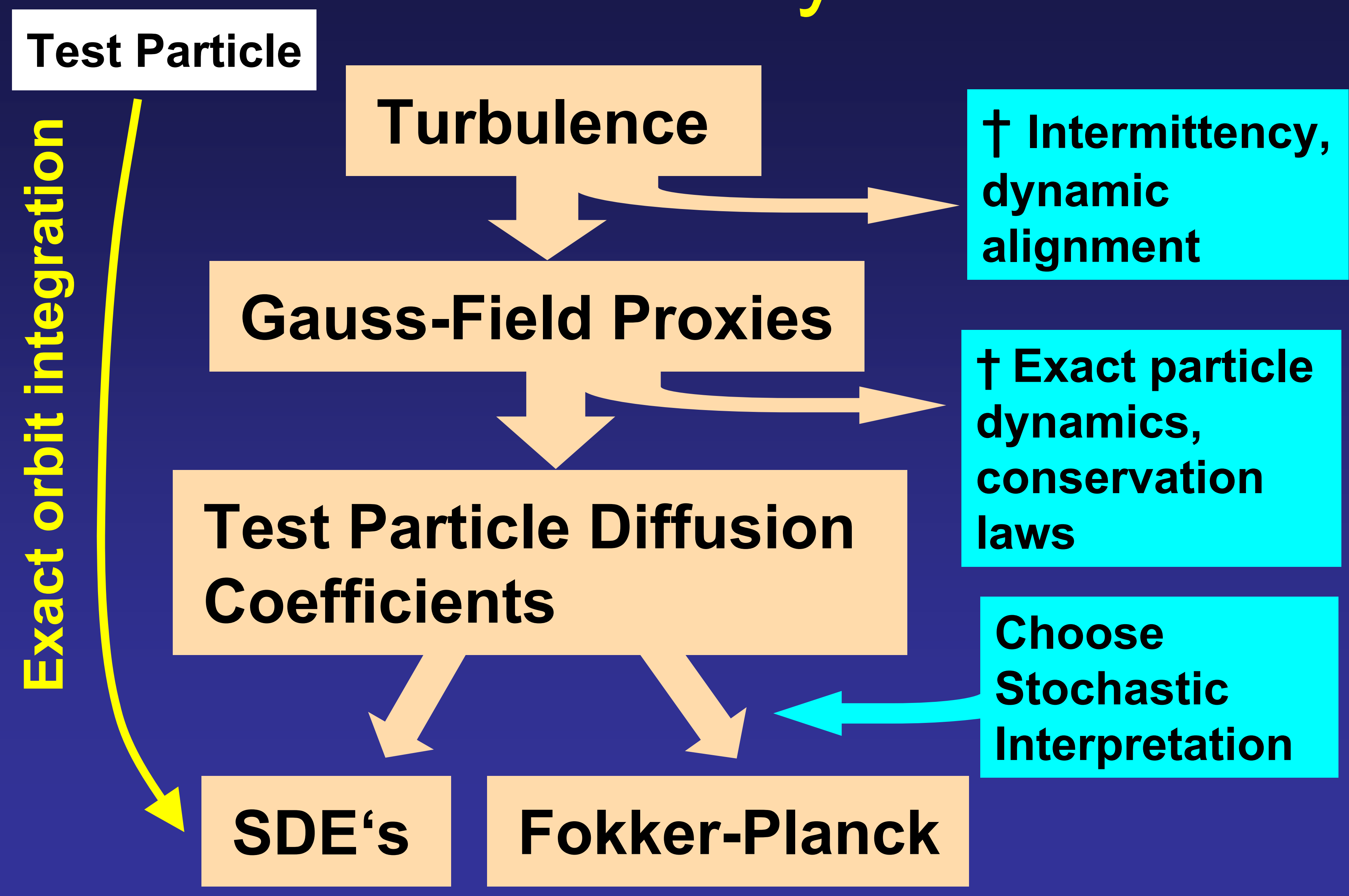
Arzner et al., 2006

Diagnostics: Radiation into vacuum from a general orbit x(t)

$$|E(\omega)|^2 = \int \langle v_\perp(0).v_\perp(\tau)\, e^{i\omega\tau - ik.x(\tau)}\rangle\, d\tau$$

... Inj this way, the particle two-point functon maps onto the power spectral density of the observed electromagnetic modes.

Summary


Test Particle
Exact orbit integration
Turbulence
† Intermittency, dynamic alignment
Gauss-Field Proxies
† Exact particle dynamics, conservation laws
Test Particle Diffusion Coefficients
Choose Stochastic Interpretation
SDE's
Fokker-Planck