Code performances in subsonic flows of an SPH scheme based on a matrix approach (Integral Approximation)

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HOT SPOTS IN THE XMM SKY : Cosmology from X-ray to Radio

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Introduction

- In the hierarchical scenario galaxy clusters are the most recent and massive objects known in the universe
- The dependency of their properties on cosmological parameters renders these objects very useful as cosmological probes
- Upcoming X-ray (eROSITA) and SZ cluster surveys (ACT/SPT,Planck) will provide multi-wavelength informations complementary to CMB and SNe data
- It will be possible to fully exploit these data and to put constraints on cosmological parameters (dark energy EOS ,m_ν , ...) provided that N-body/hydrodynamical codes can be faithfully used to model their evolution
- However, it is well known that there are inconsistencies between the results (core entropies) of non-radiative simulations predicted using SPH and AMR codes

DIFFICULTIES WITH SPH

- These discrepancies are part of a more general set of problems which are present in standard SPH

- In several hydrodynamical test cases the SPH results are inconsistent with those found using mesh-based codes (see refs in Hopkins 15)

There are two distint problems which affects standard SPH

- Suppression of fluid-mixing at fluid interfaces $\rightarrow$ unable to follow the growth of Kelvin-Helmholtz (KH) instabilities and others

- Zeroth-order errors in the momentum equation due to discreteness effects $\rightarrow$ noise $\rightarrow$ relevant in subsonic flows

Because SPH is a Lagrangian code it possesses very good conservation and resolution properties

IS IT POSSIBILE TO RESCUE SPH ?

A partial list of recently proposed SPH variants:

Alternative (pressure based) SPH equatons:
Read +10, Hopkins 13, Saitoh & Makino 13, Hu +14

Godunov-SPH :
Inutsuka 02, Cha +10, Murante + 11

Alternative switches: Read & Hayfield 10

Artificial Conductivity (AC):
Wadsley +08, Price 08, V12, Huber +11, Beck + 16

SPH COMPETITORS:

Moving-mesh codes (Arepo): Springel 10, Pakmor +16

Lagrangian – Godunov FV (Gizmo) : Hopkins 15

AMR & Garlekin : Schaal +15

CONSISTENCY ERRORS IN SPH

In SPH a field quantity is smoothed through a convolution with a kernel function $W(\vec{x})$

$$A_s(\vec{x}) = \int A(\vec{x}')W(|\vec{x}' - \vec{x}|, h)d^3x'$$

For a set of N points integrals are replaced by summations:

$$A_d(\vec{x}_i) = \sum_k \frac{m_k}{\rho_k} A_k W(|\vec{x}_k - \vec{x}_i|, h_i)$$

N particles , N_{nb} neighhbors, $W = 0 \;\; if \;\; |\vec{x}_k - \vec{x}_i| > \zeta h_i$

The continuum is recovered in the limit $N \to \infty \;\; N_{nb} \to \infty \;\; h_i \to 0$
(Zhu + 15)

Discreteness errors are introduced for finite values of N and N_{nb}

We Taylor expand $A(\vec{x}_k)$ around $\vec{x}_i$:

$$A_i = \sum_k \frac{m_k}{\rho_k} A_k W_{ik} = A_i \sum_k \frac{m_k}{\rho_k} W_{ik} + \vec{\nabla} A_i \cdot \sum_k \frac{m_k}{\rho_k} (\vec{x}_k - \vec{x}_i) W_{ik} + O(h^2)$$

For a costant function, the conditions

$$\left\{ \begin{array}{rcl} \sum_k \frac{m_k}{\rho_k} W_{ik} & \simeq & 1 \\ \sum_k \frac{m_k}{\rho_k} (\vec{x}_k - \vec{x}_i) W_{ik} & \simeq & 0 \end{array} \right.$$

must be satisfied

The errors depend on the particle distribution and impact on the Euler equations, this is the so called E_0 error (Read + 10)

In SPH the Euler equations become

$$\begin{cases} \dfrac{d\rho_i}{dt} & \simeq & -\rho_i(\mathcal{R}_i\vec{\nabla})\cdot\vec{v}_i + O(h) \\ \dfrac{d\vec{v}_i}{dt} & \simeq & -\dfrac{P_i}{h\rho_i}\vec{E}_i^0 - \dfrac{(\mathcal{V}_i\vec{\nabla})P_i}{\rho_i} + O(h) \end{cases}$$

where $\mathcal{R}_i,\ \mathcal{V}_i \simeq \mathcal{I}$ and

$$\vec{E}_i^{\,0} = \sum_j \frac{m_j}{\rho_j}\left[\frac{\rho_i}{\rho_j} + \frac{\rho_j}{\rho_i}\right] h_i \vec{\nabla}_i \bar{W}_{ij}$$

If N_{nb} is kept fixed when $N \rightarrow \infty$ the errors will produce noise which will dominate in the presence of cold flows

A naive approach would be to modify the equations so to have 'exact' gradients, but this will destroy the conservation properties of the Lagrangian (Price 12)

If N_{nb} becomes large, for certain kernel choices there will be instabilities (B-splines). It is now common practice to use Wedland kernels

GRADIENTS THROUGH INTEGRALS (Garcia-Senz +12)

Let us define

$$I_\alpha = \int \left[f(\vec{x}') - f(\vec{x})\right] (\vec{x}' - \vec{x})_\alpha W(|\vec{x}' - \vec{x}|, h) d^3x'$$

by Taylor expanding $f(\vec{x}\,')$

$$f(\vec{x}') - f(\vec{x}) \simeq (\vec{x}' - \vec{x}) \cdot \vec{\nabla} f + h.o.t.$$

we define

$$\tau_{\alpha\beta} = \int (x'_\alpha - x_\alpha)(x'_\beta - x_\beta) W(|\vec{x}' - \vec{x}|, h) d^3x'$$

so that we have now

$$\boxed{I_\alpha \simeq \tau_{\alpha\beta}(\vec{\nabla} f)_\beta \rightarrow (\vec{\nabla} f)_\alpha = [\tau]^{-1}_{\alpha\beta}\, I_\beta}$$

In SPH the above equations become

$$\begin{aligned}\tau_{\alpha\beta}(i) &= \sum_j \frac{m_j}{\rho_j}(x_j - x_i)_\alpha (x_j - x_i)_\beta W_{ij}(h_i) \\ &\equiv \sum_j \frac{m_j}{\rho_j}\Delta_\alpha^{ji}\Delta_\beta^{ji} W_{ij}(h_i)\end{aligned}$$

To evaluate I_α we now assume

$$Q_2 : \quad \sum_k m_k(\vec{x}_k - \vec{x}_i)W_{ik}(h_i) \simeq 0$$

In such a case

$$I_\beta(i) = \sum_k \frac{m_k}{\rho_k} f_k \Delta_\beta^{ki} W_{ik}(h_i)$$

It can be shown that standard SPH equations are replaced according to the prescriptions

$$\begin{cases} [\nabla_i W_{ik}(h_i)]_\alpha & \rightarrow \quad \sum_\beta C_{\alpha\beta}(i)\Delta^{ki}_\beta W(r_{ik}, h_i) \\ [\nabla_i W_{ik}(h_k)]_\alpha & \rightarrow \quad \sum_\beta C_{\alpha\beta}(k)\Delta^{ki}_\beta W(r_{ik}, h_k) \end{cases}$$

where $\mathcal{C} = \mathcal{T}^{-1}$

Because of the approximation Q2 for linear functions gradient estimates are no longer exact , but Q2 is crucial to ensure gradient antisymmetry in the pair ij and thus conservation properties

The new scheme has been carefully tested by Garcia-Senz + 12, Rosswog 15

Here we present a suite of hydrodynamical tests aimed at exploring the performances of IA-SPH in subsonic flows

Specifically, we consider:

- The Gresho-Chan vortex problem
- Driven subsonic turbulence

The Gresho-Chan vortex problem

A fluid is set in differential rotation with $\rho = 1$
Pressure gradients are balanced by centrifugal forces
The fluid is stationary with $V_\varphi(R)$

Sampling (E0) errors $\longrightarrow$ noise $\longrightarrow$ particle disorder $\longrightarrow$ AV $\longrightarrow$ transport of angular momentum $\longrightarrow$ very difficult for SPH to keep $V_\varphi(R)$ unaltered

Initial conditions set-up : NxNx16 particles arranged in a HCP lattice - periodic BC $0 \le x, y < 1$ $\quad r = \sqrt{x^2 + y^2}$

(AV=Artificial Viscosity)

The velocity and pressure profiles are

$$v_\phi(r) = \begin{cases} 5r & 0 \le r \le 0.2 \\ 2 - 5r & 0.2 \le r \le 0.4 \\ 0 & 0.4 \le r \end{cases}$$

$$\begin{cases} P_0 = (\gamma M^2)^{-1} \\ \gamma = 5/3 \\ M = 0.34 \quad P_0 \simeq 5 \end{cases}$$

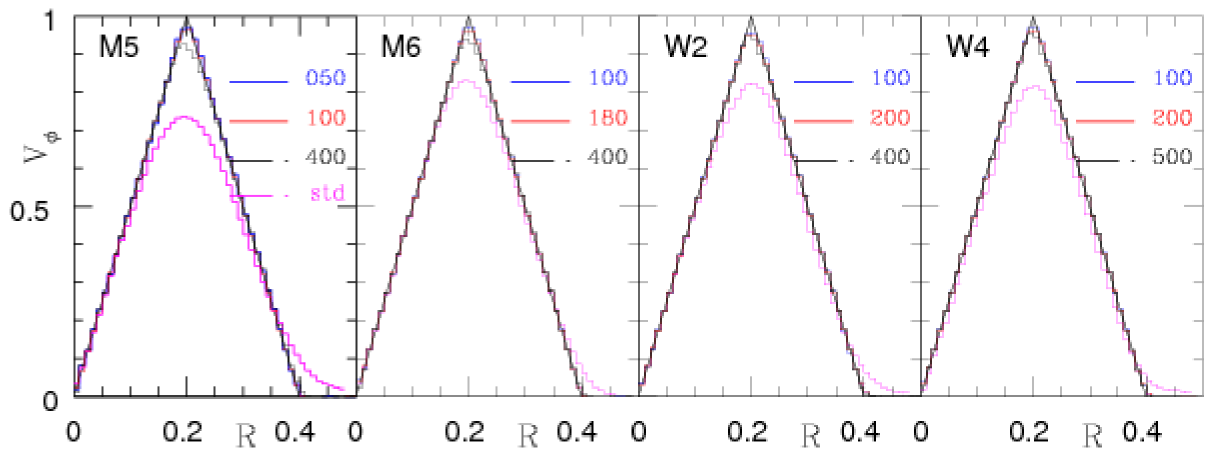
$$P(r) = P_0 + \begin{cases} 12.5r^2 & 0 \le r \le 0.2 \\ 12.5r^2 + 4 - 20r + 4ln(5r) & 0.2 \le r \le 0.4 \\ 2(ln(2) - 1) & 0.4 \le r \end{cases}$$

The standard case is M=0.34 – errors are quantified using the L1 norm

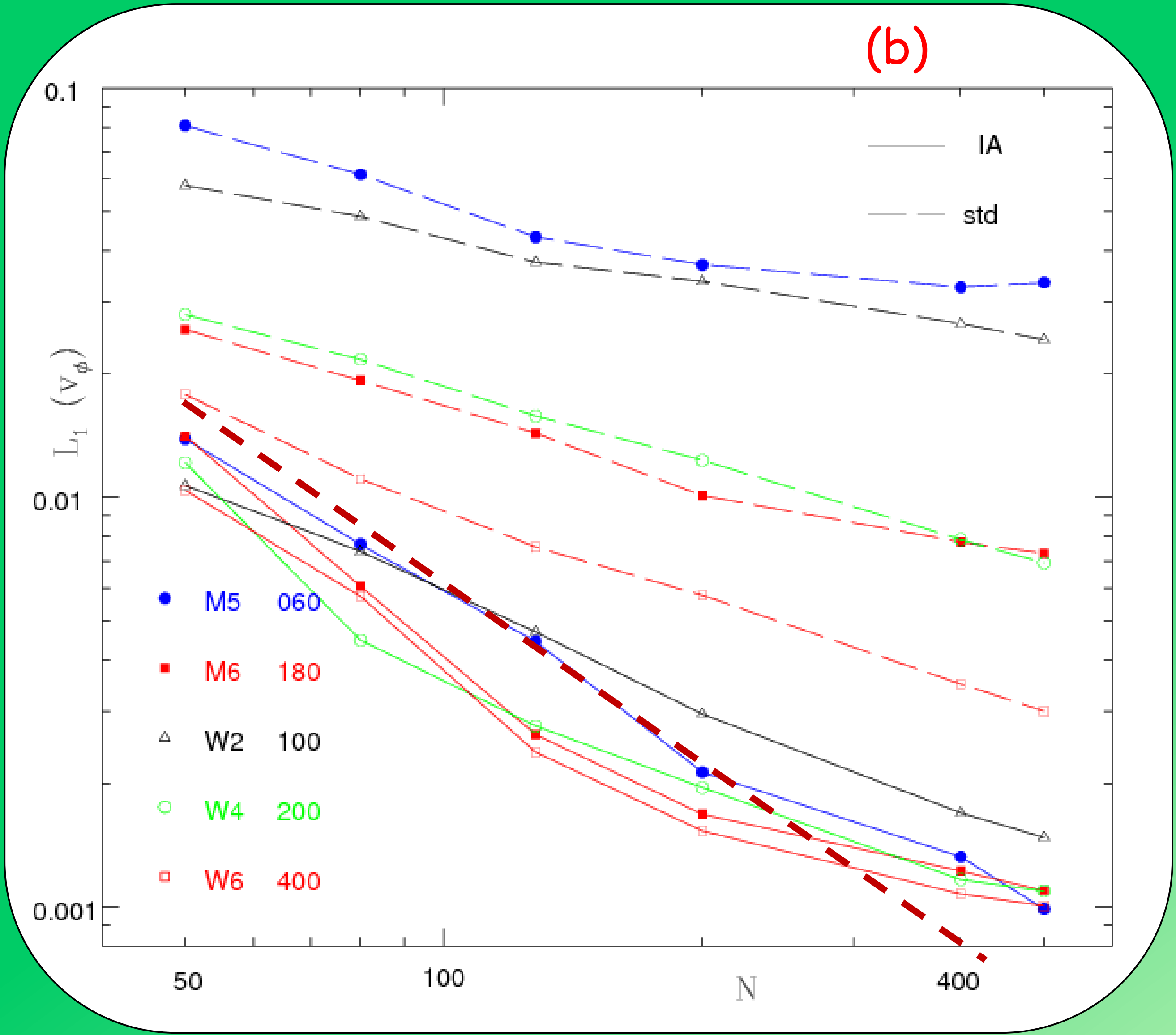
$$L1(v_\phi) = \frac{1}{N_b} \sum_i^{N_b} |\overline{v}_\phi(i) - v_\phi(r_i)|$$

We next show :

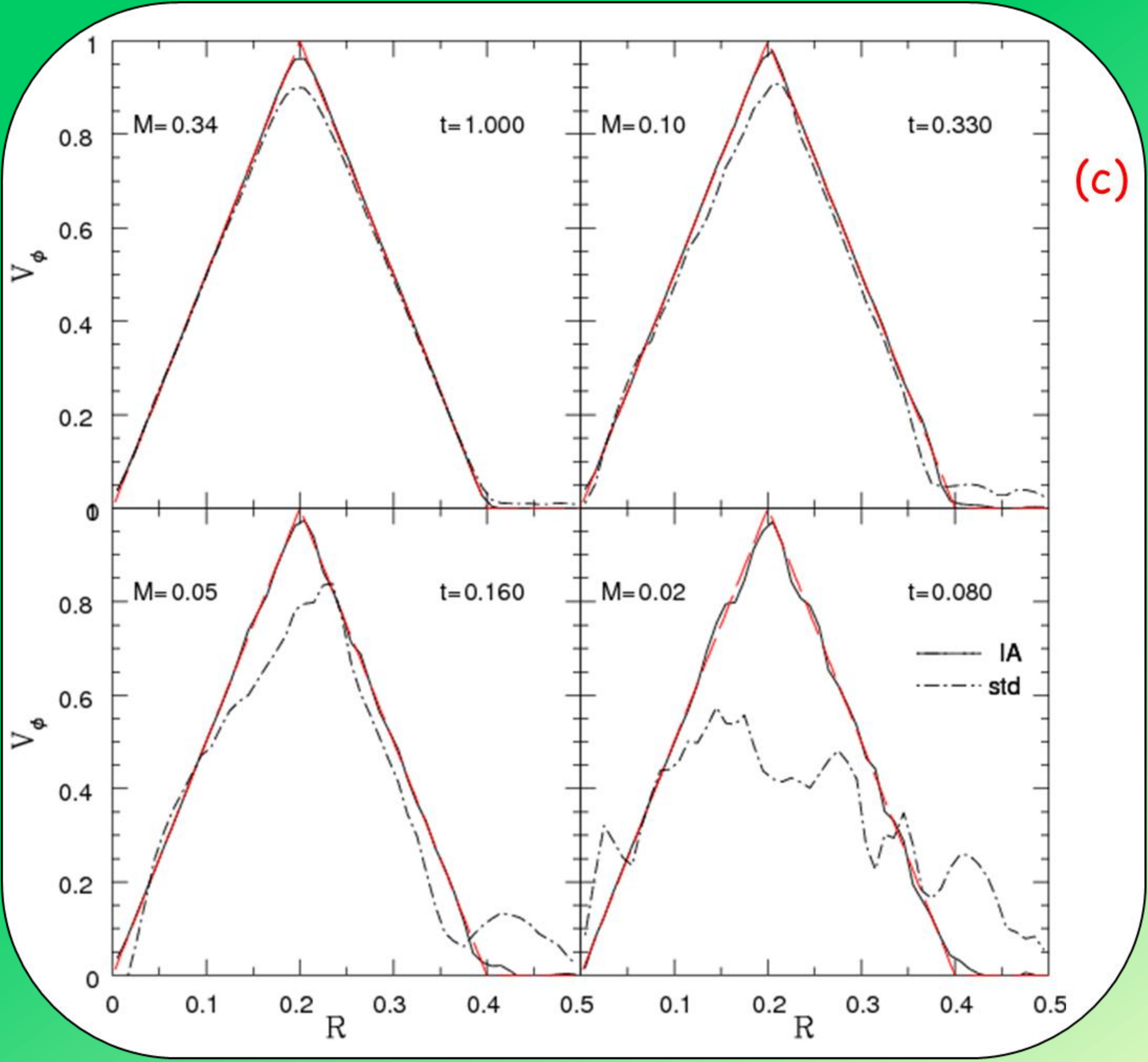
- Velocity profiles at t=1 M=0.34 N=128 for different kernels (a)
- Convergence rate L1 vs N , t=1 M=0.34 - std SPH + IA (b)
- Velocity profiles at t=3M for $M = \{0.02,\ 0.05,\ 0.1,\ 0.34\}$ (c)



(a)



Arepo (Springel 10)



(c)
M=0.34
t=1.000
M=0.10
t=0.330
M=0.05
t=0.160
M=0.02
t=0.080
IA
std
V_ϕ
R
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5

MAIN RESULTS

- IA-SPH : $V_{\varphi}(R)$ ~ analytical solution even for M <<1
- convergence rate L1(N) ~ 1/N much better than standard and close to that of mesh codes

SUBSONIC TURBULENCE

SPH simulations of driven subsonic turbulence show velocity power spectra P(k) with a very narrow inertial range ($P(k) \propto k^{-5/3}$, Kolgomorov scaling) when compared against spectra extracted from Eulerian simulations (Bauer & Springel 12 =BS12) or Lagrangian-FV codes (Hopkins 15, Gizmo)

Given the importance of subsonic turbulence in many astrophysical problems, this is a serious shortcoming of SPH

Improving the AV scheme alleviates the problem but does not solve it (Price 12)

The origin of the difficulties is due to SPH gradient errors $\longrightarrow$ velocity noise $\longrightarrow$ higher impact as Mach number $\longrightarrow$ 0

Here we run a set of simulations using the same IC of previous authors (BS12, Price 12, Hopkins 15, Zhu+15)

We consider an isothermal gas in a periodic 3D box with

$$\begin{cases} L = 1 & \rho = 1 \\ \gamma = 1 & c_s = 1 \\ \vec{V}_{in} = 0 & \end{cases}$$

Turbulence is driven by adding to the momentum equation an external stochastic force $\vec{a}_{stir}$ at each step with power spectrum

$$\begin{array}{ccc} P(k)\,[\vec{a}_{stir}] & \propto & k^{-5/3} \\ P(k) \neq 0 & & k_{min} = 2\pi/L \leq k \leq 2k_{min} \end{array}$$

The phases are drawn from an Ornstein-Uhlenbeck (UO) process

$$\begin{array}{ccc} x_{n+1} & = & f x_n + \sigma\sqrt{1-f^2}\, z_n \\ < x_n >= 0 & & < x_{n+1} x_n >= \sigma^2 f \\ f = e^{-dt/t_s} & & z_n \quad Gaussian \end{array}$$

We enforce a pure solenoidal driving by applying an Helmholtz decomposition

$$\vec{a}(k,t) = \vec{b}(k,t) - \vec{k}(\vec{b}\cdot\vec{k})/k^2 \qquad \vec{a}\cdot\vec{k} = 0$$

For the driving parameters see BS12

The power spectrum is normalized so that M ~ 0.25 – 0.3 after steady-state is reached (t ≥ 5)

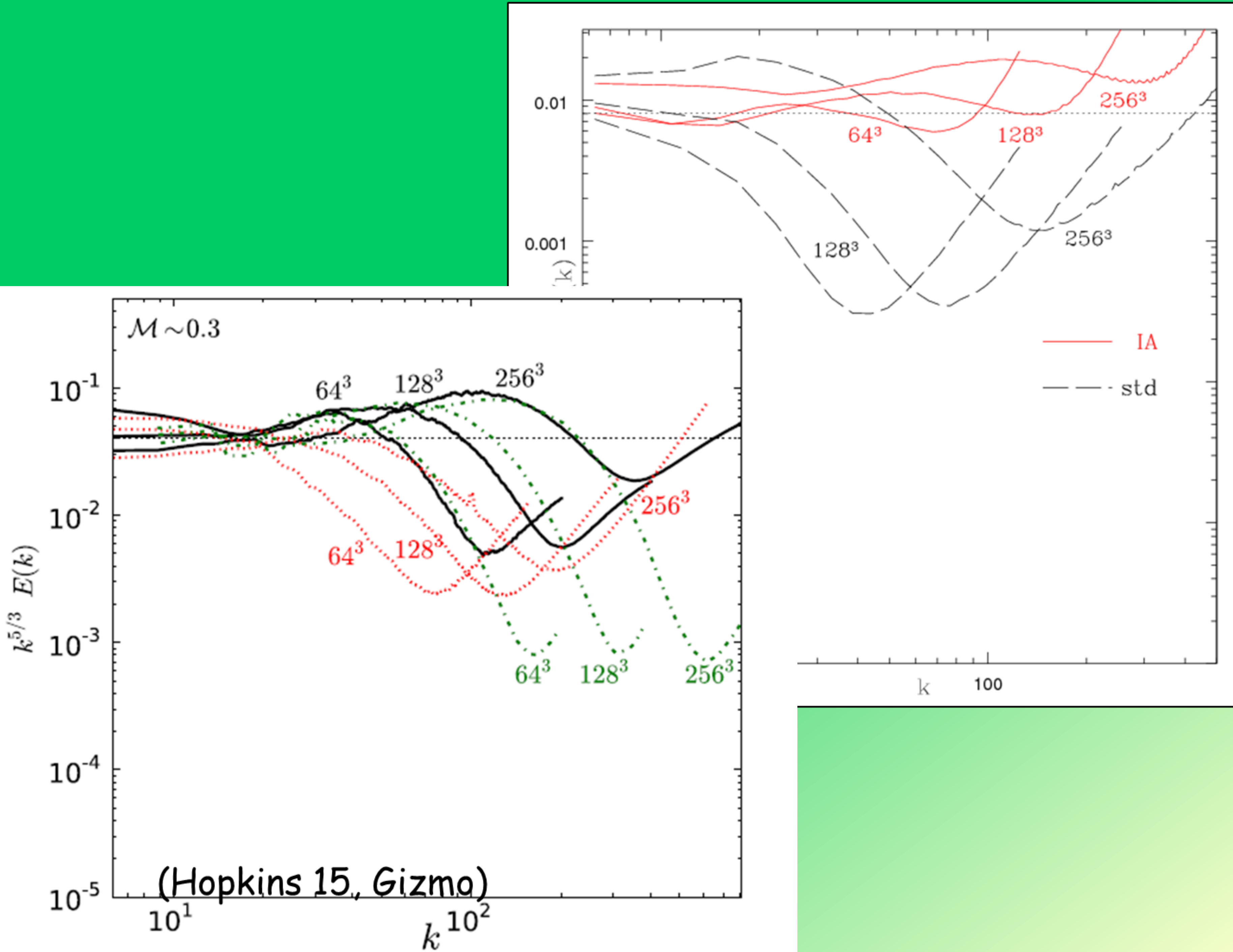
Our IC set-up consists of a HCP lattice with $N^3 = 64^3,\ 128^3,\ 256^3$

We measure the velocity power spectrum

$$E(k) = 2\pi k^2 \mathcal{P}(k) \qquad < \vec{\tilde{v}}^{\dagger}(\vec{k}')\vec{\tilde{v}}(\vec{k}) >= \delta_D(\vec{k}' - \vec{k})\mathcal{P}(k)$$

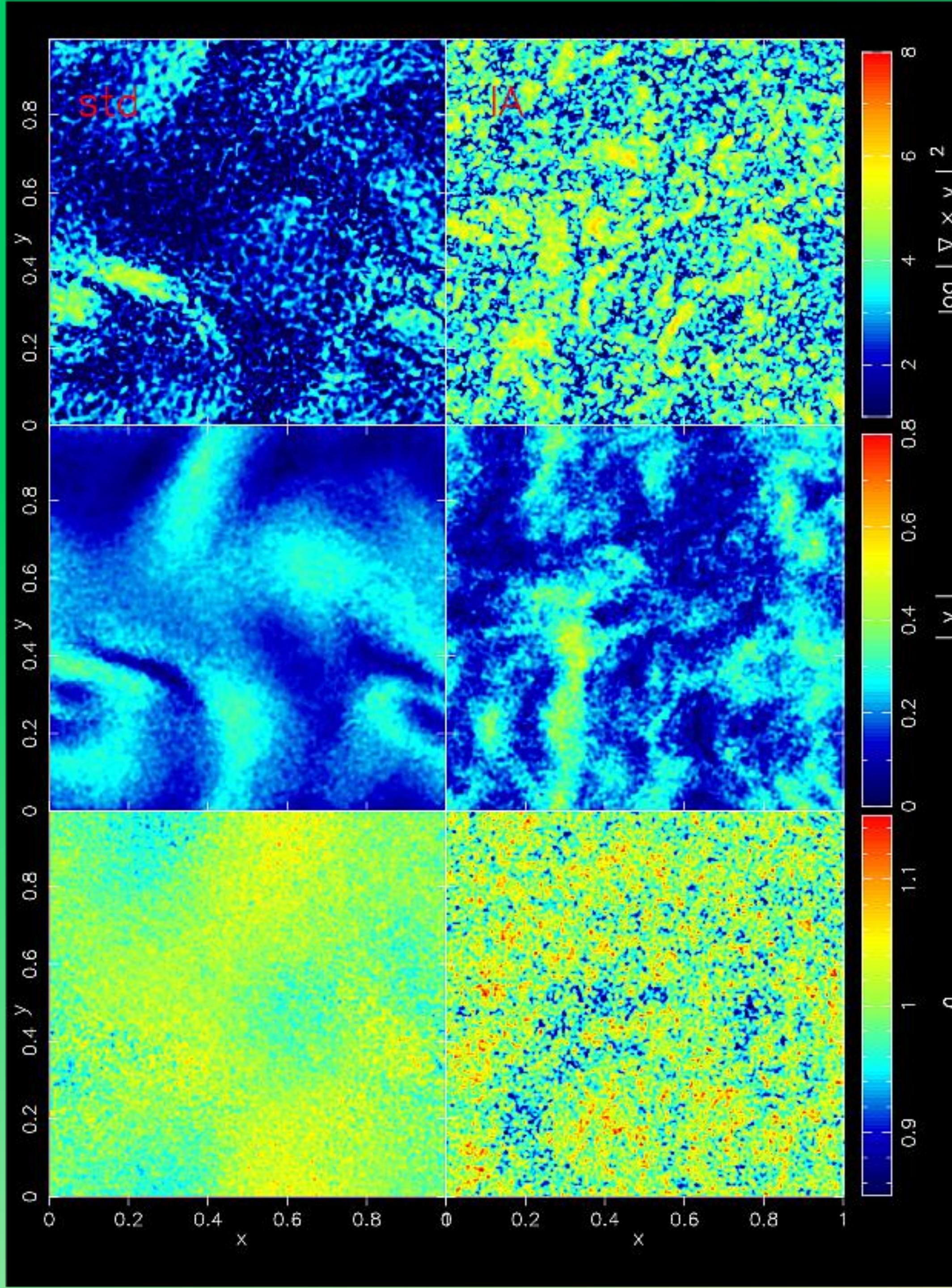
by averaging between t=5 and t=25

For incompressible turbulence we expect $E(k) \propto k^{-5/3}$



(Hopkins 15, Gizmo)

2D maps of density (bottom), velocity (middle), ensthropy (top)
N=128 t=50 z=L/2
left :std SPH right : IA



RESULTS

- standard SPH: E(k) behaviour in line with previous results
  - very small inertial range
  - decline as k gets higher , presence of a minimum
  - turnaround and increase as $k \geq k_{turn}(N)$
- By contrast, IA-SPH exhibits an intertial range which covers almost a decade in k
  - spectra are now in accord with results from mesh runs
  - the improvement is dramatic and confirms the effectiveness of the IA approach to almost eliminate gradient errors

CONCLUSIONS

- The results of our tests demonstrate that incorporating the IA method in SPH drastically reduces zeroth-order errors
- We find the IA-SPH formulation to give very good result in the modeling of subsonic flows, outperforming standard SPH
- Moreover, for the tests presented here, in terms of accuracy the code behavior can be considered competitive to that of other numerical schemes recently proposed
- The new IA-SPH scheme can then be used in many astrophysical problems where subsonic flows have a significant impact
- For example in galaxy clusters, where turbulence adds a contribution to the ICM pressure and affects mass estimates